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Schauder bases for the space of continuous functions on an n-dimensional cube

1. Introduction. The purpose of this paper is to describe explicitly three constructions of Schauder bases for $C(I^n)$ (two of which are known) and generalize a theorem of Ciesielski [3] to the case of n variables. Ciesielski's theorem reads as follows: Let $(\varphi_n)_{n=1}^{\infty}$ be the classical basis for $C(I)$ constructed by Schauder [7] for the dyadic partition of $I = [0, 1]$. Let $0 < a < 1$. If $f = \sum a_n \varphi_n \in C(I)$, then the following conditions are equivalent:

(i) $a_n = O(1/n^a)$,

(ii) $\|f - S_n\| = O(1/n^a)$,

(iii) the function f satisfies the Hölder condition with the exponent a i.e. $\omega_f(\delta) = O(\delta^a)$.

(Actually, Ciesielski formulated the theorem in a more general way replacing the function δ^a by a function $\omega(\delta)$ satisfying conditions (3.1), (4.1), (4.2) written in Sections 3 and 5 below.)

Using this theorem Ciesielski constructed an isomorphism $f \mapsto (a_n \cdot n^a)_{n=1}^{\infty}$ between the space H_a of Hölder functions on I and the space l_∞ of bounded sequences. He also proved that, under this isomorphism, the subspace c_0 is the image of the subspace Λ_a of the functions in H_a satisfying the condition

$$|f(x_1) - f(x_2)| = o(|x_1 - x_2|^a).$$

About 1960 Ciesielski raised the question whether an analogical characterization of Hölder functions is true for functions of several variables. In 1969 Ciesielski and Gęba solved the problem in positive, but they did not published the solution, as one of the main corollaries of the theorem — the existence of an isomorphism from the space $H_a^{(n)}$ of Hölder functions of n variables onto l_∞ — was obtained by Bonic, Frampton and Tromba [2], independently and at about the same time.

In this paper the theorem of Ciesielski will be proved for an n-dimensional cube I^n and for a certain class of bases in $C(I^n)$ which contains the squew pyramidal basis considered by Ciesielski and Gęba, and the regular pyramidal basis obtained by generalizing of construction Ellis and Kuehner [5]. The proof is analogous to that of Ciesielski; the only essential difference is in lemma (4.9) for which it was necessary to modify the estimations involved. It is worth mentioning that this theorem is not a direct consequence of results of [2].

Three types of Schauder bases for the space $C(I^n)$ are constructed bellow. Two of them consist of piecewise affine functions (i.e. each function is affine on each simplex belonging to a certain sequence of triangulations of the cube I^n). The third basis (the cube basis) consists of piecewise n-linear functions, the supports of which are cubes. The first two bases were known, but their constructions were not published explicitly. The fourth known basis type, the product basis consisting of functions of the form

$$\varphi_{m_1,\ldots,m_n}(x_1, \ldots, x_n) = \varphi_{m_1}(x_1)\cdot \ldots \cdot\varphi_{m_n}(x_n)$$

is not considered here. Let us note that all these bases are interpolating in the sense of Semadeni [8]. The basis described by Vaher [9] in her proof of existence a basis in $C(Q)$ (where $Q \subset I^\omega$), the basis constructed by Bessaga [1] for $C(I^\omega\|H) = \{f\in C(I^\omega)\colon f|H = 0\}$ and the interpolating basis of Gurarij [6] also consist of piecewise affine functions and are analogous to the pyramidal bases.

2. Preliminaries. A sequence $(e_i)_{i=1}^\infty$ in a Banach space X is called a *Schauder basis* (or simply *basis*) if for every x in X there exists a unique sequence of scalars $(a_i(f))_{i=1}^\infty$ such that the sequence of partial sums $\sum_{i=1}^{n} a_i(f)e_i$ converges strongly to x. $C(K)$ will denote the space of all real-valued continuous functions on K with $\|f\| = \sup\{|f(t)|\colon t\in K\}$. A basis $(e_i)_{i=1}^\infty$ is called *interpolating with nodes* $(t_n)_{n=1}^\infty$ if for each function f in $C(K)$ and each n we have

$$\sum_{i=1}^{n} a_i(f)e_i(t_m) = f(t_m) \quad (\text{for } m = 1, \ldots, n).$$

(2.1) Lemma. *Let K be a compact metric space and let $(e_i)_{i=1}^\infty$ be a sequence of functions in $C(K)$ satisfying the following conditions*:

(i) *for any f in $C(K)$ there exists a sequence $(a_i)_{i=1}^\infty$ of scalars such that*

$$f = \sum_{i=1}^{\infty} a_i e_i, \tag{1}$$

(ii) *there exists a sequence* $(t_j)_{j=1}^{\infty}$ *in* K *such that*

$$e_i(t_j) = \delta_{ij} \quad (\text{for } j = 1, \ldots, i).$$

Then the sequence $(e_i)_{i=1}^{\infty}$ *is an interpolating Schauder basis for* $C(K)$ *with nodes* $(t_j)_{j=1}^{\infty}$.

Proof. Let $f = \sum_{i=1}^{\infty} b_i e_i$. Then we have

$$f(t_1) = b_1,$$

$$f(t_n) = \sum_{i=1}^{\infty} b_i e_i(t_n) = b_n + \sum_{i=1}^{n-1} b_i e_i(t_n) \qquad (\text{for } n = 2, \ldots).$$

Consequently, the sequence $(a_i)_{i=1}^{\infty}$ is unique.

I^n will denote the n-dimensional cube (the Cartesian product of n copies of I). For x, y in I^n we define

$$d(x, y) = \max\{|x_i - y_i|: i = 1, \ldots, n\}.$$

If $f \in C(I^n)$, $\omega_f(\delta)$ will denote the modulus of continuity of f i.e.

$$\omega_f(\delta) = \sup\{|f(x) - f(y)|: d(x, y) \leqslant \delta\}.$$

By a simplex we mean a closed simplex in $\boldsymbol{R}^n$. By a triangulation of the cube K we mean a set of simplexes $\{T_i\}_{i=1}^{k}$ such that $\bigcup_{i=1}^{k} T_i = K$ and $T_i \cap T_j$ is a face of each of simplexes T_i, T_j.

3. Bases of piecewise affine functions (pyramidal bases). Let $(R^{(i)})_{i=0}^{\infty}$ be a sequence of triangulations of the cube I^n and let $W^{(i)}$ be a set of all vertices of simplexes of the triangulation $R^{(i)}$. Let us suppose that the sequences $(R^{(i)})_{i=0}^{\infty}$, $(W^{(i)})_{i=0}^{\infty}$ satisfy the following conditions:

(3.1) *The triangulation* $R^{(i+1)}$ *is a subdivision of a triangulation* $R^{(i)}$ *i.e. for every simplex* T *in* $R^{(i+1)}$ *there exists a simplex* T_1 *in* $R^{(i)}$ *which contains the simplex* T *as a proper subset* (*hence* $W^{(i+1)} \supset W^{(i)}$)

(3.2) $$\sup\{\operatorname{diam} T: T \in R^{(i)}\} = d_i \to 0 \text{ as } i \to \infty.$$

Now, let $V^{(i)} = W^{(i)} \setminus W^{(i-1)}$ (for $i = 1, 2, \ldots$); $V^{(0)} = W^{(0)}$; let $v \in V^{(i)}$ and let φ_v be a function in $C(I^n)$ satisfying the conditions: $\varphi_v(v) = 1$, and $\varphi_v(w) = 0$ (for $w \in W^{(i)} \setminus \{v\}$) and φ_v is affine on each simplex of the triangulation $R^{(i)}$.

The set $W = \bigcup_{i=0} W^{(i)}$ will be ordered as follows. If $v \in W^{(i)}$ and $w \in W \setminus W^{(i)}$, then $v < w$. In each set $V^{(i)}$ we introduce the lexicographical order. This yields a sequence $w_1, w_2, \ldots$ of elements of W.

(3.3) PROPOSITION. *The sequence of functions* $(\varphi_{w_n})_{n=1}^{\infty}$ *is an interpolating Schauder basis with nodes* $w_1, w_2, \ldots$

Proof. Let $f \in C(I^n)$. Write

$$a_v = a_v(f) = f(v) \quad (\text{for } v \in V^{(0)}),$$
$$S_v = \sum_{w \leqslant v}' a_w \varphi_w, \qquad S^{(i)} = \sum_{w \in W^{(i)}} a_w \varphi_w,$$
$$a_v = f(v) - S^{(i)}(v) \quad (\text{for } v \in V^{(i)}).$$

We shall prove (by induction respect to i) that

$$(2) \qquad f(w) = S_v(w) \quad (\text{for } w \leqslant v,\ v \in V^{(i)}).$$

If $i = 0$, then

$$S_v(w) = \sum_{u \leqslant v} a_u \varphi_u(w) = f(w).$$

Let us suppose that (2) is true for some i. If $v \in V^{(i+1)}$ and $w \leqslant v$, then we have

$$S_v(w) = S^{(i)}(w) + f(w) \quad (\text{for } w \in V^{(i+1)}),$$
$$S_v(w) = S^{(i)}(w) = f(w) \quad (\text{for } w \in W^{(i)})$$

(by induction hypothesis).

Let us note that the function $S^{(i)}$ is affine on each simplex of the triangulation $R^{(i)}$. Hence

$$|f(x) - S^{(i)}(x)| \leqslant |f(x) - f(w)| + |f(w) - S^{(i)}(w)| + |S^{(i)}(w) - S^{(i)}(x)|$$
$$\leqslant \omega_f(d_i) + 0 + \omega_f(d_i),$$

where w is any vertex of the simplex T in $R^{(i)}$ containing x. Consequently,

$$|a_v| \leqslant 2\omega_f(d_i) \quad (\text{for } v \in V^{(i+1)})$$

and

$$|f(x) - S_v(x)| \leqslant |f(x) - S^{(i)}(x)| + \sum_{w \in V^{(i+1)}} (i+1) a_w \varphi_w(x)$$
$$\leqslant 2(n+2)\,\omega_f(d_i) \quad (\text{for } v \in V^{(i+1)}).$$

The last inequality means that the sequence of partial sums S_v converges to f uniformly on I^n. From Lemma (2.1) it follows that the sequence $(\varphi_{w_n})_{n=1}^{\infty}$ is a Schauder basis for $C(I^n)$.

4. The Hölder functions. For any n-dimensional simplex T and any $a \in \boldsymbol{R}^n \setminus \{0\}$ let

$$a(a, T) = \sup\{d(x, y):\ x, y \in T \text{ and } x - y = \lambda a \ (\text{for some } \lambda \in \boldsymbol{R})\},$$
$$a(T) = \inf\{a(a, T):\ a \in \boldsymbol{R}^n \setminus \{0\}\}.$$

Let us assume that $(R^{(i)})_{i=0}^{\infty}$ is a sequence of the triangulations of I^n satisfying (3.1) and the following conditions

(4.1) $\sup\{\operatorname{diam} T\colon T\in R^{(i)}\} = d_i \leqslant A\cdot 2^{-i}$,

(4.2) $\inf\{a(T)\colon T\in R^{(i)}\} \geqslant B\cdot 2^{-i}$.

(4.3) Lemma. *There exists a positive integer n_0 such that for any points x, y in I^n and any i the condition $d(x,y)\leqslant 2^{-i}$ implies the existence of a sequence of points $z_0, \ldots, z_{n_0}$ satisfying the following conditions: $x = z_0$, $y = z_{n_0}$ and for each $k = 1, \ldots, n_0$ there is simplex T_k of the triangulation $R^{(i)}$ such that both z_{k-1}, z_k belong to T_k.*

Proof. If $d(x,y)\leqslant 2^{-i}$, then x and y belong to a cube

$$\Gamma = [a_1, b_1]\times\ldots\times[a_n, b_n], \qquad \text{where } b_i - a_i = 2^{-i},$$

Let $\bar{\Gamma} = \{z\in I^n\colon d(z,\Gamma)\leqslant A\cdot 2^{-i}\}$, $\tilde{\Gamma} = \{T\in R^{(i)}\colon T\cap\Gamma\neq\emptyset\}$. The volume of $\bar{\Gamma}$ is not greater than $[(2A+1)\cdot 2^{-i}]^n$ and, by (4.2), is not less than $(B\cdot 2^{-i})^n$. From (4.1) it follows that

$$\bigcup_{T\in\tilde{\Gamma}} T\subset\bar{\Gamma}.$$

Hence the number of elements of $\tilde{\Gamma}$ is not greater than

$$n_0 = \left[\left(\frac{2A+1}{B}\right)^n\right].$$

We now arrange the elements of $\tilde{\Gamma}$ in a sequence $T_1, \ldots, T_{n_0}$ (repetitions being admitted) such that $x\in T_1$, $y\in T_{n_0}$, $T_k\cap T_{k+1}\neq\emptyset$ (for $k = 1, \ldots, n_0$) and we choose any point of $T_k\cap T_{k+1}$ as z_k.

Let ω be a positive function defined on I. We shall consider the following conditions

(4.4) *ω is non-decreasing in I and there is a constant K such that*

$$\omega(2t)\leqslant K\cdot\omega(t) \qquad (\text{for } 0\leqslant t\leqslant \tfrac{1}{2}).$$

(4.5) *There is a constant L such that*

$$\int_0^{\delta}\frac{\omega(t)}{t}\,dt\leqslant L\cdot\omega(\delta) \qquad (\text{for } 0<\delta\leqslant 1).$$

(4.6) *There is a constant M such that*

$$\delta\int_{\delta}^{1}\frac{\omega(t)}{t^2}\,dt\leqslant M\cdot\omega(\delta) \qquad (\text{for } 0<\delta\leqslant 1).$$

Note that the function $\omega(\delta) = \delta^{\alpha}$ satisfies (4.4), (4.5), (4.6).

(4.7) THEOREM. *Let ω satisfy conditions* (4.4), (4.5) (4.6) *and let* $(R^{(i)})_{i=0}^{\infty}$ *satisfy conditions* (3.1), (4.1), (4.2). *Then for every f in $C(I^n)$ the following conditions are equivalent*:

(i) $\sup_{v\in V^{(i)}} |a_v(f)| = O[\omega(2^{-i})]$,

(ii) $\sup_{v\in V^{(i)}} \|f-S_v(f)\| = O[\omega(2^{-i})]$,

(iii) $\omega_f(\delta) = O[\omega(\delta)]$,

where $a_v(f)$, $S_v(f)$ are as in (3.3).

This theorem is a consequence of the following lemmas, which are somewhat more general and more explicit versions of the implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii). Implication (iii) $\Rightarrow$ (i) follows from an estimation of the coefficients $a_v(f)$ shown in the proof of (3.3).

(4.8) LEMMA. *Let ω satisfy conditions* (4.4) (4.5), *let $f\in C(I^n)$ and let k be an integer such that*

$$\sup_{v\in V^{(i)}} |a_v(f)| \leqslant \omega(2^{-i}) \quad (\textit{for } i = k, k+1, \ldots).$$

Then

$$\sup_{v\in V^{(i)}} \|f-S_v(f)\| \leqslant 2K\cdot L\omega(2^{-i}) \quad (\textit{for } i = k, k+1, \ldots).$$

Proof. Let $v\in V^{(i)}$ $(i = k, k+1, \ldots)$, and $x\in I^n$. Then

$$|(f-S_v)(x)| \leqslant \sum_{j=i}^{\infty} \sum_{v\in V^{(j)}} |a_w|\varphi_w(x) \leqslant \sum_{j=i}^{\infty} \sup_{w\in V^{(j)}} |a_w|\cdot \sum_{w\in V^{(j)}} \varphi_w(x)$$
$$\leqslant \sum_{j=i}^{\infty} \omega(2^{-j}) \leqslant 2\cdot \sum_{j=i}^{\infty} 2^{-j}\frac{\omega(2^{-j})}{2^{-j+1}} \leqslant 2\cdot \sum_{j=i} \int_{2^{-j}}^{2^{-j+1}} \frac{\omega(t)}{t}\,dt$$
$$\leqslant 2\int_{0}^{2^{-i+1}} \frac{\omega(t)}{t}\,dt \leqslant 2L\omega(2^{-i+1}) \leqslant 2\cdot K\cdot L\omega(2^{-i}).$$

(4.9) LEMMA. *Let ω be a non-decreasing positive function in I, let $f\in C(I^n)$ and let k be an integer such that*

$$\sup_{v\in V^{(i)}} \|f-S_v(f)\| \leqslant \omega(2^{-i}) \quad (\textit{for } i = k+1, \ldots).$$

Then there exists a positive constant D such that

$$\omega_f(\delta) \leqslant D\cdot\delta\cdot\left(2\omega(1) + \int_{\delta}^{1} \frac{\omega(t)}{t^2}\,dt\right) \quad (\textit{for } \delta \leqslant 2^{-k-1}).$$

Proof. Let us assume that x, y belong to the same simplex T of the triangulation $R^{(i)}$ $(i > k)$. For any $j \leqslant i$ let T_j be a simplex of $R^{(j)}$ such that $T \subset T_j$ and v be a vertex of T_j which belongs to $V^{(j)}$. By (4.2) there exist points x_j, y_j $(x_j, y_j \in T_j)$ such that $d(x_j, y_j) \geqslant B \cdot 2^{-j}$ and $x_j - y_j$ is parallel to $x - y$. We also have

$$d(x, y) \leqslant A \cdot 2^{-i} \quad \text{and} \quad |\varphi_v(x_j) - \varphi_v(y_j)| \leqslant 1.$$

Hence (from similarity of respective triangles)

$$|\varphi_v(x) - \varphi_v(y)| = \frac{d(x, y)}{d(x_1, y_1)} |\varphi_v(x_j) - \varphi_v(y_j)| \leqslant \frac{A}{B} \cdot 2^{j-i} \quad (\text{for } j = k+1, \ldots, i),$$

$$|S^{(k)}(x) - S^{(k)}(y)| = \frac{d(x, y)}{d(x_k, y_k)} |S^{(k)}(x_k) - S^{(k)}(y_k)|$$

$$\leqslant \frac{A}{B} \cdot 2^{k-i} \cdot \omega_f(A \cdot 2^{-k}).$$

Let us also note that

$$\sup_{v \in V^{(j+1)}} |a_v| \leqslant \|f - S^{(j)}\| \leqslant \omega(2^{-j}) \quad (\text{for } j = k, k+1, \ldots).$$

Using the above inequalities we obtain

$$|f(x) - f(y)| \leqslant |S^{(k)}(x) - S^{(k)}(y)| + \sum_{j=k+1}^{i} \sum_{v \in V^{(j)}} |a_v| \cdot |\varphi_v(x) - \varphi_v(y)| + 2\|f - S^{(i)}\|$$

$$\leqslant \frac{A}{B} \cdot 2^{k-i} \omega_f(A \cdot 2^{-k}) + \sum_{j=k+1}^{i} (\sup_{v \in V^{(j)}} |a_v|) \sum_{v \in V^{(j)}} |\varphi_v(x) - \varphi_v(y)| + 2\omega(2^{-i})$$

$$\leqslant \frac{A}{B} 2^{k-i} \omega_f(A \cdot 2^{-k}) + \frac{A}{B} \cdot 2^{-i} \cdot (n+1) \cdot \sum_{j=k+1}^{i} 2^j \omega(2^{-j+1}) + 2\omega(2^{-i})$$

$$\leqslant D_0 \cdot 2^{-i} \sum_{j=0}^{i} 2^{j-1} \cdot \omega(2^{-j}) \leqslant D_0 \cdot 2^{-i} \cdot \sum_{j=1}^{2^i} \omega(j^{-1}).$$

Now let x^0, y^0 be two distinct points of I^n. Then there exists an integer i_0 such that

$$2^{-i_0-1} < d(x^0, y^0) \leqslant 2^{-i_0}.$$

Let us assume that $i_0 > k$. By Lemma (4.3) we obtain

$$|f(x^0)-f(y^0)| \leqslant n_0 \cdot D_0 \cdot 2^{-i_0} \cdot \sum_{j=1}^{2^{i_0}} \omega(j^{-1}) \leqslant 2n_0 D_0 \cdot 2^{-i_0} \cdot \Bigg[\omega(1) + $$

$$+ \sum_{j=1}^{2^{i_0}-1} \int_{(j+1)^{-1}}^{j^{-1}} \frac{\omega(t)}{t^2}\, dt\Bigg] \leqslant 2n_0 D_0 \cdot 2^{-i_0}\Bigg[\omega(1) + \int_{2^{-i_0}}^{1} \frac{\omega(t)}{t^2}\, dt\Bigg]$$

$$\leqslant 4n_0 D_0 d(x^0, y^0)\Bigg[\omega(1) + \int_{d(x^0,y^0)}^{1} \frac{\omega(t)}{t^2}\, dt\Bigg].$$

Since the function $\left[t \cdot \int_t^1 \frac{\omega(u)}{u^2}\, du + \int_0^t \omega(u)\, du\right]$ is non-decreasing in I we have

$$|f(x^0)-f(y^0)| \leqslant D \cdot \left[\delta\left(\omega(1) + \int_\delta^1 \frac{\omega(t)}{t^2}\, dt\right) + \int_0^\delta \omega(t)\, dt\right]$$

$$\leqslant D \cdot \delta \cdot \left[2\omega(1) + \int_\delta^1 \frac{\omega(t)}{t^2}\, dt\right] \qquad (\text{for } \delta \geqslant d(x^0, y^0))$$

Thus the lemma is proved.

5. The squew pyramidal basis. We shall now present a construction of the sequence $(R^{(i)})_{i=0}^{\infty}$. If $g_1, \ldots, g_n$ is any orthogonal basis in $\boldsymbol{R}^n$ satisfying the condition $\|g_1\| = \ldots = \|g_n\|$, and g_0 is an arbitrary element of $\boldsymbol{R}^n$, then $K(g_0, g_1, \ldots, g_n) = \{x = g_0 + \sum_{k=1}^{n} x_k g_k \in \boldsymbol{R}^n \colon 0 \leqslant x_k \leqslant 1\}$ will denote the cube spanned by the elements $g_0, g_1, \ldots, g_n$. Let S_n be the set of all permutations of the set $\{1, \ldots, n\}$, let $r = (i_1, \ldots, i_n)$ belong to S_n and let $K = K(g_0, \ldots, g_n)$. Let $T_r(K)$ be the simplex of the vertices $g_0, g_0 + g_{i_1}, g_0 + g_{i_1} + g_{i_2}, \ldots, g_0 + g_{i_1} + \ldots + g_{i_n}$. The triangulation $\Delta(K) = \{T_r(K)\}_{r \in S_n}$ will be used for the construction of the desired basis. Let $x = g_0 + \sum_{k=1}^{n} x_k g_k \in K$ and let $1 \geqslant x_{i_1} \geqslant \ldots \geqslant x_{i_n} \geqslant 0$. We have

$$x = g_0 + \sum_{k=1}^{n} x_k g_k = (1 - x_{i_1}) g_0 + \sum_{k=1}^{n-1} (x_{i_k} - x_{i_{k-1}}) \cdot (g_0 + g_{i_1} + \ldots + g_{i_k}) +$$

$$+ x_{i_n} \cdot (g_0 + g_{i_1} + \ldots + g_{i_n}).$$

Denoting $t_0 = 1 - x_{i_1}$, $t_k = x_{i_k} - x_{i_{k-1}}$, $t_n = x_{i_n}$ we obtain $\sum_{k=0}^{n} t_k = 1$, $t_k \geqslant 0$. This means that x belongs to the simplex $T_{(i_1, \ldots, i_n)}(K)$. The point x belongs to the interior of $T_{(i_1, \ldots, i_n)}(K)$ if and only if all the coefficients $t_0, \dots, t_n$ are positive (i.e. $x_i \neq x_j$ for $i \neq j$). There is exactly one way to arrange such a sequence $x_1, \dots, x_n$ into a non-increasing sequence $x_{i_1}, \dots, x_{i_n}$; therefore the point x may belong to the interior of only one simplex.

The cube K is now divided into 2^n cubes $K_1, \dots, K_{2^n}$ of faces parallel to those of K (consider hyperplanes H_i, where $H_i \perp g_i$ and $g_0 + \frac{1}{2}g_i \in H_i$). $Q(K)$ will denote the set $\{K_1, \dots, K_{2^n}\}$. $\{T_r(K_i)\}_{r \in S_n} = \Delta(K_i)$ is a triangulation of the cube K_i. We shall demonstrate that for each simplex $T_r(K_i)$ there exists a simplex $T_{r_1}(K)$ (in the triangulation of the cube K) which contains $T_r(K_i)$. It is enough to show that the union of the faces of simplexes $T_r(K)$ is contained in the union of the faces of the simplexes $T_r(K_i)$ $(i = 1, \dots, 2^n)$. The point $x = g_0 + \sum_{k=1}^{n} x_k g_k$ belongs to a face of a simplex $T_r(K)$ if and only if $x_i = x_j$ for some $i \neq j$, or $x_j = 0$, or $x_j = 1$. If $x_j = 0$ of $x_j = 1$ the point x belongs to a face of cube K hence it belongs to a face of cube K_{i_0} and to a face of a simplex $T_{r_0}(K_{i_0})$ as well. Let $x_k > 0$ (for $k = 1, \dots, n$), $x_i = x_j$. The point x may be written as

$$x = g + \sum_{k=1}^{n} \varepsilon_k g_k + \tfrac{1}{2} \sum_{k=1}^{n} y_k g_k, \qquad \text{where } \varepsilon_k = 0 \text{ or } \varepsilon_k = \tfrac{1}{2},\ 0 < y_k \leqslant 1$$

(i.e. $y_k = 2x_k - 2\varepsilon_k$). If $\varepsilon_i = \varepsilon_j$, then $y_i = y_j$ and x belongs to a face of a simplex $T_{r_1}(K_{i_1})$. If $\varepsilon_i \neq \varepsilon_j$, then $|y_i - y_j| = 2|\varepsilon_i - \varepsilon_j| = 1$ this contradicts the condition $0 < y_i \leqslant 1$, $0 \leqslant y_j \leqslant 1$.

Let us write

$$D_k = \{(2p-1)2^{-k}\}_{p=1,\dots,2^{k-1}}; \qquad D_0 = \{0, 1\}; \qquad D = \bigcup_{k=0}^{\infty} D_k,$$

$$D_k^n = \Big\{\tau = (\tau_1, \dots, \tau_n) \in D^n \colon\ \tau_i \in \bigcup_{j=0}^{k} D_j \text{ and } \underset{i_0}{\exists}\ \tau_{i_0} \in D_k\Big\}.$$

Thus $D^n = \bigcup_{k=0}^{\infty} D_k^n$; if $k_1 \neq k_2$, then $D_{k_1}^n \cap D_{k_2}^n = \emptyset$. We shall need two transformations:

$$^{+}\colon D^n \setminus D_0^n \to D^n \quad \text{and} \quad ^{-}\colon D^n \setminus D_0^n \to D^n.$$

If $\tau = (\tau_1, \dots, \tau_n) \in D_k^n$, then $\tau^+ = (\tau_1^+, \dots, \tau_n^+)$ and $\tau^- = (\tau_1^-, \dots, \tau_n^-)$, where

$$\tau_i^{\pm} = \begin{cases} \tau_i \pm 2^{-k} & \text{for } \tau_i \in D_k, \\ \tau_i & \text{for } \tau_i \in \bigcup_{j=0}^{k-1} D_j. \end{cases}$$

Now $Q^{(0)}, Q^{(1)}, \ldots$ will denote a sequence of sets of cubes defined by induction:

$$Q^{(0)} = \{I^n\},$$
$$Q^{(k+1)} = \bigcup_{K \in Q^{(k)}} Q(K) \quad (\text{for } k = 0, 1, \ldots).$$

Let $U^{(k)} = \bigcup_{K \in Q^{(k)}} \Delta(K)$. The set $\bigcup_{i=0}^{k} D_i^n$ is a set of all vertices of the division $Q^{(k)}$, hence also of $U^{(k)}$. The sets D_0^2, D_1^2, D_2^2 and the triangulations $U^{(0)}, U^{(1)}, U^{(2)}$ are shown in Fig. 1. If $T_1, T_2 \in U^{(k)}$, $T_1 \neq T_2$, then

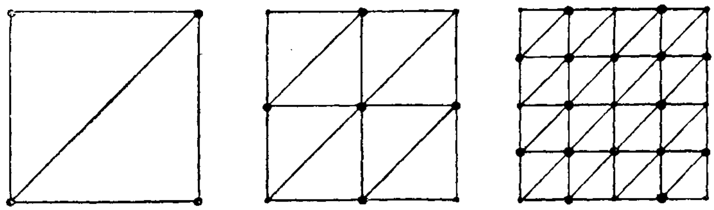

Fig. 1

$\operatorname{int} T_1 \cap \operatorname{int} T_2 = \emptyset$. For any integer $k \geqslant 1$ and for any simplex $T \in U^{(k)}$ there exists a simplex $T_1 \in U^{(k-1)}$ containing T. Let us note that for each $\tau \in D_k^n$ the points τ^+, τ^- are vertices of the same simplex in $U^{(k-1)}$. Obviously τ^+, τ^- are vertices of the same cube $K \in Q^{(k-1)}$. Let the cube K be spanned by elements $e, 2^{-k+1} \cdot e_1, \ldots, 2^{-k+1} \cdot e_n$ and let

$$\tau_{i_s}^+ \neq \tau_{i_s}^- \quad (\text{for } s = 1, \ldots, q), \qquad \tau_i^+ = \tau_i^- \quad (\text{for } i \notin \{i_1, \ldots, i_q\}),$$
$$\tau^- = e + \sum_{s=1}^{p} 2^{-k+1} e_{j_s}.$$

It is clear that $\{i_1, \ldots, i_q\} \cap \{j_1, \ldots, j_p\} = \emptyset$. Let $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_q, j_1, \ldots, j_p\} = \{l_1, \ldots, l_{n-p-q}\}$ and let r be a permutation $(j_1, \ldots, j_p, i_1, \ldots, i_q, l_1, \ldots, l_{n-p-q})$. The points τ^+ and τ^- are vertices of the simplex $T_r(K)$.

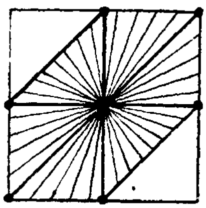

Fig. 2

If we assume that $W^{(k)} = \bigcup_{j=0}^{k} D_j$, $R^{(k)} = U^{(k)}$ and the functions φ_τ are defined as in (3.3), then by Proposition (3.3) the sequence $(\varphi_\tau)_{\tau \in D^n}$ is a basis for $C(I^n)$. The support of φ_{τ_7} is shown in Fig. 2.

The simplexes of U are similar to the respective simplexes of $U^{(0)}$. Hence if we write

$$\alpha(n) = \inf\{\alpha(T):\ T \in U^{(0)}\},$$

then

$$\inf\{\alpha(T):\ T \in U^{(k)}\} = 2^{-k}\cdot\alpha(n), \qquad \sup\{\operatorname{diam} T:\ T \in U^{(k)}\} = 2^{-k},$$

i.e. conditions (4.1) and (4.2) are satisfied. Hence the basis $(\varphi_\tau)_{\tau\in D^n}$ satisfies the assumption of Theorem (4.7).

We are going to compute the coefficients $a_\tau(f)$. Let $\tau \in D^n_k$ (for $k = 1, \ldots$)

$$\begin{aligned} a_\tau &= f(\tau) - S^{(k-1)}(\tau) = f(\tau) - S^{(k-1)}\big(2^{-1}(\tau^+ + \tau^-)\big) \\ &= f(\tau) - 2^{-1}\big(S^{(k-1)}(\tau^+) + S^{(k-1)}(\tau^-)\big) = f(\tau) - \tfrac{1}{2}\big(f(\tau^+) + f(\tau^-)\big). \end{aligned}$$

Let us note that if we change the succession of the coordinate axes, then the triangulations $U^{(k)}$ and the functions φ_τ also change.

6. The regular pyramidal basis. Let K be any n-dimensional cube. We shall define its triangulation $V(K)$ into $2^{n-1}\cdot n!$ simplexes by induction respect to n. For an interval J (a 1-dimensional cube) we define $V(J) = \{J\}$. Now let K be an $(n+1)$-dimensional cube. The n-dimensional faces L_i (for $i = 1, \ldots, 2(n+1)$) of a cube K may be regarded as n-dimensional cubes. By induction hypothesis we have triangulations $V(L_i)$ (for $i = 1, \ldots, 2(n+1)$). Let s be the barycenter of the cube K. The set of simplexes $V(K) = \{\operatorname{conv}(T, s):\ T \in V(L_i),\ i = 1, \ldots, 2(n+1)\}$ is the desired triangulation.

Let us denote

$$E_n^{(i)} = \bigcup_{K\in Q^{(i)}} V(K) \qquad (\text{for } i = 0, 1, \ldots),$$

where $Q^{(k)}$ are as in Section 5. Obviously $E_n^{(i)}$ are triangulations of the cube I^n. We can prove that the triangulation $E_n^{(i)}$ is a subdivision of the trian-

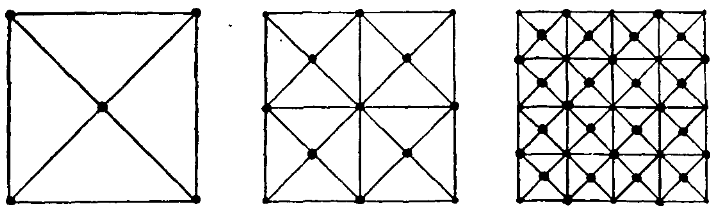

Fig. 3

gulation $E_n^{(i-1)}$. Denoting $R^{(i)} = E_n^{(i)}$ we obtain, just as in preceding section, a sequence of triangulations of I^n satisfying conditions (3.1), (4.1), (4.2). Hence the basis coresponding to the sequence $E_n^{(i)}$ satisfies the assumption of Theorem (4.7). Let us note that for this basis the triangulations do not depend on the order of coordinate axes. The triangulations $E_2^{(0)}, E_2^{(1)}, E_2^{(2)}$ are shown in Fig. 3.

7. The cube basis. Let ψ be a function defined by

$$\psi(x) = \begin{cases} x+1 & \text{for } -1 \leqslant x < 0, \\ 1-x & \text{for } 0 \leqslant x \leqslant 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

For $\tau = (\tau_1, \ldots, \tau_n) \in D_k^n$ we define the function $\psi_\tau \in C(I^n)$

$$\psi_\tau(x) = \prod_{i=1}^{n} \psi\big(2^k(x_i - \tau_i)\big), \quad \text{where } x = (x_1, \ldots, x_n).$$

Let us note that the function ψ_τ is n-linear on each cube K in $Q^{(k)}$ and its support is a cube.

(7.1) PROPOSITION. *The sequence* $(\psi_\tau)_{\tau \in D^n}$ *is a basis for* $C(I^n)$.

Proof. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$; $\tau = (\tau_1, \ldots, \tau_n) \in D_k^n$,

$$\tau^\varepsilon = (\tau_1^\varepsilon, \ldots, \tau_n^\varepsilon), \quad \text{where} \quad \tau_i^\varepsilon = \begin{cases} \tau_i + \varepsilon_i \cdot 2^{-k} & \text{for } \tau_i \in D_k, \\ \tau_i & \text{for } \tau_i \in \bigcup_{j=0}^{k-1} D_j. \end{cases}$$

The coefficients $c_\tau(f) = c_\tau$ are defined as

$$c_\tau = f(\tau) \qquad \text{for } \tau \in D_0^n,$$

$$c_\tau = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} \big(f(\tau) - f(\tau^\varepsilon)\big) \qquad \text{for } \tau \in \bigcup_{j=1}^{\infty} D_j^n.$$

We shall prove that the series $\sum_{\tau \in D^n} c_\tau \psi_\tau$ converges to the function f uniformly. Let

$$S^{(k)}(x) = \sum_{j=0}^{k} \sum_{\tau \in D_j^n} c_\tau \psi_\tau(x).$$

We claim that

$$S^{(k)}(\tau) = f(\tau) \quad \text{for } \tau \in \bigcup_{j=0}^{k} D_j^n. \tag{3}$$

Indeed, for $k = 0$ and $\tau^0 \in D_0^n$ we have

$$S^{(0)}(\tau^0) = \sum_{\tau \in D_0^n} c_\tau \psi_\tau(\tau^0) = c_{\tau^0} = f(\tau^0).$$

Let (3) hold for some k and let $\tau \in D_{k+1}^n$. Then

$$S^{(k)}(\tau) = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} S^{(k)}(\tau^\varepsilon) = 2^{-n} \sum_{\varepsilon \in \{-1,1\}^n} f(\tau^\varepsilon),$$

$$S^{(k+1)}(\tau) = S^{(k)}(\tau) + c_\tau = f(\tau).$$

Now, let $x \in I^n$. Then for each k there exists a cube K in $Q^{(k)}$ such that $x \in K$. Let V denote the set of all vertices of the cube K. Then

$$\inf\{S^{(k)}(\tau):\ \tau \in V\} \leqslant S^{(k)}(x) \leqslant \sup\{S^{(k)}(\tau):\ \tau \in V\}$$

and

$$\psi_\tau(x) = 0, \quad \text{where } \tau \in D_k^n \setminus V \ (\operatorname{card} V = 2^n),$$

and

$$|f(x) - S^{(k)}(x)| \leqslant |f(x) - f(\tau)| + |S^{(k)}(\tau) - S^{(k)}(x)| \leqslant 2\omega\ (2^{-k})$$

where $\tau \in V$. Hence for $\tau \in D_k^n$ we have $|c_\tau| \leqslant 2\omega_f(2^{-k})$.

Let $\sigma \in D_{k+1}^n$. We obtain

$$\Big|f(x) - \sum_{\tau \leqslant \sigma} c_\tau \psi_\tau(x)\Big| \leqslant |f(x) - S^{(k)}(x)| + \sum_{\tau \in D_{k+1}^n} |c_\tau| \psi_\tau(x)$$
$$\leqslant 2\cdot\omega_f(2^{-k}) + 2\cdot 2^n \cdot \omega_f(2^{-k-1}).$$

This shows that the series $\sum_{\tau \in D^n} c_\tau \psi_\tau$ is uniformly convergent and we may apply Lemma (2.1).

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