On a modular space of infinitely differentiable functions

This paper contains generalizations of results contained in papers [1], [4] and [6]. In place of spaces $D_M$ considered in the above mentioned papers, there is considered a more general space $D_\varrho$ of infinitely differentiable functions which is defined by means of a modular $\varrho$. The problem of the dual of $D_\varrho$ considered in papers [6] and [7] in the special case of $D_M$, is not investigated here.

1. First, we give the definition of a modular $\varrho$, the space $D_\varrho$ and of convergence in $D_\varrho$.

Let $\mathcal{E}$ be the space of all real infinitely differentiable functions in the real $n$-dimensional space $\mathbb{R}^n$. Let an extended real-valued functional $\varrho$, called a modular, be defined on $\mathcal{E}$, satisfying the following conditions:

(A.1) $0 \leq \varrho(\varphi) \leq \infty$; $\varrho(\varphi) = 0$, if and only if, $\varphi = 0$,

(A.2) $\varrho(\alpha \varphi_1 + \beta \varphi_2) \leq \alpha \varrho(\varphi_1) + \beta \varrho(\varphi_2)$
for all $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta = 1$,

(A.3) $\varrho(\varphi_1) \leq \varrho(\varphi_2)$ for $|\varphi_1| \leq |\varphi_2|$,

(A.4) if the support $S_\varphi$ of a function $\varphi \in \mathcal{E}$ is compact, then $\varrho(\varphi) < \infty$,

(A.5) $\varrho(\varphi_1 + \varphi_2) = \varrho(\varphi_1) + \varrho(\varphi_2)$ for functions $\varphi_1$ and $\varphi_2$ such that $S_{\varphi_1} \cap S_{\varphi_2} = \emptyset$,

(A.6) if $\lim_{i \to \infty} \varphi_i \in \mathcal{E}$, then $\varrho(\lim_{i \to \infty} \varphi_i) \leq \lim_{i \to \infty} \varrho(\varphi_i)$.

We shall say that a sequence $\{\varphi_i\}, \varphi_i \in \mathcal{E}$ is $\varrho$-convergent to zero (we write $\varphi_i \xrightarrow{\varrho} 0$), if $\varrho(\lambda \varphi_i) \to 0$ as $i \to \infty$ for some $\lambda > 0$. By $D_\varrho$ we denote the subspace of $\mathcal{E}$ which consists of all functions $\varphi \in \mathcal{E}$ such that for every multiindex $p = (p_1, p_2, \ldots, p_n)$ ($p_i \geq 0$) there exists a constant $\lambda_p > 0$ satisfying the condition $\varrho(\lambda_p D^p \varphi) < \infty$, where $D^p$ is the operator of differentiation:

$$D^p = \partial^{p_1 + \cdots + p_n} / \partial x_1^{p_1} \cdots \partial x_n^{p_n}.$$
In the space $D^e$ we define a sequence of pseudonorms

$$\|\varphi\|^m = \sup_{|p| \leq m} \|D^p\varphi\|_{(e)}, \quad m = 0, 1, 2, \ldots,$$

where $|p| = p_1 + \ldots + p_n$ and

$$\|\varphi\|_{(e)} = \inf \left\{ k : \varphi \left( \frac{\varphi}{k} \right) \leq 1 \right\}.$$

Obviously, the topology defined in $D^e$ by means of the base of neighbourhoods of zero

$$V(m, \varepsilon) = \{ \varphi \in D^e : \|\varphi\|^m \leq \varepsilon \}, \quad m = 0, 1, 2, \ldots, \varepsilon > 0,$$

is equivalent to the topology defined in $D^e$ by means of the $F$-norm

$$\|\varphi\|_e = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\|\varphi\|^m}{1 + \|\varphi\|^m}.$$

A sequence $\{\varphi_i\}, \varphi_i \in D^e$ will be called $\tilde{\varepsilon}$-convergent to zero (we write $\varphi_i \xrightarrow{\tilde{\varepsilon}} 0$) if $D^p\varphi_i \xrightarrow{\tilde{\varepsilon}} 0$ for every $p$.

We shall say that the modular $\varphi$ satisfies condition (a) (we write $\varphi \in (a)$), if for every sequence $\{\varphi_i\}, \varphi_i \in D^e$ such that $\varphi_i \xrightarrow{\tilde{\varepsilon}} 0$ there exists a constant $a > 0$ satisfying the inequality $|\varphi_i(t)| \leq a$ for every $t \in \mathbb{R}^n$ and every $i$.

Now, we formulate some lemmas needed in the sequel.

**Lemma 1.** Let $\varphi \in \mathcal{E}$. Then $\varphi(\varphi) < \infty$, if and only if,

$$\lim_{k \to \infty} \varphi \left( \varphi \left( \sum_{j=k}^{\infty} \omega_j \right) \right) = 0,$$

where the functions $\omega_j \in \mathcal{E}$, $\omega_j$ of compact support, which give a partition of unity, are defined as in [4], Lemma 1.

**Proof.** Let $\varphi(\varphi) < \infty$, and let $\varphi_i = \sum_{k=1}^{i} \varphi \omega_k$. Since $\varphi(\varphi_i) \leq \varphi(\varphi)$ and $\lim_{i \to \infty} \varphi_i = \varphi$, we have

$$\varphi(\varphi) \leq \lim_{i \to \infty} \varphi(\varphi_i) \leq \varphi(\varphi),$$

i.e.

$$\lim_{i \to \infty} \varphi(\varphi_i) = \varphi(\varphi).$$

Let us choose an arbitrary $\varepsilon > 0$. Then there exists an $i_0$ such that

$$\varphi(\varphi) - \varphi(\varphi_i) < \varepsilon \quad \text{for} \quad i \geq i_0.$$
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Hence
\[ e(\varphi) \geq \varphi \left( \sum_{k=1}^{t} \varphi \omega_k + \sum_{k=t+2}^{\infty} \varphi \omega_k \right) = e(\varphi_t) + e \left( \sum_{k=t+2}^{\infty} \varphi \omega_k \right), \]
and so we get
\[ e \left( \sum_{k=t+2}^{\infty} \varphi \omega_k \right) \leq e(\varphi) - e(\varphi_t) < \varepsilon \quad \text{for } i \geq i_0. \]
This proves (1.1).

Now, let us suppose
\[ e \left( \varphi \sum_{j=k_0}^{\infty} \omega_j \right) < \varepsilon \quad \text{for an } \varepsilon > 0 \text{ and a } k_0 > 1. \]
Let a number \( a > 0 \) be chosen in such a manner that \( \omega_{k_0}(t) = 1 \) for \( |t| = a \).
Writing
\[ A = \{t: |t| \leq a\}, \quad B = \{t: |t| > a\}, \]
we obtain
\[ S_{k_0-1} \subset A \quad \text{and} \quad \sum_{j=k_0}^{\infty} \omega_j \chi_B = \chi_B. \]
Hence, one may construct non-negative functions \( \omega_{j,i} \in \mathcal{E} \) such that
\[ S_{\omega_{j,i}} \subset \left\{ t \in R^n: (j-1) \frac{a}{i} < |t| < j \frac{a}{i} + \frac{a}{2^i} \right\}, \]
where \( j = 1, 2, \ldots, i = 1, 2, \ldots, \) and \( \sum_{j=1}^{\infty} \omega_{j,i} = 1 \) for \( i = 1, 2, \ldots \)

Let us define two sequences of functions
\[ \varphi_{1,i} = \varphi \sum_{j=1}^{i} \omega_{j,i}, \quad \varphi_{2,i} = \varphi \sum_{j=i+1}^{\infty} \omega_{j,i}. \]
It is easily observed that for each \( i \),
\[ A \subset S_{\varphi_{1,i}}, \quad S_{\varphi_{1,i}} \subset B, \quad S_{\varphi_{1,i}} \cap S_{\varphi_{2,i}} = \emptyset. \]
Since \( \lim_{t \to \infty} \varphi_{1,i} = \varphi \chi_A \) and \( \lim_{t \to \infty} \varphi_{2,i} = \varphi \chi_B \), we get \( \lim_{t \to \infty} (\varphi_{1,i} + \varphi_{2,i}) = \varphi \).
Hence we obtain
\[ e(\varphi) \leq \lim_{t \to \infty} [e(\varphi_{1,i} + \varphi_{2,i})] = \lim_{t \to \infty} [e(\varphi_{1,i}) + e(\varphi_{2,i})]
\leq \lim_{t \to \infty} [e(\varphi \sum_{j=1}^{k_0+1} \omega_j) + e \left( \varphi \sum_{i=k_0}^{\infty} \omega_j \right)] < \infty. \]
This proves the lemma.
Lemma 2. If \( q \in (a) \), then every \( \tilde{q} \)-convergent to zero sequence \( \{q_i\} \) is convergent almost uniformly to zero.

Proof. Let \( q_i \xrightarrow{\tilde{q}} 0 \). By the condition \((a)\), there exists a sequence of positive numbers \( \{a_p\} \) such that the inequality \(|D_p q_i(t)| \leq a_p \) holds for every \( t \in \mathbb{R}^n \) and each \( i \). Let \( p^j = (p_1, p_2, \ldots, p_{j-1}, p_j+1, p_{j+1}, \ldots, p_n) \).

By the mean-value theorem,

\[
|D_p q_i(t) - D_p q_i(t')| = \left| \sum_{j=1}^{n} D_p q_i(\xi_j)(t_j - t'_j) \right| \leq a_p' \sum_{j=1}^{n} |t_j - t'_j|,
\]

where \( a_p' = \max_{1 \leq j \leq n} a_p^j \). Thus the functions \( D_p q_i \) satisfy Lipschitz condition, and so they are equicontinuous functions for each fixed \( p \), uniformly bounded according to the assumption. Applying Arzelà theorem, the diagonal method yields a subsequence \( \{q_{i_k}\} \) such that \( D_p q_{i_k} \rightarrow q_p \) almost uniformly for every \( p \) separately. It is easily seen that

(2.1) \( \quad D_p q_{i_k} \rightarrow D_p q \) almost uniformly,

where \( q = q_a \) and \( \varphi \in \mathcal{E} \). Next, we have \( \varphi(\lambda \varphi) = \lim_{t \to \infty} \varphi(\lambda \varphi_{i_k}) = 0 \), i.e. \( \varphi = 0 \). From this and from formula (2.1) it follows that

(2.2) \( \quad q_{i_k} \rightarrow 0 \) almost uniformly.

It is easily observed that the sequence \( \{q_i\} \) is also almost uniformly convergent to zero. Thus, we proved the lemma.

Lemma 3. If \( q \in (a) \), then the space \( D_q \) is complete.

Proof. Let \( \{q_i\} \) be a Cauchy sequence in \( D_q \), i.e. for each \( p \) and every constant \( \lambda > 0 \),

\[
\varphi[\lambda(D_p q_{i+k} - D_p q_i)] \rightarrow 0
\]

uniformly with respect to \( k \). By the condition \((a)\), we have

(3.1) \( \quad |D_p q_{k}(t) - D_p q_{0}(t)| \leq a_p \)

for every \( k \). Now, we prove the function \( D_p q_0 \) to be bounded. Indeed, let \( \varphi(\lambda D_p q_0) < \infty \); then, by Lemma 1,

\[
\lim_{t \to \infty} \varphi(\lambda D_p q_0 \sum_{k=1}^{\infty} \omega_k) = 0.
\]

Hence, by \((a)\), there exists a constant \( \beta_p > 0 \) such that \( |D_p q_0 \sum_{k=1}^{\infty} \omega_k| \leq \beta_p \)
for each \( i \), that is \(|D^p\varphi_0| \leq \beta_p\). From this and from (3.1) it follows that

\[ |D^p\varphi_k| \leq a_p + \beta_p = K_p. \]

Arguments analogous to those applied in the proof of Lemma 2 show that for every \( p \), \( D^p\varphi_{ik} \to D^p\varphi \) almost uniformly, where \( \varphi \in \mathcal{S} \). Consequently, applying Cauchy condition, we have for fixed \( p \)

\[
\varrho[\lambda(D^p\varphi_i - D^p\varphi)] = \varrho[\lim_{s \to \infty} \lambda(D^p\varphi_{i+k+s} - D^p\varphi_{i})] \\
\leq \lim_{s \to \infty} \varrho[\lambda(D^p\varphi_{i+k+s} - D^p\varphi_{i})] \leq \varepsilon
\]

for \( i \geq i_k \) and for every constant \( \lambda > 0 \). Hence \( \varphi_i \to \varphi \). It is also easily seen that \( \varphi \in D_\varepsilon \), and the proof is finished.

**Lemma 4.** Let \( \varrho(\lambda_p D^p \varphi) < \infty \) for every \( p \) and for a fixed sequence of positive numbers \( \{\lambda_p\} \). Let

\[
\tilde{\varrho}(\{\lambda\}, \varphi) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\sup_{|p| \leq m} \varrho(\lambda_p D^p \varphi)}{1 + \sup_{|p| \leq m} \varrho(\lambda_p D^p \varphi)};
\]

then the condition \( \varphi_i \xrightarrow{i \to \infty} 0 \) is equivalent to the condition

\[
(4.1) \quad \lim_{i \to \infty} \tilde{\varrho}(\{\lambda_p\}, \varphi_i) = 0.
\]

**Proof.** Let \( \varphi_i \xrightarrow{i \to \infty} 0 \), i.e. \( \varrho(\lambda_p D^p \varphi_i) \to 0 \), and let us choose an \( \varepsilon > 0 \). We take \( k_0 \) so large that \( \sum_{m=k_0+1}^{\infty} \frac{1}{2^m} < \frac{\varepsilon}{4} \) and we choose \( i_0 \) in such a manner that \( \varrho(\lambda_p D^p \varphi_i) < \frac{\varepsilon}{8} \) for \( |p| \leq k_0 \) and \( i \geq i_0 \). Then

\[
\tilde{\varrho}(\{\lambda\}, \varphi_i) \leq \frac{\varepsilon}{2} \sum_{m=1}^{k_0} \frac{1}{2^m} + \sum_{m=k_0+1}^{\infty} \frac{1}{2^m} \leq \varepsilon
\]

for \( i \geq i_0 \). Consequently, \( \lim_{i \to \infty} \tilde{\varrho}(\{\lambda_p\}, \varphi_i) = 0 \). The converse implication is obvious.

**2.** In order to formulate the fundamental theorems concerning spaces \( D_\varepsilon \), we shall need some further notation and terminology.

A set \( A \subset D_\varepsilon \) is called bounded in \( D_\varepsilon \), if there exists a sequence of positive numbers \( \{K_p\} \) such that the inequality \( \|D^p \varphi\|_{(\varepsilon)} \leq K_p \) holds for every function \( \varphi \in A \) and for each \( p \).

A modular \( \varrho_1 \) is called non-stronger than a modular \( \varrho_2 \) (we write \( \varrho_1 \preceq \varrho_2 \)), if for every non-negative sequence of numbers \( \{\lambda_p\} \) there exists a constant \( k > 0 \) such that for every function \( \varphi \in \mathcal{S} \) satisfying the condition \( \varrho_1(\lambda_p D^p \varphi) \leq 1 \) for every \( p \), there holds the inequality \( \varrho_2(k\varphi) < 1 \).
We shall say that a modular \( q \) satisfies the condition \( (A_2) \) (we write \( q \in (A_2) \), if for every sequence of positive numbers \( \{\lambda_p\} \) there exists a constant \( a > 0 \) such that for every function \( \varphi \) satisfying the condition \( \check{q}(\{\lambda_p\}, \varphi) < a \) there holds the inequality \( q(2\varphi) < 1 \).

Now, some important theorems concerning spaces \( D_\varepsilon \) will be given.

**Theorem 5.** If \( q \in (\alpha) \), then for every sequence \( \{\varphi_i\} \), \( \varphi_i \in D_\varepsilon \) such that the supports of \( \varphi_i, S_{\varphi_i} \subset A \), where \( A \) is a compact set, the condition \( q \in (A_2) \) is equivalent to the condition \( q \in (A_2) \).

**Proof.** Let \( q \in (A_2) \) uniformly on a compact set \( A \), and let \( k_0 \) be so large that \( A \subset S_{k_0} \), where \( \omega_j \) are the same as in Lemma 1. Let \( \epsilon \sum_{j=1}^{k_0+1} \omega_j \) satisfy the inequalities \( 0 < \epsilon < 1 \), then there exists an \( i_0 \) such that \( |D^p\varphi_i| < \epsilon \) for \( i \geq i_0 \), i.e. \( q(\lambda D^p\varphi_i) \leq \varphi(\epsilon \sum_{j=1}^{k_0} \omega_j) \leq \epsilon M, \) where \( \lambda \) is an arbitrary positive number. From the last inequality it follows that \( \varphi_i \in (A_2) \).

Converse implication follows from Lemma 2, immediately.

**Theorem 6.** Let \( \varphi \in S \). Then \( \varphi \in D_\varepsilon \), if and only if, \( \sum_{j=k}^{\infty} q\omega_j \rightarrow 0 \), \( \omega_j \) being defined as in Lemma 1. Moreover, \( q(\lambda D^p\varphi) < \infty \) for every \( p \) and for every \( \lambda > 0 \), if and only if,

\[
\sum_{j=k}^{\infty} q\omega_j \rightarrow 0.
\]

Thus, the subspace \( D \) of functions from \( S \) of compact support, is dense in \( D_\varepsilon \) in the sense of \( \check{q} \)-convergence.

**Proof.** Let \( q(\lambda_p D^p\varphi) < \infty \). By Lemma 1, there holds \( \lim_{k \to \infty} q(\lambda_p D^p\varphi \sum_{j=k}^{\infty} \omega_j) = 0 \) for every \( p \). Now, we prove that for every function \( \varphi \in S \) such that \( q(\lambda \varphi) < \infty \) for some \( \lambda > 0 \), there holds

\[
\varphi D^p \left( \sum_{j=k}^{\infty} \omega_j \right) \rightarrow 0.
\]

Indeed, by [4], Lemma 1, there exists a sequence of numbers \( \{K_p\} \) such that \( |D^p(\sum_{j=k}^{\infty} \omega_j)| \leq K_p \). Then

\[
q \left[ \frac{\lambda}{K_p} \varphi D^p \left( \sum_{j=k}^{\infty} \omega_j \right) \right] = q \left[ \frac{\lambda}{K_p} \varphi D^p \left( \sum_{j=k}^{\infty} \omega_j \right) \sum_{j=k-1}^{\infty} \omega_j \right] \leq q \left[ \lambda \varphi \sum_{j=k-1}^{\infty} \omega_j \right].
\]
Hence, Lemma 1 implies
\[
\lim_{k \to \infty} \varrho \left[ \frac{\lambda}{K_p} \psi D^p \left( \sum_{j=k}^{\infty} \omega_j \right) \right] = 0
\]
and this proves (6.2). Now Leibniz formula gives
\[
D^p \left( \varphi \sum_{j=k}^{\infty} \omega_j \right) = \sum_{0 \leq |q| \leq |p|} \left( \begin{array}{c} p_1 \\ v_1 \\ \vdots \\ v_n \end{array} \right) D^{p_0} \varphi \psi D^p \left( \sum_{j=k}^{\infty} \omega_j \right),
\]
where \( p = (p_1, \ldots, p_n) \), \( v = (v_1, \ldots, v_n) \), \( p^v = (p_1 - v_1, \ldots, p_n - v_n) \).

Let
\[
\theta_p = \min_{0 \leq |q| \leq |p|} \lambda_q \left( \sum_{0 \leq |v| \leq |p|} \left( \begin{array}{c} p_1 \\ v_1 \\ \vdots \\ v_n \end{array} \right) \right)^{-1}.
\]

Then, by (6.2), we obtain
\[
\lim_{k \to \infty} \varrho \left[ \theta_p D^p \left( \varphi \sum_{j=k}^{\infty} \omega_j \right) \right] = \lim_{k \to \infty} \varrho \left[ \sum_{0 \leq |v| \leq |p|} \lambda_q D^{p_0} \varphi D^p \left( \sum_{j=k}^{\infty} \omega_j \right) \right] = 0.
\]

Now, let \( \lim_{k \to \infty} \varrho \left[ \lambda_p D^p \left( \sum_{j=k}^{\infty} \varphi \omega_j \right) \right] = 0 \); then
\[
\lim_{k \to \infty} \varrho \left[ \lambda_p D^p \varphi \sum_{j=k+1}^{\infty} \omega_j \right] = \lim_{k \to \infty} \varrho \left[ \lambda_p D^p \left( \varphi \sum_{j=k}^{\infty} \omega_j \right) \right] = 0.
\]

Hence, Lemma 1 shows that \( \varphi \in D_{\varrho} \). Converse statement is proved analogously.

**Theorem 7.** Let us consider the following conditions:

(A) if a set \( A \) is bounded in \( D_{\varrho_1} \), then \( A \) is bounded in \( D_{\varrho_2} \);

(B) if \( \varphi_i \to 0 \), then \( \varphi_i \to 0 \);

(C) \( \varrho_1 \rightarrow \varrho_2 \);

(D) \( D_{\varrho_1} \subseteq D_{\varrho_2} \).

Conditions (A), (B) and (C) are mutually equivalent. Condition (C) implies condition (D). Moreover, if \( \varrho_1 \in \alpha \) and \( \varrho_2 \in \alpha \), then all conditions (A), (B), (C) and (D) are mutually equivalent.

**Proof.** Let (A) be satisfied and let \( \varphi_i \to 0 \). In particular, \( \| \varphi_i \|_{(\varrho_1)} \to 0 \); hence there exists a subsequence \( \{ \varphi_{i_k} \} \) such that \( \| k \varphi_{i_k} \|_{(\varrho_1)} \to 0 \). Let us arrange the systems \( p \) in a sequence \( \{ P_s \} \), where \( p_0 = (0, 0, \ldots, 0) \). Since \( \| D^p \varphi_{i_k} \|_{(\varrho_1)} \to 0 \), from the sequence \( \{ \varphi_{i_k} \} \) one may extract a subsequence
{\varphi_{i_k}} for which \( \|D^p \varphi_{i_k}\|_{(q_1)} \to 0 \). Proceeding further in the same manner, we get (applying the diagonal method) a subsequence \{\varphi_{i_s}\} such that
\[ \|sD^p \varphi_{i_s}\|_{(q_1)} \leq K_p. \]
By (A), there exists a sequence of numbers \{K'_p\} satisfying the inequality \( \|sD^p \varphi_{i_s}\|_{(q_2)} \leq K'_p \), i.e.

(7.1) \[ \|D^p \varphi_{i_s}\|_{(q_2)} \to 0. \]

Hence it follows easily that \( \|D^p \varphi_i\|_{(q_2)} \to 0. \)

Now, let condition (B) be satisfied, and let us suppose that \( \varepsilon_1 \to \varepsilon_2 \) does not hold, i.e. \( \varepsilon_1(\lambda_p D^p \varphi_v) \leq 1 \) and \( \varepsilon_2(v^{-2} \varphi_v) \geq 1 \) for every \( p \), and for a sequence of positive numbers and a sequence of functions \{\varphi_v\}, \( \varphi_v \in \mathcal{S} \).

Let \( \varphi_v = v^{-2} \varphi_v \) and let \( \lambda \) be an arbitrary positive number. Taking \( p \) fixed, we choose an index \( v_0 \) such that \( v^{-1} \lambda \leq \lambda_p \) for \( v \geq v_0 \). Then

\[ \varepsilon_1(\lambda D^p \varphi_v) \leq \frac{1}{v} \varepsilon_1\left(\frac{\lambda}{v} D^p \varphi_v\right) \leq \frac{1}{v} \varepsilon_1\left(\lambda D^p \varphi_v\right) \leq \frac{1}{v} \]

for \( v \geq v_0 \), i.e. \( \varepsilon_2(v D^p \varphi_v) \to 0 \). Since \( \varepsilon_2(\varphi_v) = \varepsilon_2(v^{-2} \varphi_v) \geq 1 \), the condition \( \varphi_v \to 0 \) does not hold, a contradiction to the assumption. Thus, condition (C) is satisfied.

Let \( \varepsilon_1 \to \varepsilon_2 \) and let \( A \) be a bounded set in \( D_{\varepsilon_1} \), i.e. \( \|K_p^{-1} D^p \varphi\|_{(q_1)} \leq 1 \) for every \( \varphi \in A \). Hence \( \varepsilon_1(K_p^{-1} D^p \varphi) \leq 1 \), and so \( \varepsilon_2(k \varphi) < 1 \), where \( k \) is the constant in (C) chosen with respect to the sequence of numbers \( \lambda_p = K_p^{-1} \). Thus, we obtain \( \|\varphi\|_{(q_2)} \leq 1/k \). Repeating the argument in case of the functions \( D^p \varphi \) in an analogous manner, we get condition (A). Hence, we proved the equivalency of conditions (A), (B) and (C).

The fact that (C) implies (D) is obvious.

Now, let \( \varepsilon_1 \in (a) \), \( \varepsilon_2 \in (a) \) and \( D_{\varepsilon_1} \subset D_{\varepsilon_2} \). Let \( T \) be the operation of embedding of \( D_{\varepsilon_1} \) into \( D_{\varepsilon_2} \), and let \( \varphi_i \to \varphi \), \( T(\varphi_i) = \varphi_i \to \varphi \). By Lemma 2, \( \varphi_i \to \varphi \to 0 \) and \( \varphi_i \to \varphi \to 0 \), both almost uniformly. Hence \( T(\varphi) = \varphi \), and the closed graph theorem implies condition (B).

**Theorem 8.** Let us consider the following conditions:

(a) \( \varepsilon \in (A_2) \),

(b) if \( \varphi_i \to \varphi \to 0 \), then \( \varphi_i \to 0 \),

(c) \( D_g = \{\varphi \in \mathcal{S} : \varphi(\lambda D^p \varphi) < \infty \text{ for every } p \text{ and every } \lambda > 0\} \),

(d) the space \( D \) is dense in \( D_g \).

Then conditions (a), (b) are equivalent, conditions (c), (d) are equivalent, and condition (a) implies condition (c). Moreover, if \( \varepsilon \in (a) \), then all conditions (a), (b), (c), (d) are mutually equivalent.

**Proof.** Let \( \varepsilon \in (A_2) \) and \( \varphi_i \to 0 \). By Lemma 4, \( \sim_1(\{\lambda_p\}, \varphi_i) \to 0 \) for a sequence of numbers \( \{\lambda_p\} \). To this sequence we choose a number \( a_1 \) from the condition \( (A_2) \). Then there exists an index \( i_1 \) for which \( \sim_1(\{\lambda_p\}, \varphi_{i_1}) < a_1 \).
and the condition \((A_2)\) gives \(g(2\varphi_{i_k}) < 1\). Now, we choose to the sequence \(\{a_0\}\) a number \(a_2\) from the condition \((A_2)\). Then there exists an index \(i_2\) such that \(g(\{a_2\}_j, 2\varphi_{i_2}) < a_2\), whence \(g(2\cdot 2\varphi_{i_2}) < 1\), etc. In this manner we may define a subsequence \(\{\varphi_{i_k}\}\) for which \(g(2^{k-1}2\varphi_{i_k}) < 1\); hence \(\lim_{k \to \infty} g(2\varphi_{i_k}) = 0\). An analogous argumentation enables us to extract a subsequence \(\{\varphi_{i_{k'}}\}\) from the sequence \(\{\varphi_{i_k}\}\) for which \(\lim_{k \to \infty} g(2\varphi_{i_{k'}}) = 0\), where \(p_1\) is the first of all multiindices \(p_i \neq (0, 0, \ldots, 0)\) put in a sequence \(p_1, p_2, \ldots\). Repeating the same argumentation and applying the diagonal method, one may finally define a subsequence \(\{\varphi_{i_j}\}\) such that \(\lim_{k \to \infty} g(2^{p_1(p)}\varphi_{i_j}) = 0\) for each \(p\). It is easily seen that also \(\lim_{k \to \infty} \mathcal{Q}(\varphi, \mathcal{B}^p, \varphi) = 0\) for each \(p\) and for every \(\lambda > 0\). But this proves that \(\varphi_{i_j} \to 0\), and it is easily concluded that \(\varphi_{i_j} \to 0\). Since the converse implication is obvious, the equivalence of (a) and (b) is proved.

Now, let us suppose (c) to be satisfied, and let \(\varphi \in D_e\). We take a sequence of numbers \(\{K_p\}\) such that \(|D^p \varphi| < K_p\) for every \(j\), where \(\omega_j\) are defined as in Lemma 1. Then \(|D^p\left(\sum_{j=1}^{k} \omega_j\right)| < K_p\) for every \(k\). Hence we have for an arbitrary \(\lambda > 0\)

\[
\mathcal{Q}\left[\lambda \varphi D^p \left(\sum_{j=1}^{k} \omega_j\right)\right] = \mathcal{Q}\left[\left(\lambda \varphi D^p \sum_{j=1}^{k} \omega_j\right) \sum_{j=1}^{\infty} \omega_j\right] < \mathcal{Q}\left[\lambda K_p \varphi \sum_{j=1}^{\infty} \omega_j\right].
\]

But, by condition (c) and Lemma 1, the expression on the right-hand side of the last inequality tends to zero as \(k \to \infty\). Hence

(8.1) \[
\lim_{k \to \infty} \mathcal{Q}(\varphi D^p \left(\sum_{j=1}^{k} \omega_j\right)) = 0
\]

for an arbitrary \(\varphi \in D_e\). Applying (8.1) and the Leibniz formula we obtain easily that

\[
\lim_{k \to \infty} \mathcal{Q}(\varphi \sum_{j=1}^{k} \omega_j - D^p \varphi) = 0
\]

i.e. \(\varphi_k = \sum_{j=1}^{k} \varphi \omega_j \to \varphi\), and \(\varphi_k \in D\). Hence \(D\) is dense in \(D_e\). Conversely, let us suppose condition (d), and let \(\varphi \in D_e\).

We take a fixed \(p\), and we choose an arbitrary positive \(\lambda\). Then

\[
\mathcal{Q}(\lambda D^p \varphi) \leq \mathcal{Q}(2\lambda D^p \varphi \sum_{j=1}^{k} \omega_j) + \mathcal{Q}(2\lambda D^p \varphi \sum_{j=k+1}^{\infty} \omega_j).
\]

Since \(D\) is dense in \(D_e\), there exists a sequence \(\{\varphi_k\}, \varphi_k \in D\) such that \(\varphi_k \to \varphi\),
i.e. \( \lim g[2\lambda D^p \varphi - 2\lambda D^p \varphi_i] = 0 \). Let \( i_0 \) and \( k_0 \) be two indices chosen in such a manner that

\[
g[2\lambda D^p \varphi - 2\lambda D^p \varphi_i] < 1 \quad \text{and} \quad S_{\varphi_i} \cap S \bigcap_{j=k_0+1}^\infty = \emptyset.
\]

Then we have

\[
g(\lambda D^p \varphi) \leq g\left[2\lambda D^p \varphi \sum_{j=1}^{k_0} \omega_j \right] +
\]

\[
+ g\left[2\lambda D^p \varphi \sum_{j=k_0+1}^\infty \omega_j + 2\lambda D^p (\varphi - \varphi_i) \sum_{j=1}^{k_0} \omega_j \right] < \infty.
\]

Hence condition (c) is satisfied.

Next, let us suppose \( g \in (A_p) \) and \( \varphi \in D_g \), i.e. \( g(\lambda D^p \varphi) < \infty \) for a sequence of numbers \( \{\lambda_p\} \). By Theorem 6, \( \varphi \sum_{j=k}^\infty \omega_j \rightarrow 0 \). We choose a number \( a \) from the condition \( (A_p) \), corresponding to the sequence \( \{\lambda_p\} \). Then \( g(\{\lambda_p\}, \varphi \sum_{j=k}^\infty \omega_j) < a \) for \( k \geq k_0 \), and \( (A_p) \) implies \( g(2\varphi \sum_{j=k}^\infty \omega_j) < 1 \). Hence \( g(2\varphi) < \infty \). The proof for functions \( D^p \varphi \) and constants \( \lambda \) of the form \( \lambda = 2^k \) is performed analogously. Thus, we proved (c).

Finally, let \( g \in (A) \), and let condition (c) be satisfied. Let \( \{\varphi_i\} \) be \( \tilde{g} \)-convergent to zero, i.e. \( \lim g(\lambda D^p \varphi_i) = 0 \) for a sequence of numbers \( \{\lambda_p\} \).

Since \( \lim_{s \to \infty} g(\lambda D^p \varphi_i \sum_{j=s}^\infty \omega_j) = 0 \) for an arbitrary \( \lambda > 0 \), we obtain

\[
\lim_{s \to \infty} \|\varphi_i \sum_{j=s}^\infty \omega_j\|_e = 0.
\]

Hence there exists a sequence of indices \( \{s_i\} \) for which

\[
(8.2) \quad \lim_{i \to \infty} \|\varphi_i \sum_{j=s_i}^\infty \omega_j\|_e = 0.
\]

By Lemma 2, the sequence \( \{\varphi_i\} \) is uniformly convergent on every compact set. Let \( r_1 \) be an arbitrary positive integer. Then the sequence \( \{\varphi_i \sum_{j=1}^{r_1} \omega_j\} \)

is convergent in the sense of convergency in \( D \). By Theorem 5, \( \lim_{i \to \infty} \|\varphi_i \sum_{j=1}^{r_1} \omega_j\|_e = 0 \); hence there exists an index \( i_1 \) such that \( r_1 < s_{i_1} \) and \( \|\varphi_{i_1} \sum_{j=1}^{r_1} \omega_j\|_e \leq 1 \).

Let \( r_2 \) be an arbitrary positive integer such that \( r_2 > s_{i_1} \). Then \( \lim_{i \to \infty} \|\varphi_i \sum_{j=1}^{r_2} \omega_j\|_e = 0 \). Hence there exists an index \( i_2 \) such that \( r_2 < s_{i_2} \) and \( \|\varphi_{i_2} \sum_{j=1}^{r_2} \omega_j\|_e \)
Infinitely differentiable functions

Continuing this procedure, we may define two sequences of indices \( \{i_m\} \) and \( \{r_m\} \) such that \( s_{i_{m-1}} < r_m < s_{i_m} \) and \( \|q_{i_m} \sum_{j=1}^{r_m} \omega_j\|_q \leq 1/m \). Then

\[
\lim_{m \to \infty} \left\| q_{i_m} \sum_{j=1}^{r_m} \omega_j \right\|_q = 0.
\]

Since \( \lim_{m \to \infty} q(\lambda_0 q_{i_m}) = 0 \), there exists a subsequence \( \{i_{m_0,k}\} \) of the sequence of indices \( \{i_m\} \) such that

\[
q(\lambda_0 q_{i_{m_0,k}} \sum_{j=r_{m_0,k}}^{s_{i_{m_0,k}}} \omega_j) \leq \frac{1}{2^k}.
\]

Let \( p_1, p_2, \ldots \) be the sequence of all multiindices \( p \neq (0, 0, \ldots, 0) \). Then, in particular, \( \lim q(\lambda_1 D^{p_1} q_{i_{m_0,k}}) = 0 \). Hence there is a subsequence \( \{i_{m_1,k}\} \) of the sequence \( \{i_{m_0,k}\} \) for which

\[
q \left[ \lambda_1 D^{p_1} \left( q_{i_{m_1,k}} \sum_{j=r_{m_1,k}}^{s_{i_{m_1,k}}} \omega_j \right) \right] \leq \frac{1}{2^k},
\]

etc. Continuing this procedure and applying the diagonal method we obtain a sequence \( \{i_{m_{k,k}}\} \) such that

\[
q \left[ \lambda_p D^p \left( q_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}} \omega_j} \right) \right] \leq \frac{1}{2^k}
\]

for \( k \geq k_p \); it follows from the construction that the supports of functions on the left-hand side of this inequality are disjoint for different \( k \). Let us define

\[
q = \sum_{k=0}^{\infty} \left( q_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}} \omega_j} \right).
\]

Then

\[
q(\lambda_p D^p q) = \sum_{k=0}^{\infty} q \left[ \lambda_p D^p \left( q_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}} \omega_j} \right) \right] \leq 1,
\]

i.e. \( q \in D_\phi \). According to the assumption we obtain \( q(\lambda D^p q) < \infty \) for each \( p \) and for an arbitrary \( \lambda > 0 \). Hence we get

\[
\lim_{k \to \infty} q \left[ \lambda D^p \left( q_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}} \omega_j} \right) \right] = 0
\]
for an arbitrary \( \lambda > 0 \), i.e.

\[
(8.4) \quad \lim_{k \to \infty} \left\| q_{\lambda k, k} \sum_{j \in r_{m k, k}} \omega_j \right\| = 0.
\]

From (8.3), (8.4) and (8.2) we conclude \( \lim_{k \to \infty} \| q_{\lambda k, k} \| \leq 0 \), that is \( q_{\lambda k, k} \to 0 \). This implies easily \( q_{i} \to 0 \), and the proof is finished.

6. In the following, we shall construct some examples of spaces \( D_c \). First, we give the definition of an \( M \)-function, and we prove a lemma useful in our further considerations. This lemma generalizes a lemma given in [1].

A function \( M(t, u) = M(t_1, t_2, \ldots, t_n, u) \) defined on \( R^{n+1} \) will be called an \( M \)-function in the variable \( u \), if

1. for every \( u \), the function \( M(t, u) \) is a continuous function of the variable \( t \);
2. \( M(t, u) \geq 0 \); if \( u = 0 \), then \( M(t, u) = 0 \);
3. \( M(t, u) = M(t, -u) \);
4. \( M(t, \alpha u_1 + \beta u_2) \leq \alpha M(t, u_1) + \beta M(t, u_2) \) for \( \alpha, \beta \geq 0 \), \( \alpha + \beta = 1 \);
5. for every \( t \), there holds

\[
\lim_{u \to 0} \frac{M(t, u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{M(t, u)}{u} = \infty.
\]

Let \( P \) denote the set of all multiindices \( p = (p_1, \ldots, p_n) \) such that \( p_i = 0 \) for \( i = i_k \), where \( \{i_k\} \) are indices extracted from 1, 2, \ldots, \( n \), \( p_i = 1 \) for the remaining \( i \), and let \( |p| = p_1 + p_2 + \cdots + p_n \). Then we denote by \( \partial^p M(t, u) \) the derivative \( \partial^{p_1} M(t_1, \ldots, t_n, u) / \partial t_1^{p_1} \cdots \partial t_n^{p_n} \); \( \partial^p M(t, u(t)) \) will stand for the derivative \( \partial^{p_1} M(t_1, \ldots, t_n, u) / \partial t_1^{p_1} \cdots \partial t_n^{p_n} \) calculated at the point \( u = u(t_1, \ldots, t_n) \).

**Lemma 9.** Let \( \partial^p M(t, u) \) be \( M \)-functions in the variable \( u \) and let \( \varphi \) possess continuous derivatives \( \partial^p \varphi \) in the set \( \{ t : |t| > |y| \} \) for all \( p \in P \). Then for every \( x \) such that \( |x| > |y| \) there holds the inequality

\[
M[x, \varphi(x)] \leq \sum_{p + q \in P, |t| > |x|} \int \partial^p M[t, 2^{n-|p|+q} \partial^q \varphi(t)] dt.
\]

**Proof.** Let \( C(x, r) \) be one of the \( 2^n \) \( n \)-dimensional closed cubes with a vertex at \( x \) and with sides of length \( r \) parallel to the coordinate axes, and let \( |x| > |y| \). Moreover, let \( J_k \) be the projection of the cube \( C(x, r) \) on the axis \( x_k \); we fix the cube \( C(x, r) \) in such a manner that

\[
J_k = \{ t_k : x_k \leq t_k \leq x_k + r \} \quad \text{for} \quad x_k > 0,
\]

\[
J_k = \{ t_k : x_k - r \leq t_k \leq x_k \} \quad \text{for} \quad x_k < 0,
\]
where \( x = (x_1, \ldots, x_n) \). Thus, \( C(x, r) = J_1 \times J_2 \times \ldots \times J_n \). Let \( P_k = \{ p \in P : p = (p_1, \ldots, p_{k-1}, 0, \ldots, 0) \} \), \( P'_k = \{ p' \in P : p' = (0, \ldots, 0, p_k, \ldots, p_n) \} \). Let us denote for every \( 1 \leq k \leq n \) and every \( p \)
\[
S^p_k[\varphi] = \int_{J_1} \cdots \int_{J_n} \partial^p M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)] dt.
\]
Then the following inequality holds for every \( p' \in P'_k \):
\[
S^p_k[\varphi] \leq \sum_{p+q=p'} \left( 1 + \frac{1}{r} \right)^{k-1-[p+q]} S^{p+q} [2^{k-1-[p+q]} D^q \varphi].
\]
The proof of formula (9.1) will be performed by induction.
For \( k = 1 \) we have
\[
S^p_1[\varphi] = \int_{J_1} \cdots \int_{J_n} \partial^p M [t, \varphi(t)] dt = \int_{C(x, r)} \partial^p M [t, \varphi(t)] dt,
\]
and formula (9.1) is true. Let us suppose (9.1) to be true for \( 1 \leq k < n \).
The mean-value theorem gives
\[
\frac{1}{r} \int_{J_k} \partial^p M \left[ x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n) \right] dt_k
\]
\[
= \partial^p M \left[ x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n) \right],
\]
where \( \theta_k = \theta_k(x_1, \ldots, x_{k-1}, t_{k+1}, \ldots, t_n) \in J_k \). Hence, we have for every \( p' \in P'_{k+1} \):
\[
S^{p'}_{k+1}[\varphi] = \int_{J_{k+1}} \cdots \int_{J_n} \left\{ \int_{J_k} \left( \frac{\partial}{\partial t_k} \right) \partial^{p'} M \left[ x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n) \right] dt_k \right\} dt + \int_{J_{k+1}} \cdots \int_{J_n} \partial^{p'} M \left[ x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n) \right] dt.
\]
Denoting by \( \overline{p} \) multiindex, obtained from the multiindex \( p' \in P'_{k+1} \) by putting 1 in place of 0 at the \( k \)-th place, we obtain
\[
S^{p'}_{k+1}[\varphi] = \int_{J_{k+1}} \cdots \int_{J_n} \partial^{p'} M \left[ x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n) \right] dt +
\]
\[
+ \int_{J_{k+1}} \cdots \int_{J_n} \left\{ \left( \frac{\partial}{\partial \varphi} \right) \partial^{p'} M \left[ x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi \right] \left|_{\varphi=q(x_1,\ldots,x_{k-1},t_k,\ldots,t_n)} \right. \times
\]
\[
\left. \times \left( \frac{\partial}{\partial t_k} \right) \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n) \right\} dt +
\]
\[
+ \int_{J_{k+1}} \int_{J_k} \frac{1}{r} \int_{J_k} \partial^{p'} M \left[ x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n) \right] dt.
\]
We apply the following inequality (see [1])
\[ \frac{\partial M[x, f(t)]}{\partial t} \leq M[x, 2f(t)] + M[x, \partial f(t)/\partial t] \]
to the integrand in the second term on the right-hand side of the above inequality. We obtain

\[ S_{k+1}^p[f] \leq \int_{J_k} \ldots \int_{J_n} \partial^p M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi (x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)] dt + \]
\[ + \int_{J_k} \ldots \int_{J_n} \partial^p M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, 2\varphi (x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)] dt + \]
\[ + \int_{J_k} \ldots \int_{J_n} \partial^p M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \frac{\partial}{\partial t_k} \varphi (x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)] dt + \]
\[ + \frac{1}{r} \int_{J_k} \ldots \int_{J_n} \partial^p M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, 2\varphi (x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)] dt \]
\[ = S_{k}^p \left[ \frac{\partial}{\partial t_k} \varphi \right] + \left( 1 + \frac{1}{r} \right) S_{k+1}^p[2\varphi] + S_{k}^\xi[f]. \]

By (9.1), we get

\[ S_{k+1}^p[f] \leq \sum_{p+q \in P_k} \left( 1 + \frac{1}{r} \right)^{k-1-|p+q|} S_{1}^{p+q} \left[ 2^{k-1-|p+q|} D^{q} \left( \frac{\partial}{\partial t_k} \varphi \right) \right] + \]
\[ + \sum_{p+q \in P_k} \left( 1 + \frac{1}{r} \right)^{k-1-|p+q|} S_{1}^{p+q} \left[ 2^{k-1-|p+q|} D^{q} (2\varphi) \right] + \]
\[ + \sum_{p+q \in P_k} \left( 1 + \frac{1}{r} \right)^{k-1-|p+q|} S_{1}^{p+q} \left[ 2^{k-1-|p+q|} D^{q} \varphi \right]. \]

Let \( p \) and \( q \) be arbitrary multiindices belonging to \( P_{k+1} \), and let \( \bar{p} \) and \( \bar{q} \) be obtained from \( p \) and \( q \), respectively, by replacing 0 by 1 at the \( k \)-th place. Then

\[ S_{k+1}^p[f] \leq \sum_{p+q \in P_{k+1}} \left( 1 + \frac{1}{r} \right)^{(k+1)-1-|p+q|} S_{1}^{p+q} \left[ 2^{(k+1)-1-|p+q|} D^{q} \varphi \right] + \]
\[ + \sum_{p+q \in P_k} \left( 1 + \frac{1}{r} \right)^{(k+1)-1-|p+q|} S_{1}^{p+q} \left[ 2^{(k+1)-1-|p+q|} D^{q} \varphi \right] + \]
\[ + \sum_{\bar{p}+\bar{q} \in P_{k+1}} \left( 1 + \frac{1}{r} \right)^{(k+1)-1-|\bar{p}+\bar{q}|} S_{1}^{\bar{p}+\bar{q}} \left[ 2^{(k+1)-1-|\bar{p}+\bar{q}|} D^{q} \varphi \right] \]
\[ = \sum_{p+q \in P_{k+1}} \left( 1 + \frac{1}{r} \right)^{(k+1)-1-|p+q|} S_{1}^{p+q} \left[ 2^{(k+1)-1-|p+q|} D^{q} \varphi \right]. \]

Thus, (9.1) is proved for \( 1 \leq k < n \).
Applying (9.1), we prove now the following inequality

\[(9.2) \quad M(x, \varphi(x)) \leq \sum_{p+q \in P_\lambda} \left(1 + \frac{1}{r}\right)^{n-1-|p+q|} \int_{c(x,r)} \partial^p M[t, 2^{n-1-|p+q|} D^q \varphi(t)] dt.\]

We have, by the mean-value theorem,

\[
\frac{1}{r} \int_{J_n} M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)] dt_n
\]

\[
= M[x_1, \ldots, x_{n-1}, \theta_n, \varphi(x_1, \ldots, x_{n-1}, \theta_n)],
\]

where \(\theta_n = \theta_n(x_1, \ldots, x_{n-1}) \in J_n\). Thus,

\[
M(x, \varphi(x)) = \int_{\partial_n} \left(\frac{\partial}{\partial t_n}\right) M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)] dt_n +
\]

\[
+ M[x_1, \ldots, x_{n-1}, \theta_n, \varphi(x_1, \ldots, x_{n-1}, \theta_n)].
\]

Denoting by \(p^n\) a multiindex belonging to \(P\) which has 1 only at the \(n\)-th place, we get

\[
M(x, \varphi(x)) \leq \int_{\partial_n} \partial^{p^n} M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)] dt_n +
\]

\[
+ \int_{\partial_n} M[x_1, \ldots, x_{n-1}, t_n, 2\varphi(x_1, \ldots, x_{n-1}, t_n)] dt_n +
\]

\[
+ \int_{\partial_n} M[x_1, \ldots, x_{n-1}, t_n, \frac{\partial}{\partial t_n} \varphi(x_1, \ldots, x_{n-1}, t_n)] dt_n +
\]

\[
+ \frac{1}{r} \int_{J_n} M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)] dt_n
\]

\[
\leq S_n^0 \left[\frac{\partial}{\partial t_n} \varphi\right] + \left(1 + \frac{1}{r}\right) S_n^0 [2\varphi] + S_n^{p^n} [\varphi].
\]

By (9.1), we obtain the inequality

\[
M(x, \varphi(x)) \leq \sum_{p+q \in P_\lambda} \left(1 + \frac{1}{r}\right)^{n-1-|p+q|} S_1^p \left[2^{n-1-|p+q|} D^q \left(\frac{\partial}{\partial t_n} \varphi\right)\right] +
\]

\[
+ \left(1 + \frac{1}{r}\right) \sum_{p+q \in P_\lambda} \left(1 + \frac{1}{r}\right)^{n-1-|p+q|} S_1^p \left[2^{n-1-|p+q|} D^q (2\varphi)\right] +
\]

\[
+ \sum_{p+q \in P_\lambda} \left(1 + \frac{1}{r}\right)^{n-1-|p+q|} S_1^{p+n} \left[2^{n-1-|p+q|} D^q \varphi\right].
\]
Thus, inequality (9.2) is proved.

Evidently, \( C(x, r) \subseteq \{ t : |t| \geq |x| \} \). Hence, taking \( r \to \infty \) in (9.2), we obtain

\[
C(x, r) \subseteq \bigcup_{p \in P} \{ t : |t| \geq |x| \}
\]

and the lemma is proved.

A real-valued function \( M(t, u) = M(t_1, \ldots, t_n, u) \) defined on \( \mathbb{R}^{n+1} \) will be called a \( \partial M \)-function, if

(C.1) the functions \( \partial_p M(t, u) \) are \( M \)-functions in the variable \( u \) for every \( p \in P \);

(C.2) \( M(t, u) \to \infty \) as \( u \to \infty \) uniformly with respect to the variable \( t \);

(C.3) if \( \lim_{|t| \to \infty} M(t, u) = 0 \), then \( u = 0 \);

(C.4) for every \( t \), \( M(t, u) = 0 \) implies \( u = 0 \).

If we define the modular \( \varepsilon_{\partial M} \) by the formula

\[
\varepsilon_{\partial M}(\varphi) = \sup_{p \in P} \int_{\mathbb{R}^n} \partial_p M(t, \varphi(t)) dt,
\]

it is easily verified that \( \varepsilon_{\partial M} \) satisfies all conditions (A.1)-(A.6) of the definition of the modular \( \varepsilon \). Thus \( D_{\varepsilon_{\partial M}} \) is an example of a space \( D_{\varepsilon} \). All conditions (A)-(D) of Theorem 7 are mutually equivalent, and the same holds for all conditions (a)-(d) of Theorem 8, because

**Theorem 10.** The modular \( \varepsilon_{\partial M} \) satisfies the condition (a).

**Proof.** Let \( \lim_{\varepsilon \to \infty} \varepsilon_{\partial M}(\lambda D^q \varphi_i) = 0 \); then for every \( p \in P \) and \( q \in P \)

such that \( p + q \in P \), and for every \( \varepsilon > 0 \) there exists index \( i_{p,q,\varepsilon} \) with the property that

\[
\int_{\mathbb{R}^n} \partial_p M(t, 2n^{-|p+q|} \lambda_{p,q} D^q \varphi_i(t)) dt < \frac{\varepsilon}{3^n}
\]

for all \( i \geq i_{p,q,\varepsilon} \), where \( \lambda_{p,q} = \lambda_q \cdot 2^{-(n-|p+q|)} \). Let \( i_\varepsilon = \max_{p+q \in P} i_{p,q,\varepsilon} \), and

let \( \lambda_1 = \min_{p+q \in P} \lambda_{p,q} \); then

\[
\int_{\mathbb{R}^n} \partial_p M(t, 2n^{-|p+q|} \lambda_1 D^q \varphi_i(t)) dt < \frac{\varepsilon}{3^n}
\]
for all $i \geq i_e$. By Lemma 9, we have

\[(10.1) \quad M(x, \lambda_i q_i(x)) \leq \varepsilon\]

for $i \geq i_e$ and for every $x \in \mathbb{R}^n$. Since $q_i \in D_{q_M}$ by assumption, for all $p, q \in P$ such that $p + q \in P$ and for each $i$ there exists a constant $\lambda_{q_i}$ for which

\[\int_{\mathbb{R}^n} \partial^p M[t, \lambda_{q_i} D^q q_i(t)] dt < \infty.\]

Let $\lambda_2 = \min_{p+q \in P, i \leq i_e} \lambda_{q_i} \cdot 2^{-|p+q|}$; then for $p + q \in P$ and all $i \leq i_e$ there holds

\[(10.2) \quad \int_{\mathbb{R}^n} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q q_i(t)] dt < \infty\]

i.e. there exists a constant $\Delta_{p,q,i} > 0$ such that

\[\int_{|t| \geq \Delta_{p,q,i}} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q q_i(t)] dt < \frac{\varepsilon}{3^n}.\]

Let $\Delta = \max_{n+q \in P, i \leq i_e} \Delta_{p,q,i}$; then

\[\int_{|t| \geq \Delta} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q q_i(t)] dt < \frac{\varepsilon}{3^n}.\]

Hence we get, by Lemma 9,

\[M(x, \lambda_2 q_i(x)) \leq \sum_{p+q \in P, |t| > |x|} \int_{|t| \geq \Delta} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q q_i(t)] dt \leq \varepsilon\]

for $i \leq i_e$ and $|x| \geq \Delta$. This inequality and inequality (10.1) imply the inequality

\[(10.3) \quad M(x, \lambda q_i(x)) \leq \varepsilon\]

for all $i$ and $|x| \geq \Delta$, where $\lambda = \min(\lambda_1, \lambda_2)$.

Now, let $|x| < \Delta$ and $i < i_e$. According to formula (10.2) we have

\[\int_{\mathbb{R}^n} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q q_i(t)] dt = K_{p,q,i} < \infty.\]

Applying Lemma 9, we obtain the inequality

\[M(x, \lambda_2 q_i(x)) \leq \sum_{p+q \in P} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q q_i(t)] dt \leq K_{p,q,i} \leq K < \infty,\]
where $K = 3^n \max_{p+q \in P, t \leq t_q} K_{p,q,t}$. Hence and from inequality (10.1) we obtain

\begin{equation}
M(x, \lambda \varphi_i(x)) \leq K
\end{equation}

for $|x| < \Lambda$ and for all $i$, where $\lambda = \min(\lambda_1, \lambda_2)$.

From inequalities (10.3) and (10.4) it follows that

\begin{equation}
M(x, \lambda \varphi_i(x)) \leq K
\end{equation}

for every $x \in \mathbb{R}^n$ and each $i$. From the last inequality we conclude that the functions $\varphi_i$ are uniformly bounded. Indeed, let us suppose this is not true, that is, there exists a sequence $\{x_k\}$ such that $|\varphi_{i_k}(x_k)| > k$ for a subsequence of indices $\{i_k\}$. Then $K \geq M(x_k, \lambda \varphi_{i_k}(x_k)) \geq M(x_k, \lambda k)$. Since $\lim_{k \to \infty} M(x_k, \lambda k) = \infty$, we get a contradiction. Hence $\varphi_i$ are uniformly bounded. We proved that $\partial_{\partial M}$ satisfies the condition (a).

It is easily proved that the following theorems hold for the space $D_{\partial M}$:

**Theorem 11.** If $\varphi_i \to 0$, then $\lim_{|x| \to \infty} D^q \varphi_i(x) = 0$ uniformly with respect to $i$ and the functions $D^q \varphi_i(x)$ are uniformly bounded with respect to $i$ and $x \in \mathbb{R}^n$, for each $q$ separately.

**Theorem 12.** If $\varphi \in D_{\partial M}$, then for every $q$ we have $\lim_{|x| \to \infty} D^q \varphi(x) = 0$, and the functions $D^q \varphi$ are bounded.

Now, we show a possibility of constructing the $\partial M$-functions. Let a point $t \in \mathbb{R}^n$ be given, and let $C(t) = S_1 \times \ldots \times S_n$, where we have for $i = 1, 2, \ldots, n$,

\begin{align*}
S_1 &= \{v_i : 0 \leq v_i \leq t_i\} \quad \text{for } t_i \geq 0, \\
S_2 &= \{v_i : t_i < v_i \leq 0\} \quad \text{for } t_i \leq 0.
\end{align*}

Then it is easily seen that

\[
M(t, u) = \int_{\mathcal{C}(0)} \left[ m(v, u) \, dv + m(0, u) \right]
\]

is a $\partial M$-function, of $m(t, u)$ is an $M$-function. Let us remark that if $M(u)$ is a $N$-function in the sense of [3], and $p(t)$ is a non-negative continuous function; then $m(t, u) = [M(u)]^{p(t)+1}$ is an $M$-function in the sense defined at the beginning of section 3 of this paper.

Finally, let us remark that taking here $\partial M$-function $M(t, u) = M(u)$ we obtain as a special case the space $D_M$ and theorems concerning this space given in [4] and [6].
Infinitely differentiable functions

References


