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On a certain case of asymptotic stability of the integral $y = 0$ of the differential equation $dy/dx = g(y/x)$

In [2] sufficent conditions of asymptotic stability of the integral $y = y_0$ of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

have been given. They were stated in Theorem 3 of that paper and are quoted here in the following theorem.

THEOREM 1. *Let us assume that:*

1° *function* $f(x, y)$ *is defined and continuous in the plane set*

$$D = \{(x, y): x \in \langle a, +\infty), y \in (y_0 - \alpha, y_0 + \alpha)\}$$

(*where* $\alpha > 0$; *in particular it may be* $\alpha = +\infty$);

2° $f(x, y) \leqslant 0$ *in* D_1 *and* $f(x, y) \geqslant 0$ *in* D_2, *where*

$$D_1 = \{(x, y): x \in \langle a, +\infty), y \in (y_0, y_0 + \alpha)\},$$
$$D_2 = \{(x, y): x \in \langle a, +\infty), y \in (y_0 - \alpha, y_0)\};$$

3° *there exist the limits*:

$$\lim_{\substack{x\to+\infty\\ y\searrow\bar{y}}} f(x, y) = \delta, \qquad \delta < 0 \text{ for each } \bar{y} \in (y_0, y_0 + \alpha),$$

$$\lim_{\substack{x\to+\infty\\ y\nearrow\bar{y}}} f(x, y) = \gamma, \qquad \gamma > 0 \text{ for each } \bar{y} \in (y_0 - \alpha, y_0).$$

Then $y = y_0$ *is a solution of* (1) *which is asymptotic uniformly stable with respect to the initial conditions given on every fixed segment*

$$K = \{(x, y): x = x_0, y \in \langle y_0 - \beta, y_0 + \beta\rangle\},$$

where $0 < \beta < \alpha$ *and* $x_0 \geqslant a$.

In the case when $\delta = 0$ or $\gamma = 0$ or else $\delta = \gamma = 0$ a stable solution $y = y_0$ may be or may not be asymptotic stable. Corresponding examples can be found in 2 and 3 of [2].

We shall consider here the case when both limits in 3 are equal zero and we shall give a sufficient condition, which, for homogeneous equations will decide of asymptotic stability of the stable integral $y = 0$. This stability need not be, as we shall show, uniform with respect to the initial conditions.

THEOREM 2. *If*

1° *function* $g(u)$ *is continuous in* $(-u_0, u_0)$, *where* $u_0 > 0$ (*in particular* $u_0 = +\infty$);

2° $g(u) < 0$ *for* $u \in (0, u_0)$, $g(u) > 0$ *for* $u \in (-u_0, 0)$;

3° *through every point of the set* $E_1 \cup E_2$, *where*

$$E_1 = \{(x, y):\ x \in (0, +\infty), 0 < y < u_0 x\},$$
$$E_2 = \{(x, y):\ x \in (0, +\infty), -u_0 x < y < 0\},$$

passes exactly one integral of the equation

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right); \tag{2}$$

4° *for every* $b \neq 0$

$$\lim_{\substack{x\to+\infty\\ y\to b}} x g\left(\frac{y}{x}\right) = \eta, \qquad \eta \neq 0$$

exists, then the integral $y = 0$ *of* (2) *is asymptotic stable uniformly with respect to the initial conditions given on every fixed segment*

$$K_1 = \{(x, y):\ x = x_0, y \in \langle -y_0, y_0\rangle\},$$

where $x_0 < 0$, $0 < y_0 < u_0 x_0$.

Proof. That the integral $y = 0$ of (2) is asymptotic stable follows in a simple manner from Theorem 6 of [1].

Indeed, putting in that Theorem $a = 0$ we see that integral $y = 0$ of (2) is asymptotic stable.

From condition 3° of our theorem we get that this stability is uniform with respect to initial conditions given on every fixed interval of the type K_1.

Remark 1. Theorem 1 must not be applied for investigation of asymptotic stability of equation (2) since then we have for every $\bar{y} > 0$

$$\lim_{\substack{x\to+\infty\\ y\searrow \bar{y}}} g\left(\frac{y}{x}\right) = 0$$

and for every $\bar{y} < 0$

$$\lim_{\substack{x\to+\infty\\ y\nearrow \bar{y}}} g\left(\frac{y}{x}\right) = 0.$$

We shall now show that in this case asymptotic stability of the integral $y = 0$ of (2) does not have to be uniform (with respect to the initial conditions given simultaneously on all segments K_1 for $x_0 > 0$, $0 < y_0 < u_0 x_0$).

The integral $y = 0$ of (2) is asymptotic stable uniformly with respect to the initial conditions given simultaneously on all segments of the form K_1 if

1. this integral is asymptotic stable,

2. for every $\varepsilon > 0$ there exists a constant $A > 0$ depending only of ε such that for $x > x_0 + A$ holds the inequality $|\varphi(x)| < \varepsilon$ for all integrals $\varphi(x)$ originating from any segment of the form

$$K_1 = \{(x, y): x = x_0, y \in \langle -y_0, y_0\rangle\}$$

for every $x_0 > 0$ and fixed $y_0 \in (0, u_0 x_0)$.

THEOREM 3. *Assume that $g(u)$ is a function satisfying conditions of Theorem 2, and also, let $g(u)$ be strictly decreasing in $(-u_0, u_0)$. Then asymptotic stability of the integral $y = 0$ of (2) is not uniform with respect to the initial conditions.*

Proof. We shall first show that for fixed y_0 and any $0 < \varepsilon < y_0$ there is no constant $A > 0$ depending only on ε such that $x > x_0 + A$ would imply $|\varphi(x)| < \varepsilon$ for all integrals $\varphi(x)$ of (2) originating from any segment of the form

$$K_1 = \{(x, y): x = x_0, y \in \langle -y_0, y_0\rangle\},$$

where $x_0 > 0$, $u_0 x_0 > y_0$.

To show this we shall construct a sequence of segments

$$\langle x_0, x_0 + A_0\rangle, \langle x_1, x_1 + A_1\rangle, \ldots, \langle x_n, x_n + A_n\rangle, \ldots,$$

where $A_0 > 0$, $A_1 > 0, \ldots, A_n > 0, \ldots$ and a sequence of integrals

$$\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$$

of (2), such that the following conditions hold:

$$\lim_{n\to\infty} A_n = +\infty, \tag{3}$$

$$\varphi_n(x_n) = y_0, \quad \varphi_n(x_n + A_n) = \varepsilon, \tag{4}$$

where y_0 and ε are fixed and such that $0 < \varepsilon < y_0$. We fix next a point $x_0 > 0$ such that $x_0 > \bar{x}$, where $y_0 = u_0 \bar{x}$, and $\varphi_0(x)$ will denote an integral of (2) satisfying the initial conditions

$$\varphi_0(x_0) = y_0 \tag{5}$$

and a constant $A_0 > 0$ is chosen so that

(6) $$\varphi_0(x_0+A_0) = \varepsilon.$$

Such a constant exists, since $0 < \varepsilon < y_0$ and $\varphi_0(x)$ decreases asymptotically to zero when $x \to +\infty$. Continuing our procedure we denote by x_1 a coordinate of the point in which the straight lines $y = \frac{\varepsilon}{x_0+A_0}x$ and $y = y_0$ cross, and by $\varphi_1(x)$ the integral of (2) satisfying the initial condition

(7) $$\varphi_1(x_1) = y_0.$$

The constant $A_1 > 0$ is chosen so that

(8) $$\varphi_1(x_1+A_1) = \varepsilon.$$

Since

$$\varepsilon < y_0 = \frac{\varepsilon}{x_0+A_0}x_1$$

then

(9) $$x_1 > x_0+A_0.$$

We shall now prove that

(10) $$A_1 > A_0.$$

To show this, in view of the strict monotonicity of $\varphi_1(x)$ and of the equality $\varphi_1(x_1+A_1) = \varepsilon$, it is enough to prove that

(11) $$\varphi_1(x_1+A_0) > \varepsilon.$$

It is easy to verify that the integral $y = \varphi_1(x)$ of (2) can be represented as $y = \frac{1}{c}\varphi_0(cx)$, where

(12) $$c = \frac{x_0+A_0}{x_1} = \frac{\varepsilon}{y_0}.$$

Thus

(13) $$\varphi_1(x) \equiv \frac{y_0}{\varepsilon}\varphi_0\left(\frac{x_0+A_0}{x_1}x\right).$$

Assembling (5), (6), (7), (12), (13) together we obtain

(14) $$\begin{aligned}\varphi_1(x_1+A_0) &= \frac{1}{c}\varphi_0(cx_1+cA_0)-\frac{1}{c}\varphi_0(cx_1)-[\varphi_0(x_0+A_0)-\varphi_0(x_0)]+\varepsilon\\ &= A_0\varphi_0'(cx_1+\theta_1 cA_0)-A_0\varphi_0'(x_0+\theta_2 A_0)+\varepsilon\\ &= A_0[\varphi_0'(x_0+A_0+c\theta_1 A_0)-\varphi_0'(x_0+\theta_2 A_0)]+\varepsilon\\ &= A_0[\varphi_0'(\bar{x})]-\varphi_0'(\bar{\bar{x}})]+\varepsilon,\end{aligned}$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$, $\bar{x} = x_0+A_0+c\theta_1 A_0$, $\bar{\bar{x}} = x_0+\theta_2 A_0$.

We see that $\bar{x} > \bar{\bar{x}}$ and since $\varphi_0(x)$ and $g(u)$ are strictly decreasing we get

$$\varphi_0'(\bar{x}) - \varphi_0'(\bar{\bar{x}}) = g\left(\frac{\varphi_0(\bar{x})}{\bar{x}}\right) - g\left(\frac{\varphi_0(\bar{\bar{x}})}{\bar{\bar{x}}}\right) > 0,$$

and from this last inequality combined with (14) follows (11), because, according to our assumptions $A_0 > 0$.

In general, after defining the $(n-1)$-th interval $\langle x_{n-1}, x_{n-1}+A_{n-1}\rangle$ and the $(n-1)$-th integral $\varphi_{n-1}(x)$ so that

$$\varphi_{n-1}(x_{n-1}) = y_0$$

and

$$\varphi_{n-1}(x_{n-1}+A_{n-1}) = \varepsilon$$

hold, we define next interval $\langle x_n, x_n+A_n\rangle$ and next integral $\varphi_n(x)$ in the following fashion:

Let x_n designates a coordinate of the point in which the straight

$$y = \frac{\varepsilon}{x_{n-1}+A_{n-1}}\, x$$

and $y = y_0$ meet.

Let, further, $\varphi_n(x)$ designate an integral of (2) satisfying the initial condition

$$\varphi_n(x_n) = y_0. \tag{15}$$

The constant $A_n > 0$ is chosen so that the equality

$$\varphi_n(x_n+A_n) = \varepsilon \tag{16}$$

is satisfied.

For every n such a constant A_n exists, since $0 < \varepsilon < y_0$ and $\varphi_n(x)$ asymptotically decreases to zero when $x \to +\infty$.

Since $\varepsilon < y_0 = \dfrac{\varepsilon}{x_{n-1}+A_{n-1}}\, x_n$, then

$$x_n > x_{n-1}+A_{n-1}, \tag{17}$$

$$A_n > A_{n-1}. \tag{18}$$

The proof of this last inequality is similar to that of (10), so we shall omit it here. We shall now show that condition (3) holds.

In fact, from the mean value theorem and from (15) and (16) it follows that

$$\varphi_n(x_n+A_n) - \varphi_n(x_n) = A_n\varphi_n'(\xi_n)$$

where

$$x_n < \xi_n < x_n + A_n, \tag{19}$$

and further

$$A_n = \frac{\varepsilon - y_0}{\varphi_n'(\xi_n)} = \frac{\varepsilon - y_0}{g\left(\frac{\varphi_n(\xi_n)}{\xi_n}\right)}. \tag{20}$$

(17), (18), (19) yield together a sequence of inequalities

$$\xi_n > x_n > x_{n-1} + A_{n-1} > x_{n-2} + A_{n-2} + A_{n-1} > \ldots > x_0 + A_0 + A_1 + \\ + \ldots + A_{n-1} > x_0 + nA_0,$$

from which we have $\lim_{n\to\infty} \xi_n = +\infty$, and since $\varepsilon < \varphi_n(\xi_n) < y_0$ thus

$$\lim_{n\to\infty} \frac{\varphi_n(\xi_n)}{\xi_n} = 0.$$

Now, from the above and (20) we obtain (3) since $g(u)$ is continuous in $(-u_0, u_0)$.

Condition (4) is clearly satisfied for every $n = 0, 1, 2, \ldots$ To end the proof of Theorem 3 it suffices to notice that for fixed y_0 and $0 < \varepsilon < y_0$, for every number $A > 0$, we may find, by (3) and (4) such $A_n > A$, a point x_n and an integral $\varphi_n(x)$ of (2) originating from the end of the interval

$$\tilde{K}_n = \{(x, y):\ x = x_n,\ y \in \langle -y_0, y_0 \rangle\},$$

that the following holds

$$\varphi_n(x_n + A_n) = \varepsilon.$$

From this it obviously follows

$$\varphi_n(x_n + A) > \varepsilon,$$

and so stability of the solution $y = 0$ is not uniform with respect to the initial conditions.

Remark 2. From the way the theorem 3 was proved it is evident that it may be slightly generalized if we assume that the integral $y = 0$ of (2) is asymptotic stable in place of Condition 4 of Theorem 2.

Example. Theorem 3 implies that the solution $y = 0$ of the equation

$$\frac{dy}{dx} = -\frac{y}{x}$$

is asymptotic stable in the set

$$D = \{(x, y):\ x \in \langle a, +\infty),\ y \in (-\infty, +\infty)\},$$

where $a > 0$, but this stability is not uniform with respect to the initial conditions.

References

[1] Z. Kamont and W. Pawelski *O pewnym warunku wystarczającym asymptotycznej stabilności całki $y = ax$ równania różniczkowego $dy/dx = f(y/x)$*, Zeszyty Naukowe Politechniki Gdańskiej, Matematyka 5 (1969), p. 3-18.

[2] W. Pawelski, *On a simple case of asymptotic stability*, Comm. Math. 13 (1970), p. 233-239.