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On a modular space of infinitely differentiable functions

This paper contains generalizations of results contained in papers [1], [4] and [6]. In place of spaces D_M considered in the above mentioned papers, there is considered a more general space D_ϱ of infinitely differentiable functions which is defined by means of a modular ϱ. The problem of the dual of D_ϱ considered in papers [6] and [7] in the special case of D_M, is not investigated here.

1. First, we give the definition of a modular ϱ, the space D_ϱ and of convergence in D_ϱ.

Let $\mathscr{E}$ be the space of all real infinitely differentiable functions in the real n-dimensional space R^n. Let an extended real-valued functional ϱ, called a *modular*, be defined on $\mathscr{E}$, satisfying the following conditions:

(A.1) $\quad 0 \leqslant \varrho(\varphi) \leqslant \infty;\ \varrho(\varphi) = 0$, *if and only if*, $\varphi = 0$,

(A.2) $\quad \varrho(\alpha\varphi_1 + \beta\varphi_2) \leqslant \alpha\varrho(\varphi_1) + \beta\varrho(\varphi_2)$ *for all* $\alpha \geqslant 0, \beta \geqslant 0$ *such that* $\alpha + \beta = 1$,

(A.3) $\quad \varrho(\varphi_1) \leqslant \varrho(\varphi_2)$ *for* $|\varphi_1| \leqslant |\varphi_2|$,

(A.4) $\quad$ *if the support* S_φ *of a function* $\varphi \in \mathscr{E}$ *is compact, then* $\varrho(\varphi) < \infty$,

(A.5) $\quad \varrho(\varphi_1 + \varphi_2) = \varrho(\varphi_1) + \varrho(\varphi_2)$ *for functions* φ_1 *and* φ_2 *such that* $S_{\varphi_1} \cap S_{\varphi_2} = \emptyset$,

(A.6) $\quad$ *if* $\varliminf_{i\to\infty} \varphi_i \in \mathscr{E}$, *then* $\varrho(\varliminf_{i\to\infty} \varphi_i) \leqslant \varliminf_{i\to\infty} \varrho(\varphi_i)$.

We shall say that a sequence $\{\varphi_i\}$, $\varphi_i \in \mathscr{E}$ is ϱ-convergent to zero (we write $\varphi_i \xrightarrow{\varrho} 0$), if $\varrho(\lambda\varphi_i) \to 0$ as $i \to \infty$ for some $\lambda > 0$. By D_ϱ we denote the subspace of $\mathscr{E}$ which consists of all functions $\varphi \in \mathscr{E}$ such that for every multiindex $p = (p_1, p_2, \ldots, p_n)$ $(p_i \geqslant 0)$ there exists a constant $\lambda_p > 0$ satisfying the condition $\varrho(\lambda_p D^p\varphi) < \infty$, where D^p is the operator of differentation:

$$D^p = \partial^{p_1+\ldots+p_n}/\partial x_1^{p_1} \ldots \partial x_n^{p_n}.$$

In the space D^ϱ we define a sequence of pseudonorms

$$\|\varphi\|^m = \sup_{|p|\leqslant m} \|D^p\varphi\|_{(\varrho)}, \qquad m = 0, 1, 2, \ldots,$$

where $|p| = p_1 + \ldots + p_n$ and

$$\|\varphi\|_{(\varrho)} = \inf\left\{k\colon\ \varrho\left(\frac{\varphi}{k}\right) \leqslant 1\right\}.$$

Obviously, the topology defined in D_ϱ by means of the base of neighbourhoods of zero

$$V(m, \varepsilon) = \{\varphi \in D_\varrho\colon\ \|\varphi\|^m \leqslant \varepsilon\}, \qquad m = 0, 1, 2, \ldots, \quad \varepsilon > 0,$$

is equivalent to the topology defined in D_ϱ by means of the F-norm

$$\|\varphi\|_\varrho = \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\|\varphi\|^m}{1+\|\varphi\|^m}.$$

A sequence $\{\varphi_i\}$, $\varphi_i \in D_\varrho$ will be called *$\tilde{\varrho}$-convergent to zero* (we write $\varphi_i \xrightarrow{\tilde{\varrho}} 0$) if $D^p\varphi_i \xrightarrow{\varrho} 0$ for every p.

We shall say that the modular ϱ satisfies condition (a) (we write $\varrho \in (a)$), if for every sequence $\{\varphi_i\}$, $\varphi_i \in D_\varrho$ such that $\varphi_i \xrightarrow{\tilde{\varrho}} 0$ there exists a constant $a > 0$ satisfying the inequality $|\varphi_i(t)| \leqslant a$ for every $t \in R^n$ and every i.

Now, we formulate some lemmas needed in the sequel.

LEMMA 1. *Let $\varphi \in \mathscr{E}$. Then $\varrho(\varphi) < \infty$, if and only if,*

$$\lim_{k\to\infty} \varrho\Big(\varphi \sum_{j=k}^{\infty} \omega_j\Big) = 0, \tag{1.1}$$

where the functions $\omega_j \in \mathscr{E}$, ω_j of compact support, which give a partition of unity, are defined as in [4], *Lemma* 1.

Proof. Let $\varrho(\varphi) < \infty$, and let $\varphi_i = \sum_{k=1}^{i} \varphi\omega_k$. Since $\varrho(\varphi_i) \leqslant \varrho(\varphi)$ and $\lim_{i\to\infty} \varphi_i = \varphi$, we have

$$\varrho(\varphi) \leqslant \varliminf_{i\to\infty} \varrho(\varphi_i) \leqslant \varlimsup_{i\to\infty} \varrho(\varphi_i) \leqslant \varrho(\varphi),$$

i.e.

$$\lim_{i\to\infty} \varrho(\varphi_i) = \varrho(\varphi).$$

Let us choose an arbitrary $\varepsilon > 0$. Then there exists an i_0 such that

$$\varrho(\varphi) - \varrho(\varphi_i) < \varepsilon \quad \text{for } i \geqslant i_0.$$

Hence

$$\varrho(\varphi) \geqslant \varrho\Big(\sum_{k=1}^{i} \varphi\omega_k + \sum_{k=i+2}^{\infty} \varphi\omega_k\Big) = \varrho(\varphi_i) + \varrho\Big(\sum_{k=i+2}^{\infty} \varphi\omega_k\Big),$$

and so we get

$$\varrho\Big(\sum_{k=i+2}^{\infty} \varphi\omega_k\Big) \leqslant \varrho(\varphi) - \varrho(\varphi_i) < \varepsilon \quad \text{for } i \geqslant i_0.$$

This proves (1.1).

Now, let us suppose

$$\varrho\Big(\varphi \sum_{j=k_0}^{\infty} \omega_j\Big) < \varepsilon \quad \text{for an } \varepsilon > 0 \text{ and a } k_0 \geqslant 1.$$

Let a number $a > 0$ be chosen in such a manner that $\omega_{k_0}(t) = 1$ for $|t| = a$. Writing

$$A = \{t\colon\ |t| \leqslant a\}, \qquad B = \{t\colon\ |t| > a\},$$

we obtain

$$S_{\sum_{j=1}^{k_0-1} \omega_j} \subset A \quad \text{and} \quad \sum_{j=k_0}^{\infty} \omega_j \chi_B = \chi_B.$$

Hence, one may construct non-negative functions $\omega_{j,i} \in \mathscr{E}$ such that

$$S_{\omega_{j,i}} \subset \Big\{t \in R^n\colon\ (j-1)\frac{a}{i} < |t| < j\,\frac{a}{i} + \frac{a}{2^i}\Big\},$$

where $j = 1, 2, \ldots$, $i = 1, 2, \ldots$, and $\sum_{j=1}^{\infty} \omega_{i,j} = 1$ for $i = 1, 2, \ldots$

Let us define two sequences of functions

$$\varphi_{1,i} = \varphi \sum_{j=1}^{i} \omega_{j,i}, \qquad \varphi_{2,i} = \varphi \sum_{j=i+1}^{\infty} \omega_{j,i}.$$

It is easily observed that for each i,

$$A \subset S_{\varphi_{1,i}}, \qquad S_{\varphi_{2,i}} \subset B, \qquad S_{\varphi_{1,i}} \cap S_{\varphi_{2,i}} = \emptyset.$$

Since $\lim_{i\to\infty} \varphi_{1,i} = \varphi\chi_A$ and $\lim_{i\to\infty} \varphi_{2,i} = \varphi\chi_B$, we get $\lim_{i\to\infty} (\varphi_{1,i} + \varphi_{2,i}) = \varphi$.

Hence we obtain

$$\varrho(\varphi) \leqslant \varliminf_{i\to\infty} [\varrho(\varphi_{1,i} + \varphi_{2,i})] = \varliminf_{i\to\infty} [\varrho(\varphi_{1,i}) + \varrho(\varphi_{2,i})]$$

$$\leqslant \varliminf_{i\to\infty} \Big[\varrho\Big(\varphi \sum_{j=1}^{k_0+1} \omega_j\Big) + \varrho\Big(\varphi \sum_{i=k_0}^{\infty} \omega_j\Big)\Big] < \infty.$$

This proves the lemma.

LEMMA 2. *If $\varrho \in (a)$, then every $\tilde{\varrho}$-convergent to zero sequence $\{\varphi_i\}$ is convergent almost uniformly to zero.*

Proof. Let $\varphi_i \xrightarrow{\tilde{\varrho}} 0$. By the condition (a), there exists a sequence of positive numbers $\{a_p\}$ such that the inequality $|D^p\varphi_i(t)| \leqslant a_p$ holds for every $t \in R^n$ and each i. Let $p^j = (p_1, p_2, \ldots, p_{j-1}, p_j+1, p_{j+1}, \ldots, p_n)$. By the mean-value theorem,

$$|D^p\varphi_i(t) - D^p\varphi_i(t')| = \Big|\sum_{j=1}^{n} D^{p^j}\varphi_i(\xi_j)(t_j - t_j')\Big| \leqslant a_p' \sum_{j=1}^{n} |t_j - t_j'|,$$

where $a_p' = \max\limits_{1\leqslant j\leqslant n} a_{p^j}$. Thus the functions $D^p\varphi_i$ satisfy Lipschitz condition, and so they are equicontinuous functions for each fixed p, uniformly bounded according to the assumption. Applying Arzelà theorem, the diagonal method yields a subsequence $\{\varphi_{i_k}\}$ such that $D^p\varphi_{i_k} \to \varphi_p$ almost uniformly for every p separately. It is easily seen that

$$D^p\varphi_{i_k} \to D^p\varphi \text{ almost uniformly}, \tag{2.1}$$

where $\varphi = \varphi_0$ and $\varphi \in \mathscr{E}$. Next, we have $\varrho(\lambda_0\varphi) \leqslant \varliminf\limits_{i\to\infty} \varrho(\lambda_0\varphi_{i_k}) = 0$, i.e. $\varphi = 0$. From this and from formula (2.1) it follows that

$$\varphi_{i_k} \to 0 \text{ almost uniformly}. \tag{2.2}$$

It is easily observed that the sequence $\{\varphi_i\}$ is also almost uniformly convergent to zero. Thus, we proved the lemma.

LEMMA 3. *If $\varrho \in (a)$, then the space D_ϱ is complete.*

Proof. Let $\{\varphi_i\}$ be a Cauchy sequence in D_ϱ, i.e. for each p and every constant $\lambda > 0$,

$$\varrho[\lambda(D^p\varphi_{i+k} - D^p\varphi_i)] \to 0$$

uniformly with respect to k. By the conditon (a), we have

$$|D^p\varphi_k(t) - D^p\varphi_0(t)| \leqslant a_p \tag{3.1}$$

for every k. Now, we prove the function $D^p\varphi_0$ to be bounded. Indeed, let $\varrho(\lambda D^p\varphi_0) < \infty$; then, by Lemma 1,

$$\lim_{i\to\infty} \varrho(\lambda D^p\varphi_0 \sum_{k=i}^{\infty} \omega_k) = 0.$$

Hence, by (a), there exists a constant $\beta_p > 0$ such that $|D^p\varphi_0 \sum\limits_{k=i}^{\infty} \omega_k| \leqslant \beta_p$

for each i, that is $|D^p\varphi_0| \leqslant \beta_p$. From this and from (3.1) it follows that $|D^p\varphi_k| \leqslant a_p+\beta_p = K_p$. Arguments analogous to those applied in the proof of Lemma 2 show that for every p, $D^p\varphi_{i_k} \to D^p\varphi$ almost uniformly, where $\varphi \in \mathscr{E}$. Consequently, applying Cauchy condition, we have for fixed p

$$\varrho[\lambda(D^p\varphi_i - D^p\varphi)] = \varrho[\lim_{s\to\infty} \lambda(D^p\varphi_{i_{k+s}} - D^p\varphi_i)]$$

$$\leqslant \lim_{s\to\infty} \varrho[\lambda(D^p\varphi_{i_{k+s}} - D^p\varphi_i)] \leqslant \varepsilon$$

for $i \geqslant i_\lambda$ and for every constant $\lambda > 0$. Hence $\varphi_i \xrightarrow{D\varrho} \varphi$. It is also easily seen that $\varphi \in D_\varrho$, and the proof is finished.

LEMMA 4. *Let $\varrho(\lambda_p D^p\varphi) < \infty$ for every p and for a fixed sequence of positive numbers $\{\lambda_p\}$. Let*

$$\tilde{\varrho}(\{\lambda\}, \varphi) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\sup\limits_{|p|\leqslant m} \varrho(\lambda_p D^p\varphi)}{1+\sup\limits_{|p|\leqslant m} \varrho(\lambda_p D^p\varphi)};$$

then the condition $\varphi_i \xrightarrow{\tilde{\varrho}} 0$ is equivalent to the condition

$$\lim_{i\to\infty} \tilde{\varrho}(\{\lambda_p\}, \varphi_i) = 0. \tag{4.1}$$

Proof. Let $\varphi_i \xrightarrow{\tilde{\varrho}} 0$, i.e. $\varrho(\lambda_p D^p\varphi_i) \to 0$, and let us choose an $\varepsilon > 0$. We take k_0 so large that $\sum\limits_{m=k_0+1}^{\infty} \frac{1}{2^m} < \frac{1}{2}\varepsilon$ and we choose i_0 in such a manner that $\varrho(\lambda_p D^p\varphi_i) < \frac{1}{2}\varepsilon$ for $|p| \leqslant k_0$ and $i \geqslant i_0$. Then

$$\tilde{\varrho}(\{\lambda_p\}, \varphi_i) \leqslant \frac{\varepsilon}{2} \sum_{m=1}^{k_0} \frac{1}{2^m} + \sum_{m=k_0+1}^{\infty} \frac{1}{2^m} \leqslant \varepsilon$$

for $i \geqslant i_0$. Consequently, $\lim\limits_{i\to\infty} \tilde{\varrho}(\{\lambda_p\}, \varphi_i) = 0$. The converse implication is obvious.

2. In order to formulate the fundamental theorems concerning spaces D_ϱ, we shall need some further notation and terminology.

A set $A \subset D_\varrho$ is called *bounded in D_ϱ*, if there exists a sequence of positive numbers $\{K_p\}$ such that the inequality $\|D^p\varphi\|_{(\varrho)} \leqslant K_p$ holds for every function $\varphi \in A$ and for each p.

A modular ϱ_1 is called *non-stronger than a modular* ϱ_2 (we write $\varrho_1 \prec \varrho_2$), if for every non-negative sequence of numbers $\{\lambda_p\}$ there exists a constant $k > 0$ such that for every function $\varphi \in \mathscr{E}$ satisfying the condition $\varrho_1(\lambda_p D^p\varphi) \leqslant 1$ for every p, there holds the inequality $\varrho_2(k\varphi) < 1$.

We shall say that a modular ϱ satisfies the condition (Δ_2) (we write $\varrho \in (\Delta_2)$, if for every sequence of positive numbers $\{\lambda_p\}$ there exists a constant $a > 0$ such that for every function φ satisfying the condition $\tilde{\varrho}(\{\lambda_p\}, \varphi) < a$ there holds the inequality $\varrho(2\varphi) < 1$.

Now, some important theorems concerning spaces D_ϱ will be given.

THEOREM 5. *If $\varrho \in (a)$, then for every sequence $\{\varphi_i\}$, $\varphi_i \in D_\varrho$ such that the supports of φ_i, $S_{\varphi_i} \subset A$, where A is a compact set, the condition $\varphi_i \xrightarrow{D} 0$ is equivalent to the condition $\varphi_i \xrightarrow{D_\varrho} 0$.*

Proof. Let $D^p\varphi_i \to 0$ uniformly on a compact set A, and let k_0 be so large that $A \subset S_{\sum_{j=1}^{k_0} \omega_j}$, where ω_j are the same as in Lemma 1. Let ε satisfy the inequalities $0 < \varepsilon < 1$, then there exists an i_0 such that $|D^p\varphi_i| < \varepsilon$ for $i \geqslant i_0$, i.e. $\varrho(\lambda D^p\varphi_i) \leqslant \varrho(\varepsilon\lambda \sum_{j=1}^{k_0+1} \omega_j) \leqslant \varepsilon M$, where λ is an arbitrary positive number. From the last inequality it follows that $\varphi_i \xrightarrow{D_\varrho} 0$.

Converse implication follows from Lemma 2, immediately.

THEOREM 6. *Let $\varphi \in \mathscr{E}$. Then $\varphi \in D_\varrho$, if and only if, $\sum_{j=k}^{\infty} \varphi\omega_j \xrightarrow{\tilde{\varrho}} 0$, ω_j being defined as in Lemma 1. Moreover, $\varrho(\lambda D^p\varphi) < \infty$ for every p and for every $\lambda > 0$, if and only if,*

$$\sum_{j=k}^{\infty} \varphi\omega_j \xrightarrow{D_\varrho} 0. \tag{6.1}$$

Thus, the subspace D of functions from $\mathscr{E}$ of compact support, is dense in D_ϱ in the sense of $\tilde{\varrho}$-convergence.

Proof. Let $\varrho(\lambda_p D^p\varphi) < \infty$. By Lemma 1, there holds $\lim_{k\to\infty} \varrho(\lambda_p D^p\varphi \sum_{j=k}^{\infty}\omega_j) = 0$ for every p. Now, we prove that for every function $\psi \in \mathscr{E}$ such that $\varrho(\lambda\psi) < \infty$ for some $\lambda > 0$, there holds

$$\psi D^p\Big(\sum_{j=k}^{\infty} \omega_j\Big) \xrightarrow{\varrho} 0. \tag{6.2}$$

Indeed, by [4], Lemma 1, there exists a sequence of numbers $\{K_p\}$ such that $\big|D^p\big(\sum_{j=k}^{\infty} \omega_j\big)\big| \leqslant K_p$. Then

$$\varrho\Big[\frac{\lambda}{K_p}\psi D^p\Big(\sum_{j=k}^{\infty} \omega_j\Big)\Big] = \varrho\Big[\frac{\lambda}{K_p}\psi D^p\Big(\sum_{j=k}^{\infty} \omega_j\Big)\sum_{j=k-1}^{\infty} \omega_j\Big] \leqslant \varrho\Big[\lambda\psi \sum_{j=k-1}^{\infty} \omega_j\Big].$$

Hence, Lemma 1 implies

$$\lim_{k\to\infty} \varrho\left[\frac{\lambda}{K_p}\psi D^p\Big(\sum_{j=k}^{\infty}\omega_j\Big)\right] = 0$$

and this proves (6.2). Now Leibniz formula gives

$$D^p\Big(\varphi\sum_{j=k}^{\infty}\omega_j\Big) = \sum_{0\leqslant|v|\leqslant|p|}\binom{p_1}{v_1}\cdots\binom{p_n}{v_n}D^{p^v}\varphi D^v\Big(\sum_{j=k}^{\infty}\omega_j\Big),$$

where $p = (p_1, \ldots, p_n)$, $v = (v_1, \ldots, v_n)$, $p^v = (p_1-v_1, \ldots, p_n-v_n)$.

Let

$$\vartheta_p = \min_{0\leqslant|q|\leqslant|p|}\lambda_q\Big(\sum_{0\leqslant|v|\leqslant|p|}\binom{p_1}{v_1}\cdots\binom{p_n}{v_n}\Big)^{-1}.$$

Then, by (6.2), we obtain

$$\lim_{k\to\infty}\varrho\left[\vartheta_p D^p\Big(\varphi\sum_{j=k}^{\infty}\omega_j\Big)\right] \leqslant \lim_{k\to\infty}\sum_{0\leqslant|v|\leqslant|p|}\varrho\left[\lambda_{p^v}D^{p^v}\varphi D^v\Big(\sum_{j=k}^{\infty}\omega_j\Big)\right] = 0.$$

Now, let $\lim\limits_{k\to\infty}\varrho\big[\lambda_p D^p\big(\sum\limits_{j=k}^{\infty}\varphi\omega_j\big)\big] = 0$; then

$$\lim_{k\to\infty}\varrho\left[\lambda_p D^p\varphi\sum_{j=k+1}^{\infty}\omega_j\right] \leqslant \lim_{k\to\infty}\varrho\left[\lambda_p D^p\Big(\varphi\sum_{j=k}^{\infty}\omega_j\Big)\right] = 0.$$

Hence, Lemma 1 shows that $\varphi\in D_\varrho$. Converse statement is proved analogously.

THEOREM 7. *Let us consider the following conditions:*

(A) *if a set A is bounded in D_{ϱ_1}, then A is bounded in D_{ϱ_2};*

(B) *if $\varphi_i \xrightarrow{D_{\varrho_1}} 0$, then $\varphi_i \xrightarrow{D_2} 0$;*

(C) $\varrho_1 \dashv \varrho_2$;

(D) $D_{\varrho_1}\subset D_{\varrho_2}$.

Conditions (A), (B) *and* (C) *are mutually equivalent. Condition* (C) *implies condition* (D). *Moreover, if $\varrho_1\in(a)$ and $\varrho_2\in(a)$, then all conditions* (A), (B), (C) *and* (D) *are mutually equivalent.*

Proof. Let (A) be satisfied and let $\varphi_i \xrightarrow{D_{\varrho_1}} 0$. In particular, $\|\varphi_i\|_{(\varrho_1)}\to 0$; hence there exists a subsequence $\{\varphi_{i_k}\}$ such that $\|k\varphi_{i_k}\|_{(\varrho_1)}\to 0$. Let us arrange the systems p in a sequence $\{p_s\}$, where $p_0 = (0, 0, \ldots, 0)$. Since $\|D^{p_1}\varphi_{i_k}\|_{(\varrho_1)}\to 0$, from the sequence $\{\varphi_{i_k}\}$ one may extract a subsequence

$\{\varphi_{i_{k_1}}\}$ for which $\|lD^{p_1}\varphi_{i_{k_1}}\|_{(\varrho_1)} \to 0$. Proceding further in the same manner, we get (applying the diagonal method) a subsequence $\{\varphi_{i_s}\}$ such that $\|sD^p\varphi_{i_s}\|_{(\varrho_1)} \leqslant K_p$. By (A), there exists a sequence of numbers $\{K'_p\}$ satisfying the inequality $\|sD^p\varphi_{i_s}\|_{(\varrho_2)} \leqslant K'_p$, i.e.

$$\|D^p\varphi_{i_s}\|_{(\varrho_2)} \to 0. \tag{7.1}$$

Hence it follows easily that $\|D^p\varphi_i\|_{(\varrho_2)} \to 0$.

Now, let condition (B) be satisfied, and let us suppose that $\varrho_1 \dashv \varrho_2$ does not hold, i.e. $\varrho_1(\lambda_p D^p\varphi_v) \leqslant 1$ and $\varrho_2(v^{-2}\varphi_v) \geqslant 1$ for every p, and for a sequence of positive numbers and a sequence of functions $\{\varphi_v\}$, $\varphi_v \in \mathscr{E}$. Let $\psi_v = v^{-2}\varphi_v$ and let λ be an arbitrary positive number. Taking p fixed, we choose an index v_0 such that $v^{-1}\lambda \leqslant \lambda_p$ for $v \geqslant v_0$. Then

$$\varrho_1(\lambda D^p\psi_v) \leqslant \frac{1}{v}\,\varrho_1\left(\frac{\lambda}{v}\,D^p\varphi_v\right) \leqslant \frac{1}{v}\,\varrho_1\left(\lambda_p D^p\varphi_v\right) \leqslant \frac{1}{v}$$

for $v \geqslant v_0$, i.e. $\psi_v \xrightarrow{D_{\varrho_1}} 0$. Since $\varrho_2(\psi_v) = \varrho_2(v^{-2}\varphi_v) \geqslant 1$, the condition $\varphi_v \xrightarrow{D_{\varrho_2}} 0$ does not hold, a contradiction to the assumption. Thus, condition (C) is satisfied.

Let $\varrho_1 \dashv \varrho_2$ and let A be a bounded set in D_{ϱ_1}, i.e. $\|K_p^{-1}D^p\varphi\|_{(\varrho_1)} \leqslant 1$ for every $\varphi \in A$. Hence $\varrho_1(K_p^{-1}D^p\varphi) \leqslant 1$, and so $\varrho_2(k\varphi) < 1$, where k is the constant in (C) chosen with respect to the sequence of numbers $\lambda_p = K_p^{-1}$. Thus, we obtain $\|\varphi\|_{(\varrho_2)} \leqslant 1/k$. Repeating the argument in case of the functions $D^p\varphi$ in an analogous manner, we get condition (A). Hence, we proved the equivalency of conditions (A), (B) and (C).

The fact that (C) implies (D) is obvious.

Now, let $\varrho_1 \in (a)$, $\varrho_2 \in (a)$ and $D_{\varrho_1} \subset D_{\varrho_2}$. Let T be the operation of embedding of D_{ϱ_1} into D_{ϱ_2}, and let $\varphi_i \xrightarrow{D_{\varrho_1}} \bar{\varphi}$, $T(\varphi_i) = \varphi_i \xrightarrow{D_{\varrho_2}} \bar{\bar{\varphi}}$. By Lemma 2, $\varphi_i - \bar{\varphi} \to 0$ and $\varphi_i - \bar{\bar{\varphi}} \to 0$, both almost uniformly. Hence $T(\bar{\varphi}) = \bar{\bar{\varphi}}$, and the closed graph theorem implies condition (B).

THEOREM 8. *Let us consider the following conditions*:

(a) $\varrho \in (\Delta_2)$,

(b) *if* $\varphi_i \xrightarrow{\tilde{\varrho}} 0$, *then* $\varphi_i \xrightarrow{D_\varrho} 0$,

(c) $D_\varrho = \{\varphi \in \mathscr{E}\colon \varrho(\lambda D^p\varphi) < \infty$ *for every* p *and every* $\lambda > 0\}$,

(d) *the space* D *is dense in* D_ϱ.

Then conditions (a), (b) *are equivalent, conditions* (c), (d) *are equivalent, and condition* (a) *implies condition* (c). *Moreover, if* $\varrho \in (a)$, *then all conditions* (a), (b), (c), (d) *are mutually equivalent.*

Proof. Let $\varrho \in (\Delta_2)$ and $\varphi_i \xrightarrow{\varrho} 0$. By Lemma 4, $\tilde{\varrho}(\{\lambda_p\}, \varphi_i) \to 0$ for a sequence of numbers $\{\lambda_p\}$. To this sequence we choose a number a_1 from the condition (Δ_2). Then there exists an index i_1 for which $\tilde{\varrho}(\{\lambda_p\}, \varphi_{i_1}) < a_1$

and the condition (Δ_2) gives $\varrho(2\varphi_{i_1}) < 1$. Now, we choose to the sequence $\{\frac{1}{2}\lambda_p\}$ a number α_2 from the condition (Δ_2). Then there exists an index i_2 such that $\tilde{\varrho}(\{\frac{1}{2}\lambda_p\}, 2\varphi_{i_2}) < \alpha_2$, whence $\varrho(2\cdot 2\varphi_{i_2}) < 1$, etc. In this manner we may define a subsequence $\{\varphi_{i_k}\}$ for which $\varrho(2^{k-1}2\varphi_{i_k}) < 1$; hence $\lim\limits_{k\to\infty} \varrho(2\varphi_{i_k}) = 0$. An analogous argumentation enables us to extract a subsequence $\{\varphi_{i_{k_s}}\}$ from the sequence $\{\varphi_{i_k}\}$ for which $\lim\limits_{s\to\infty} \varrho(2D^{p_1}\varphi_{i_{k_s}}) = 0$, where p_1 is the first of all multiindices $p_i \neq (0, 0, \ldots, 0)$ put in a sequence $p_1, p_2, \ldots$ Repeating the same argumentation and applying the diagonal method, one may finally define a subsequence $\{\varphi_{i_l}\}$ such that $\lim\limits_{l\to\infty} \varrho(2D^p\varphi_{i_l}) = 0$ for each p. It is easily seen that also $\lim\limits_{l\to\infty} \varrho(\lambda D^p\varphi_{i_l}) = 0$ for each p and for every $\lambda > 0$. But this proves that $\varphi_{i_l} \overset{D_\varrho}{\to} 0$, and it is easily concluded that $\varphi_i \overset{D_\varrho}{\to} 0$. Since the converse implication is obvious, the equivalence of (a) and (b) is proved.

Now, let us suppose (c) to be satisfied, and let $\varphi \in D_\varrho$. We take a sequence of numbers $\{K_p\}$ such that $|D^p\omega_j| \leqslant K_p$ for every j, where ω_j are defined as in Lemma 1. Then $\left|D^p\left(\sum\limits_{j=1}^{k} \omega_j\right)\right| \leqslant K_p$ for every k. Hence we have for an arbitrary $\lambda > 0$

$$\varrho\left[\lambda\varphi D^p\left(\sum_{j=1}^{k} \omega_j\right)\right] = \varrho\left[\left(\lambda\varphi D^p \sum_{j=1}^{k} \omega_j\right) \sum_{j=k}^{\infty} \omega_j\right] \leqslant \varrho\left(\lambda K_p\varphi \sum_{j=k}^{\infty} \omega_j\right).$$

But, by condition (c) and Lemma 1, the expression on the right-hand side of the last inequality tends to zero as $k \to \infty$. Hence

$$\lim_{k\to\infty} \left\|\varphi D^p\left(\sum_{j=1}^{k} \omega_j\right)\right\|_{(\varrho)} = 0 \tag{8.1}$$

for an arbitrary $\varphi \in D_\varrho$. Applying (8.1) and the Leibniz formula we obtain easily that

$$\lim_{k\to\infty} \left\|D^p\left(\varphi \sum_{j=1}^{k} \omega_j\right) - D^p\varphi\right\|_{(\varrho)} = 0$$

i.e. $\varphi_k = \sum\limits_{j=1}^{k} \varphi\omega_j \overset{D_\varrho}{\to} \varphi$, and $\varphi_k \in D$. Hence D is dense in D_ϱ. Conversely, let us suppose condition (d), and let $\varphi \in D_\varrho$.

We take a fixed p, and we choose an arbitrary positive λ. Then

$$\varrho(\lambda D^p\varphi) \leqslant \varrho\left(2\lambda D^p\varphi \sum_{j=1}^{k} \omega_j\right) + \varrho\left(2\lambda D^p\varphi \sum_{j=k+1}^{\infty} \omega_j\right).$$

Since D is dense in D_ϱ, there exists a sequence $\{\varphi_i\}$, $\varphi_i \in D$ such that $\varphi_i \overset{D_\varrho}{\to} \varphi$,

i.e. $\lim \varrho[2\lambda D^p\varphi - 2\lambda D^p\varphi_i] = 0$. Let i_0 and k_0 be two indices chosen in such a manner that

$$\varrho[2\lambda D^p\varphi - 2\lambda D^p\varphi_{i_0}] < 1 \quad \text{and} \quad S_{\varphi_{i_0}} \cap S_{\sum\limits_{j=k_0+1}^{\infty} \omega_j} = \emptyset.$$

Then we have

$$\varrho(\lambda D^p\ \varphi) \leqslant \varrho\Big[2\lambda D^p\varphi \sum_{j=1}^{k_0} \omega_j\Big] + \\ + \varrho\Big[2\lambda D^p\varphi \sum_{j=k_0+1}^{\infty} \omega_j + 2\lambda D^p(\varphi - \varphi_{i_0}) \sum_{j=1}^{k_0} \omega_j\Big] < \infty.$$

Hence condition (c) is satisfied.

Next, let us suppose $\varrho \in (\Delta_2)$ and $\varphi \in D_\varrho$, i.e. $\varrho(\lambda_p D^p\varphi) < \infty$ for a sequence of numbers $\{\lambda_p\}$. By Theorem 6, $\varphi \sum\limits_{j=k}^{\infty} \omega_j \xrightarrow{\tilde{\varrho}} 0$. We choose a number a from the condition (Δ_2), corresponding to the sequence $\{\lambda_p\}$. Then $\tilde{\varrho}\big(\{\lambda_p\}, \varphi \sum\limits_{j=k}^{\infty} \omega_j\big) < a$ for $k \geqslant k_0$, and (Δ_2) implies $\varrho\big(2\varphi \sum\limits_{j=k}^{\infty} \omega_j\big) < 1$. Hence $\varrho(2\varphi) < \infty$. The proof for functions $D^p\varphi$ and constants λ of the form $\lambda = 2^k$ is performed analogously. Thus, we proved (c).

Finally, let $\varrho \in (a)$, and let condition (c) be satisfied. Let $\{\varphi_i\}$ be $\tilde{\varrho}$-convergent to zero, i.e. $\lim\limits_{i\to\infty} \varrho(\lambda_p D^p\varphi_i) = 0$ for a sequence of numbers $\{\lambda_p\}$. Since $\lim\limits_{s\to\infty} \varrho\big(\lambda D^p\varphi_i \sum\limits_{j=s}^{\infty} \omega_j\big) = 0$ for an arbitrary $\lambda > 0$, we obtain $\lim\limits_{s\to\infty} \big\|\varphi_i \sum\limits_{j=s}^{\infty} \omega_j\big\|_\varrho = 0$. Hence there exists a sequence of indices $\{s_i\}$ for which

$$\lim_{i\to\infty} \Big\|\varphi_i \sum_{j=s_i}^{\infty} \omega_j\Big\|_\varrho = 0. \tag{8.2}$$

By Lemma 2, the sequence $\{\varphi_i\}$ is uniformly convergent on every compact set. Let r_1 be an arbitrary positive integer. Then the sequence $\{\varphi_i \sum\limits_{j=1}^{r_1} \omega_j\}$ is convergent in the sense of convergency in D. By Theorem 5, $\lim\limits_{i\to\infty} \big\|\varphi_i \sum\limits_{j=1}^{r_1} \omega_j\big\|_\varrho = 0$; hence there exists an index i_1 such that $r_1 < s_{i_1}$ and $\big\|\varphi_{i_1} \sum\limits_{j=1}^{r_1} \omega_j\big\|_\varrho \leqslant 1$. Let r_2 be an arbitrary positive integer such that $r_2 > s_{i_1}$. Then $\lim\limits_{i\to\infty} \big\|\varphi_i \sum\limits_{j=1}^{r_2} \omega_j\big\|_\varrho = 0$. Hence there exists an index i_2 such that $r_2 < s_{i_2}$ and $\big\|\varphi_{i_2} \sum\limits_{j=1}^{r_2} \omega_j\big\|_\varrho$

$\leqslant \frac{1}{2}$, etc. Continuing this procedure, we may define two sequences of indices $\{i_m\}$ and $\{r_m\}$ such that $s_{i_{m-1}} < r_m < s_{i_m}$ and $\|\varphi_{i_m} \sum_{j=1}^{r_m} \omega_j\|_\varrho \leqslant 1/m$. Then

$$\lim_{m\to\infty} \Big\|\varphi_{i_m} \sum_{j=1}^{r_m} \omega_j\Big\|_\varrho = 0. \tag{8.3}$$

Since $\lim_{m\to\infty} \varrho(\lambda_0 \varphi_{i_m}) = 0$, there exists a subsequence $\{i_{m_{0,k}}\}$ of the sequence of indices $\{i_m\}$ such that

$$\varrho\Big(\lambda_0 \varphi_{i_{m_{0,k}}} \sum_{r=j_{m_{0,k}}}^{s_{i_{m_{0k}}}} \omega_j\Big) \leqslant \frac{1}{2^k}.$$

Let $p_1, p_2, \ldots$ be the sequence of all multiindices $p \neq (0, 0, \ldots, 0)$. Then, in particular, $\lim \varrho(\lambda_1 D^{p_1} \varphi_{i_{m_{0,k}}}) = 0$. Hence there is a subsequence $\{i_{m_{1,k}}\}$ of the sequence $\{i_{m_{0,k}}\}$ for which

$$\varrho\Big[\lambda_1 D^{p_1}\Big(\varphi_{i_{m_{1,k}}} \sum_{j=r_{m_{1,k}}}^{s_{i_{m_{1,k}}}} \omega_j\Big)\Big] \leqslant \frac{1}{2^k},$$

etc. Continuing this procedure and applying the diagonal method we obtain a sequence $\{i_{m_{k,k}}\}$ such that

$$\varrho\Big[\lambda_p D^p\Big(\varphi_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}}} \omega_j\Big)\Big] \leqslant \frac{1}{2^k}$$

for $k \geqslant k_p$; it follows from the construction that the supports of functions on the left-hand side of this inequality are disjoint for different k. Let us define

$$\varphi = \sum_{k=0}^{\infty} \Big(\varphi_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}}} \omega_j\Big).$$

Then

$$\varrho(\lambda_p D^p \varphi) = \sum_{k=0}^{\infty} \varrho\Big[\lambda_p D^p\Big(\varphi_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}}} \omega_j\Big)\Big] \leqslant 1,$$

i.e. $\varphi \in D_\varrho$. According to the assumption we obtain $\varrho(\lambda D^p \varphi) < \infty$ for each p and for an arbitrary $\lambda > 0$. Hence we get

$$\lim_{k\to\infty} \varrho\Big[\lambda D^p\Big(\varphi_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}}} \omega_j\Big)\Big] = 0$$

for an arbitrary $\lambda > 0$, i.e.

$$\lim_{k\to\infty} \Big\| \varphi_{i_{m_{k,k}}} \sum_{j=r_{m_{k,k}}}^{s_{i_{m_{k,k}}}} \omega_j \Big\|_\varrho = 0. \tag{8.4}$$

From (8.3), (8.4) and (8.2) we conclude $\lim_{k\to\infty} \|\varphi_{i_{m_{k,k}}}\|_\varrho = 0$, that is $\varphi_{i_{m_{k,k}}} \xrightarrow{D_\varrho} 0$. This implies easily $\varphi_i \xrightarrow{D_\varrho} 0$, and the proof is finished.

6. In the following, we shall construct some examples of spaces D_ϱ. First, we give the definition of an M-function, and we prove a lemma useful in our further considerations. This lemma generalizes a lemma given in [1].

A function $M(t, u) = M(t_1, t_2, \ldots, t_n, u)$ defined on R^{n+1} will be called an *M-function in the variable u*, if

(B.1) for every u, the function $M(t, u)$ is a continuous function of the variable t;

(B.2) $M(t, u) \geqslant 0$; if $u = 0$, then $M(t, u) = 0$;

(B.3) $M(t, u) = M(t, -u)$;

(B.4) $M(t, \alpha u_1 + \beta u_2) \leqslant \alpha M(t, u_1) + \beta M(t, u_2)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$;

(B.5) for every t, there holds

$$\lim_{u\to 0} \frac{M(t, u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{M(t, u)}{u} = \infty.$$

Let P denote the set of all multiindices $p = (p_1, \ldots, p_n)$ such that $p_i = 0$ for $i = i_k$, where $\{i_k\}$ are indices extracted from $1, 2, \ldots, n$, $p_i = 1$ for the remaining i, and let $|p| = p_1 + p_2 + \ldots + p_n$. Then we denote by $\partial^p M(t, u)$ the derivative $\partial^{|p|} M(t_1, \ldots, t_n, u)/\partial t_1^{p_1} \ldots \partial t_n^{p_n}$; $\partial^p M\big(t, u(t)\big)$ will stand for the derivative $\partial^{|p|} M(t_1, \ldots, t_n, u)/\partial t_1^{p_1} \ldots \partial t_n^{p_n}$ calculated at the point $u = u(t_1, \ldots, t_n)$.

LEMMA 9. *Let $\partial^p M(t, u)$ be M-functions in the variable u and let φ possess continuous derivatives $D^p\varphi$ in the set $\{t\colon |t| \geqslant |y|\}$ for all $p \in P$. Then for every x such that $|x| \geqslant |y|$ there holds the inequality*

$$M\big(x, \varphi(x)\big) \leqslant \sum_{p+q\in P} \int_{|t|\geqslant|x|} \partial^p M\big[t, 2^{n-|p+q|} D^q \varphi(t)\big] dt.$$

Proof. Let $C(x, r)$ be one of the 2^n n-dimensional closed cubes with a vertex at x and with sides of length r parallel to the coordinate axes, and let $|x| \geqslant |y|$. Moreover, let J_k be the projection of the cube $C(x, r)$ on the axis x_k; we fix the cube $C(x, r)$ in such a manner that

$$J_k = \{t_k\colon x_k \leqslant t_k \leqslant x_k + r\} \quad \text{for } x_k > 0,$$
$$J_k = \{t_k\colon x_k - r \leqslant t_k \leqslant x_k\} \quad \text{for } x_k < 0,$$

where $x = (x_1, \ldots, x_n)$. Thus, $C(x, r) = J_1 \times J_2 \times \ldots \times J_n$. Let $P_k = \{p \in P\colon p = (p_1, \ldots, p_{k-1}, 0, \ldots, 0)\}$, $P'_k = \{p' \in P\colon p' = (0, \ldots, 0, p_k, \ldots, p_n)\}$. Let us denote for every $1 \leqslant k \leqslant n$ and every p

$$S_k^p[\varphi] = \int_{J_k} \ldots \int_{J_n} \partial^p M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)]\,dt.$$

Then the following inequality holds for every $p' \in P'_k$:

$$S_k^{p'}[\varphi] \leqslant \sum_{p+q \in P_k} \left(1 + \frac{1}{r}\right)^{k-1-|p+q|} S_1^{p+p'}[2^{k-1-|p+q|} D^q \varphi]. \tag{9.1}$$

The proof of formula (9.1) will be performed by induction.

For $k = 1$ we have

$$S_1^{p'} = \int_{J_1} \ldots \int_{J_n} \partial^{p'} M[t, \varphi(t)]\,dt = \int_{C(x,r)} \partial^{p'} M[t, \varphi(t)]\,dt,$$

and formula (9.1) is true. Let us suppose (9.1) to be true for $1 \leqslant k < n$. The mean-value theorem gives

$$\frac{1}{r} \int_{J_k} \partial^{p'} M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)]\,dt_k$$
$$= \partial^{p'} M[x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n)],$$

where $\theta_k = \theta_k(x_1, \ldots, x_{k-1}, t_{k+1}, \ldots, t_n) \in J_k$. Hence, we have for every $p' \in P'_{k+1}$:

$$S_{k+1}^{p'}[\varphi] = \int_{J_{k+1}} \ldots \int_{J_n} \partial^{p'} M[x_1, \ldots, x_k, t_{k+1}, \ldots, t_n, \varphi(x_1, \ldots, x_k, t_{k+1}, \ldots, t_n)]\,dt$$
$$= \int_{J_{k+1}} \ldots \int_{J_n} \left\{ \int_{\theta_k}^{x_k} \left(\frac{\partial}{\partial t_k}\right) \partial^{p'} M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots \right.$$
$$\left. \ldots, x_{k-1}, t_k, \ldots, t_n)]\,dt_k \right\} dt + \int_{J_{k+1}} \ldots \int_{J_n} \partial^{p'} M[x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots$$
$$\ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, \theta_k, t_{k+1}, \ldots, t_n)]\,dt.$$

Denoting by $\overline{p}'$ multiindex, obtained from the multiindex $p' \in P'_{k+1}$ by putting 1 in place of 0 at the k-th place, we obtain

$$S_{k+1}^{p'}[\varphi] = \int_{J_{k+1}} \ldots \int_{J_n} \int_{\theta_k}^{x_k} \partial^{\overline{p}'} M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)]\,dt +$$
$$+ \int_{J_{k+1}} \ldots \int_{J_n} \int_{\theta_k}^{x_k} \left\{ \left(\frac{\partial}{\partial v}\right) \partial^{p'} M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, v] \Big|_{v=\varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)} \times \right.$$
$$\left. \times \left(\frac{\partial}{\partial t_k}\right) \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n) \right\} dt +$$
$$+ \int_{J_{k+1}} \ldots \int_{J_n} \frac{1}{r} \int_{J_k} \partial^{p'} M[x_1, \ldots, x_{k-1}, t_k, \ldots, t_n, \varphi(x_1, \ldots, x_{k-1}, t_k, \ldots, t_n)]\,dt.$$

We apply the following inequality (see [1])

$$\partial M[x, f(t)]/\partial t \leqslant M[x, 2f(t)] + M[x, \partial f(t)/\partial t]$$

to the integrand in the second term on the right-hand side of the above inequality. We obtain

$$\begin{aligned}
S_{k+1}^{p'}[\varphi] \leqslant & \int_{J_k} \dots \int_{J_n} \partial^{\bar{p}'} M[x_1, \dots, x_{k-1}, t_k, \dots, t_n, \varphi(x_1, \dots, x_{k-1}, t_k, \dots, t_n)]\,dt + \\
& + \int_{J_k} \dots \int_{J_n} \partial^{p'} M[x_1, \dots, x_{k-1}, t_k, \dots, t_n, 2\varphi(x_1, \dots, x_{k-1}, t_k, \dots, t_n)]\,dt + \\
& + \int_{J_k} \dots \int_{J_n} \partial^{p'} M\left[x_1, \dots, x_{k-1}, t_k, \dots, t_n, \frac{\partial}{\partial t_k}\varphi(x_1, \dots, x_{k-1}, t_k, \dots, t_n)\right] dt + \\
& + \frac{1}{r} \int_{J_k} \dots \int_{J_n} \partial^{p'} M[x_1, \dots, x_{k-1}, t_k, \dots, t_n, 2\varphi(x_1, \dots, x_{k-1}, t_k, \dots, t_n)]\,dt \\
= & S_k^{p'}\left[\frac{\partial}{\partial t_k}\varphi\right] + \left(1 + \frac{1}{r}\right) S_k^{p'}[2\varphi] + S_k^{\bar{p}'}[\varphi].
\end{aligned}$$

By (9.1), we get

$$\begin{aligned}
S_{k+1}^{p'}[\varphi] \leqslant & \sum_{p+q \in P_k} \left(1 + \frac{1}{r}\right)^{k-1-|p+q|} S_1^{p+p'}\left[2^{k-1-|p+q|} D^q\left(\frac{\partial}{\partial t_k}\varphi\right)\right] + \\
& + \left(1 + \frac{1}{r}\right) \sum_{p+q \in P_k} \left(1 + \frac{1}{r}\right)^{k-1-|p+q|} S_1^{p+p'}[2^{k-1-|p+q|} D^q(2\varphi)] + \\
& + \sum_{p+q \in P_k} \left(1 + \frac{1}{r}\right)^{k-1-|p+q|} S_1^{p+\bar{p}'}[2^{k-1-|p+q|} D^q \varphi].
\end{aligned}$$

Let p and q be arbitrary multiindices belonging to P_{k+1}, and let $\bar{p}$ and $\bar{q}$ be obtained from p and q, respectively, by replacing 0 by 1 at the k-th place. Then

$$\begin{aligned}
S_{k+1}^{p'}[\varphi] \leqslant & \sum_{\bar{p}+q \in P_{k+1}} \left(1 + \frac{1}{r}\right)^{(k+1)-1-|p+q|} S_1^{p+p'}[2^{(k+1)-1-|p+\bar{q}|} D^{\bar{q}}\varphi] + \\
& + \sum_{p+q \in P_k} \left(1 + \frac{1}{r}\right)^{(k+1)-1-|p+q|} S_1^{p+p'}[2^{(k+1)-1-|p+q|} D^q \varphi] + \\
& + \sum_{\bar{p}+q \in P_{k+1}} \left(1 + \frac{1}{r}\right)^{(k+1)-1-|\bar{p}+q|} S_1^{\bar{p}+p'}[2^{(k+1)-1-|\bar{p}+q|} D^q \varphi] \\
= & \sum_{p+q \in P_{k+1}} \left(1 + \frac{1}{r}\right)^{(k+1)-1-|p+q|} S_1^{p+p'}[2^{(k+1)-1-|p+q|} D^q \varphi].
\end{aligned}$$

Thus, (9.1) is proved for $1 \leqslant k < n$.

Applying (9.1), we prove now the following inequality

$$(9.2)\quad M(x, \varphi(x)) \leqslant \sum_{p+q\in P}\left(1+\frac{1}{r}\right)^{n-|p+q|}\int_{C(x,r)} \partial^p M[t, 2^{n-|p+q|}D^q\varphi(t)]\,dt.$$

We have, by the mean-value theorem,

$$\frac{1}{r}\int_{J_n} M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)]\,dt_n$$
$$= M[x_1, \ldots, x_{n-1}, \theta_n, \varphi(x_1, \ldots, x_{n-1}, \theta_n)],$$

where $\theta_n = \theta_n(x_1, \ldots, x_{n-1}) \in J_n$. Thus,

$$M(x, \varphi(x)) = \int_{\theta_n}^{x_n}\left(\frac{\partial}{\partial t_n}\right) M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)]\,dt_n +$$
$$+ M[x_1, \ldots, x_{n-1}, \theta_n, \varphi(x_1, \ldots, x_{n-1}, \theta_n)].$$

Denoting by p^n a multiindex belonging to P which has 1 only at the n-th place, we get

$$M(x, \varphi(x)) \leqslant \int_{\theta_n}^{x_n} \partial^{p^n} M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)]\,dt_n +$$
$$+ \int_{\theta_n}^{x_n} M[x_1, \ldots, x_{n-1}, t_n, 2\varphi(x_1, \ldots, x_{n-1}, t_n)]\,dt_n +$$
$$+ \int_{\theta_n}^{x_n} M[x_1, \ldots, x_{n-1}, t_n, \frac{\partial}{\partial t_n}\varphi(x_1, \ldots, x_{n-1}, t_n)]\,dt_n +$$
$$+ \frac{1}{r}\int_{J_n} M[x_1, \ldots, x_{n-1}, t_n, \varphi(x_1, \ldots, x_{n-1}, t_n)]\,dt_n$$
$$\leqslant S_n^0\left[\frac{\partial}{\partial t_n}\varphi\right] + \left(1+\frac{1}{r}\right) S_n^0[2\varphi] + S_n^{p^n}[\varphi].$$

By (9.1), we obtain the inequality

$$M(x, \varphi(x)) \leqslant \sum_{p+q\in P_n}\left(1+\frac{1}{r}\right)^{n-1-|p+q|} S_1^p\left[2^{n-1-|p+q|}D^q\left(\frac{\partial}{\partial t_n}\varphi\right)\right] +$$
$$+ \left(1+\frac{1}{r}\right)\sum_{p+q\in P_n}\left(1+\frac{1}{r}\right)^{n-1-|p+q|} S_1^p[2^{n-1-|p+q|}D^q(2\varphi)] +$$
$$+ \sum_{p+q\in P_n}\left(1+\frac{1}{r}\right)^{n-1-|p+q|} S_1^{p+p^n}[2^{n-1-|p+q|}D^q\varphi]$$

$$= \sum_{p+q\in P}\left(1+\frac{1}{r}\right)^{n-|p+q|} S_1^p[2^{n-|p+q|}D^q\varphi]$$

$$= \sum_{p+q\in P}\left(1+\frac{1}{r}\right)^{n-|p+q|} \int_{C(x,r)} \partial^p M[t, 2^{n-|p+q|}D^q\varphi(t)]\,dt.$$

Thus, inequality (9.2) is proved.

Evidently, $C(x,r)\subset\{t\colon |t|\geqslant|x|\}$. Hence, taking $r\to\infty$ in (9.2), we obtain

$$M\big(x,\varphi(x)\big)\leqslant \sum_{p+q\in P}\int_{|t|\geqslant|x|} \partial^p M[t, 2^{n-|p+q|}D^q\varphi(t)]\,dt,$$

and the lemma is proved.

A real-valued function $M(t,u)=M(t_1,\ldots,t_n,u)$ defined on R^{n+1} will be called a *∂M-function*, if

(C.1) the functions $\partial^p M(t,u)$ are M-functions in the variable u for every $p\in P$;

(C.2) $M(t,u)\to\infty$ as $u\to\infty$ uniformly with respect to the variable t;

(C.3) if $\lim\limits_{|t|\to\infty} M(t,u)=0$, then $u=0$;

(C.4) for every t, $M(t,u)=0$ implies $u=0$.

If we define the modular $\varrho_{\partial M}$ by the formula

$$\varrho_{\partial M}(\varphi)=\sup_{p\in P}\int_{R^n}\partial^p M\big(t,\varphi(t)\big)\,dt,$$

it is easily verified that $\varrho_{\partial M}$ satisfies all conditions (A.1)-(A.6) of the definition of the modular ϱ. Thus $D_{\varrho_{\partial M}}$ is an example of a space D_ϱ. All conditions (A)-(D) of Theorem 7 are mutually equivalent, and the same holds for all conditions (a)-(d) of Theorem 8, because

THEOREM 10. *The modular $\varrho_{\partial M}$ satisfies the condition* (α).

Proof. Let $\lim\limits_{i\to\alpha}\varrho_{\partial M}(\lambda_q D^q\varphi_i)=0$; then for every $p\in P$ and $q\in P$ such that $p+q\in P$, and for every $\varepsilon>0$ there exists index $i_{p,q,\varepsilon}$ with the property that

$$\int_{R^n}\partial^p M[t, 2^{n-|p+q|}\lambda_{p,q}D^q\varphi_i(t)]\,dt<\frac{\varepsilon}{3^n}$$

for all $i\geqslant i_{p,q,\varepsilon}$, where $\lambda_{p,q}=\lambda_q\cdot 2^{-(n-|p+q|)}$. Let $i_\varepsilon=\max\limits_{p+q\in P} i_{p,q,\varepsilon}$, and let $\lambda_1=\min\limits_{p+q\in P}\lambda_{p,q}$; then

$$\int_{R^n}\partial^p M[t, 2^{n-|p+q|}\lambda_1 D^q\varphi_i(t)]\,dt<\frac{\varepsilon}{3^n}\quad .$$

for all $i \geqslant i_\varepsilon$. By Lemma 9, we have

$$M\big(x, \lambda_1 \varphi_i(x)\big) \leqslant \varepsilon \tag{10.1}$$

for $i \geqslant i_\varepsilon$ and for every $x \in R^n$. Since $\varphi_i \in D_{e_{\partial M}}$ by assumption, for all $p, q \in P$ such that $p+q \in P$ and for each i there exists a constant $\lambda_{q,i}$ for which

$$\int_{R^n} \partial^p M[t, \lambda_{q,i} D^q \varphi_i(t)]\, dt < \infty.$$

Let $\lambda_2 = \min\limits_{p+q \in P, i \leqslant i_\varepsilon} \lambda_{q,i} \cdot 2^{-(n-|p+q|)}$; then for $p+q \in P$ and all $i \leqslant i_\varepsilon$ there holds

$$\int_{R^n} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q \varphi_i(t)]\, dt < \infty \tag{10.2}$$

i.e. there exists a constant $\Delta_{p,q,i} > 0$ such that

$$\int_{|t| \geqslant \Delta_{p,q,i}} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q \varphi_i(t)]\, dt < \frac{\varepsilon}{3^n}.$$

Let $\Delta = \max\limits_{p+q \in P, i \leqslant i_\varepsilon} \Delta_{p,q,i}$; then

$$\int_{|t| \geqslant \Delta} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q \varphi_i(t)]\, dt < \frac{\varepsilon}{3^n}.$$

Hence we get, by Lemma 9,

$$M\big(x, \lambda_2 \varphi_i(x)\big) \leqslant \sum_{p+q \in P} \int_{|t| \geqslant |x|} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q \varphi_i(t)]\, dt \leqslant \varepsilon$$

for $i \leqslant i_\varepsilon$ and $|x| \geqslant \Delta$. This inequality and inequality (10.1) imply the inequality

$$M\big(x, \lambda \varphi_i(x)\big) \leqslant \varepsilon \tag{10.3}$$

for all i and $|x| \geqslant \Delta$, where $\lambda = \min(\lambda_1, \lambda_2)$.

Now, let $|x| < \Delta$ and $i < i_\varepsilon$. According to formula (10.2) we have

$$\int_{R^n} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q \varphi_i(t)]\, dt = K_{p,q,i} < \infty.$$

Applying Lemma 9, we obtain the inequality

$$\begin{aligned} M\big(x, \lambda_2 \varphi_i(x)\big) &\leqslant \sum_{p+q \in P} \partial^p M[t, \lambda_2 2^{n-|p+q|} D^q \varphi_i(t)]\, dt \\ &\leqslant \sum_{p+q \in P} K_{p,q,i} \leqslant K < \infty, \end{aligned}$$

where $K = 3^n \max_{p+q\in P, i<i_g} K_{p,q,i}$. Hence and from inequality (10.1) we obtain

$$M\big(x, \lambda\varphi_i(x)\big) \leqslant K \tag{10.4}$$

for $|x| < \Delta$ and for all i, where $\lambda = \min(\lambda_1, \lambda_2)$.

From inequalities (10.3) and (10.4) it follows that

$$M\big(x, \lambda\varphi_i(x)\big) \leqslant K \tag{10.5}$$

for every $x \in R^n$ and each i. From the last inequality we conclude that the functions φ_i are uniformly bounded. Indeed, let us suppose this is not true, that is, there exists a sequence $\{x_k\}$ such that $|\varphi_{i_k}(x_k)| > k$ for a subsequence of indices $\{i_k\}$. Then $K \geqslant M\big(x_k, \lambda\varphi_{i_k}(x_k)\big) \geqslant M(x_k, \lambda k)$. Since $\lim_{k\to\infty} M(x_k, \lambda k) = \infty$, we get a contradiction. Hence φ_i are uniformly bounded. We proved that $\varrho_{\partial M}$ satisfies the condition (a).

It is easily proved that the following theorems hold for the space $D_{\varrho_{\partial M}}$:

THEOREM 11. *If* $\varphi_i \xrightarrow{\tilde{\varrho}_{\partial M}} 0$, *then* $\lim_{|x|\to\infty} D^q\varphi_i(x) = 0$ *uniformly with respect to* i *and the functions* $D^q\varphi_i(x)$ *are uniformly bounded with respect to* i *and* $x \in R^n$, *for each* q *separately.*

THEOREM 12. *If* $\varphi \in D_{\varrho_{\partial M}}$, *then for every* q *we have* $\lim_{|x|\to\infty} D^q\varphi(x) = 0$, *and the functions* $D^q\varphi$ *are bounded.*

Now, we show a possibility of constructing the ∂M-functions. Let a point $t \in R^n$ be given, and let $C(t) = S_1 \times \ldots \times S_n$, where we have for $i = 1, 2, \ldots, n$,

$$S_i = \{v_i\colon\ 0 \leqslant v_i \leqslant t_i\} \quad \text{for } t_i \geqslant 0,$$

$$S_i = \{v_i\colon\ t_i \leqslant v_i \leqslant 0\} \quad \text{for } t_i \leqslant 0.$$

Then it is easily seen that

$$M(t, u) = \int_{C(t)} m(v, u)\,dv + m(0, u)$$

is a ∂M-function, of $m(t, u)$ is an M-function. Let us remark that if $M(u)$ is a N-function in the sense of [3], and $p(t)$ is a non-negative continuous function; then $m(t, u) = [M(u)]^{p(t)+1}$ is an M-function in the sense defined at the beginning of section 3 of this paper.

Finally, let us remark that taking here ∂M-function $M(t, u) = M(u)$ we obtain as a special case the space D_M and theorems concerning this space given in [4] and [6].

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