

ZBIGNIEW JERZY JAKUBOWSKI (Łódź)

On some properties of extremal functions of Carathéodory

1. Let Q denote the class of regular functions

$$(1.1) \quad q(z) = \sum_{k=1}^{\infty} q_k z^k$$

defined in the circle $K = \{z : |z| < 1\}$ and such that $|q(z)| < 1$ for $z \in K$. As it is known functions of the family Q have the following properties [1], [5]:

$$(1.2) \quad |q(z)| \leq |z| \quad \text{for } z \in K,$$

$$(1.3) \quad |q_n| \leq 1 \quad \text{for } n = 1, 2, \dots,$$

$$(1.4) \quad \sum_{k=1}^{\infty} |q_k|^2 \leq 1,$$

$$(1.5) \quad |q'(z)| \leq (1 - |q(z)|^2)(1 - |z|^2)^{-1} \quad \text{for } z \in K.$$

In estimations (1.2) (for $z \neq 0$), (1.5) and (1.3) (for $n = 1$) equality is the case if and only if $q(z) = \varepsilon z$, $|\varepsilon| = 1$. If there exists $n_0 > 1$ such that $|q_{n_0}| = 1$, then it follows from (1.4) that $q_n = 0$ for $n \neq n_0$ and function (1.1) is then of the form εz^{n_0} , $|\varepsilon| = 1$.

2. Let \mathfrak{P} denote the family of regular functions

$$(2.1) \quad p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

defined in the circle K and satisfying in this circle the condition $\operatorname{re} p(z) > 0$ [2]. It is known that if $p \in \mathfrak{P}$ the following sharp estimations hold

$$(2.2) \quad |p_n| \leq 2 \quad \text{for } n = 1, 2, \dots$$

$$(2.3) \quad (1-r)(1+r)^{-1} \leq |p(z)| \leq (1+r)(1-r)^{-1},$$

$$(2.4) \quad |p'(z)| \leq 2(1-r)^{-2},$$

$$(2.5) \quad |p'(z)/p(z)| \leq 2(1-r^2)^{-1},$$

where $|z| = r$, $0 < r < 1$.

These estimations may be found by various methods (in the case of result (2.2) see e.g. [2], [4], [7]-[10]). In particular, estimations (2.3)-(2.5) result easily from the following one to one relation between the functions of the classes \mathcal{Q} and \mathfrak{P} : $p(z) = (1+q(z))/(1-q(z))$ and from properties (1.2) and (1.5).

In this paper, employing an analogous method, we find some estimations in the family \mathfrak{P} and some of its subclasses. First we will generalize the results (2.4), (2.5) by obtaining the sharp estimation of the functional $|p'(z)/p^s(z)|$, $p^s(0) = 1$, when $p \in \mathfrak{P}$, where s is an arbitrary fixed real number.

THEOREM 2.1. *If $p \in \mathfrak{P}$, then the following sharp estimations*

$$(2.6) \quad |p'(z)/p^s(z)| \leq 2(1+r)^{s-2}(1-r)^{-s} \quad \text{when } s \geq 1,$$

and

$$(2.7) \quad |p'(z)/p^s(z)| \leq 2(1+r)^{-s}(1-r)^{s-2} \quad \text{when } s < 1,$$

where $|z| = r$, $0 < r < 1$ and s is an arbitrary real number, hold.

Proof. Since $p'(z) = 2q'(z)/(1-q(z))^2$

$$|p'(z)/p^s(z)| = 2|q'(z)||1-q(z)|^{s-2}|1+q(z)|^{-s}$$

whence by (1.5) we have

$$|p'(z)/p^s(z)| \leq 2 \frac{1-|q(z)|^2}{1-|z|^2} \frac{|1-q(z)|^{s-1}}{|1+q(z)|^{s-1}} \frac{1}{|1-q^2(z)|}$$

thus

$$|p'(z)/p^s(z)| \leq \frac{2}{1-|z|^2} \left| \frac{1-q(z)}{1+q(z)} \right|^{s-1}.$$

Consequently if $s \geq 1$, then

$$|p'(z)/p^s(z)| \leq \frac{2}{1-|z|^2} \left(\frac{1+|q(z)|}{1-|q(z)|} \right)^{s-1}.$$

Whence because of inequality (1.2) we obtain estimation (2.6). If, on the other hand, $s < 1$, then from the inequality

$$|p'(z)/p^s(z)| \leq \frac{2}{1-|z|^2} \left(\frac{1+|q(z)|}{1-|q(z)|} \right)^{1-s}$$

and condition (1.2) estimation (2.7) results.

Since the function $p^*(z) = (1+\varepsilon z)(1-\varepsilon z)^{-1}$, $|\varepsilon| = 1$, belongs to the family \mathfrak{P} and $p^{*'}(z) = 2\varepsilon(1-\varepsilon z)^{-1}$ estimations (2.6) and (2.7) are sharp.

Accepting in Theorem 2.1 $s = 0$ or $s = 1$ we obtain the known results (2.4) and (2.5).

3. Consider the subclass \mathfrak{P}_M , $M \geq 1$, of these functions of the family \mathfrak{P} , which satisfy the condition $|p(z) - M| < M$ in the circle Goel [4] has proved that every function of the family \mathfrak{P}_M may be represented in the form

$$(3.1) \quad p(z) = (1 + q(z)) / (1 - bq(z)),$$

where $b = 1 - M^{-1}$, $q \in Q$. Moreover, he has proved that if $p \in \mathfrak{P}_M$, then the following sharp estimations hold

$$(3.2) \quad |p_n| \leq 1 + b, \quad n = 1, 2, \dots$$

$$(3.3) \quad (1 - r)(1 + br)^{-1} \leq |p(z)| \leq (1 + r)(1 - br)^{-1}, \quad |z| = r, 0 < r < 1,$$

$$(3.4) \quad |p'(z)/p(z)| \leq (1 + b)(1 - r)^{-1}(1 + br)^{-1}, \quad |z| = r, 0 < r < 1.$$

Consider the following functional defined in the family \mathfrak{P}_M

$$(3.5) \quad U(p) = |p'(z)/p^s(z)|,$$

where s is an arbitrary fixed real number and $p^s(0) = 1$.

We introduce the following denotations

$$(3.6) \quad r_1(s) = [2b + s(1 - b)][2 - s(1 - b)]^{-1}, \quad 0 < s < 1,$$

$$(3.7) \quad r_2(s) = [2b - s(1 + b)][2 - s(1 + b)]^{-1}, \quad s \leq 0$$

and

$$(3.8) \quad \begin{aligned} E_1 &= \{(s, r): 1 \leq s, 0 < r < 1\}, \\ E_2 &= \{(s, r): 0 < s < 1, 0 < r < r_1(s)\}, \\ E_3 &= \{(s, r): 0 < s < 1, r_1(s) \leq r < 1\}, \\ E_4 &= \{(s, r): s \leq 0, 0 < r < r_2(s)\}, \\ E_5 &= \{(s, r): s \leq 0, r_2(s) \leq r < 1\}. \end{aligned}$$

We shall prove

THEOREM 3.1. *If $p \in \mathfrak{P}_M$, $M \geq 1$, then for an arbitrary fixed z , $|z| = r$, $0 < r < 1$ we have*

$$(3.9) \quad U(p) \leq (1 + b)(1 + br)^{s-2}(1 - r)^{-s} \quad \text{when } (s, r) \in E_1,$$

$$(3.10) \quad U(p) = (1 + b)(1 - br)^{s-2}(1 - r)^{-s} \quad \text{when } (s, r) \in E_2,$$

$$(3.11) \quad U(p) \leq 2^{2-s}(1 + b)^s(1 - b)^{-1}(1 - r^2)^{-1}(1 - s)^{1-s}(2 - s)^{s-2} \\ \text{when } (s, r) \in E_3,$$

$$(3.12) \quad U(p) \leq (1 + b)(1 - br)^{s-2}(1 + r)^{-s} \quad \text{when } (s, r) \in E_4$$

and

$$(3.13) \quad U(p) \leq 2^{2-s}(1-b)^{s-1}(1-r^2)^{-1}(1-s)^{1-s}(2-s)^{-2+s} \\ \text{when } (s, r) \in E_5.$$

Estimations (3.9) and (3.12) are sharp.

Proof. Let z , $|z| = r$, $0 < r < 1$ be an arbitrary fixed point. Then from (3.1) and (3.5) we get

$$U(p) = (1+b)|q'(z)||1-bq(z)|^{s-2}|1+q(z)|^{-s}$$

whence by (1.5) we have

$$(3.14) \quad U(p) \leq \frac{1+b}{1-r^2} \frac{1-|q(z)|^2}{|1-bq(z)|^{2-s}|1+q(z)|^s}, \quad |z| = r.$$

Now the functional

$$(3.15) \quad V(q) = \frac{1-|q(z)|^2}{|1-bq(z)|^{2-s}|1+q(z)|^s}$$

is to be estimated in the class Q .

To this end consider three cases.

1° Let $s \geq 1$. Since for $q \in Q$ and $0 \leq b < 1$ we have

$$|(1+q(z))(1-bq(z))| \geq (1-|q(z)|)(1+b|q(z)|)$$

thus

$$V(q) \leq \frac{(1-|q(z)|^2)(1+b|q(z)|)^{2(s-1)}}{(1+b|q(z)|)^s(1-|q(z)|)^s}$$

whence in the case considered we find

$$V(q) \leq \frac{1+|q(z)|}{1+b|q(z)|} \left(\frac{1+b|q(z)|}{1-|q(z)|} \right)^{s-1}.$$

But $|q(z)| \leq r$ (comp. (1.2)), thus $(1+|q(z)|)(1+b|q(z)|)^{-1} \leq (1+r)(1+br)^{-1}$, thus

$$V(q) \leq (1+r)(1+br)^{s-2}(1-r)^{1-s}$$

whence by (3.15) and (3.14) we obtain estimation (3.9).

2° If $0 < s < 1$, then from (3.15) we find

$$(3.16) \quad V(q) \leq g(|q(z)|),$$

where

$$(3.17) \quad g(x) = (1+x)(1-x)^{1-s}(1-bx)^{s-2}, \quad 0 \leq x \leq r.$$

Since

$$g'(x) = (1-x)^{-s}(1-bx)^{s-3}[x(s-sb-2) + s-sb+2b] \\ = [2-s(1-b)](1-x)^{-s}(1-bx)^{s-3}[r_1(s)-x],$$

then

$$(3.18) \quad \max_{0 \leq x \leq r} g(x) = g(r) \quad \text{when } r < r_1(s)$$

and

$$(3.19) \quad \max_{0 \leq x \leq r} g(x) = g(r_1(s)) \quad \text{when } r_1(s) \leq r.$$

Thus from formulas (3.6), (3.8), (3.14)-(3.19) estimations (3.10), (3.11) result.

3° If $s < 0$, then from (3.15) we obtain

$$(3.20) \quad V(q) \leq h(|q(z)|),$$

where

$$(3.21) \quad h(x) = (1-x)(1+x)^{1-s}(1-bx)^{s-2}, \quad 0 \leq x \leq r.$$

Since

$$h'(x) = [2-s(1+b)](1-x)^{-s}(1-bx)^{s-3}(r_2(s)-x)$$

thus

$$(3.22) \quad \max_{0 \leq x \leq r} h(x) = h(r) \quad \text{when } r < r_2(s)$$

and

$$(3.23) \quad \max_{0 \leq x \leq r} h(x) = h(r_2(s)) \quad \text{when } r_2(s) \leq r.$$

Thus from formulas (3.7), (3.8), (3.14), (3.15) and (3.20)-(3.23) we get estimations (3.12) and (3.13).

The function

$$p^*(z) = (1+\varepsilon z)(1-b\varepsilon z)^{-1}, \quad |\varepsilon| = 1, b = 1-M^{-1}$$

belongs to the family \mathfrak{P}_M and

$$U(p^*) = (1+b)|1+\varepsilon z|^{-s}|1-b\varepsilon z|^{s-2}$$

estimations (3.9) and (3.12) are thus sharp.

Accepting in estimation (3.9) $s = 1$ we get result (3.4). If $M = 1$ ($b = 0$), we have from Theorem 3.1

COROLLARY 3.1. *If $p \in \mathfrak{P}_1$, then*

$$(3.24) \quad \begin{aligned} U(p) &\leq (1-r)^{-s} && \text{when } (r, s) \in E_1 \cup E_2, \\ U(p) &\leq 2^{2-s}(1-s)^{1-s}(2-s)^{s-2}(1-r^2)^{-1} && \text{when } (r, s) \in E_3 \cup E_5, \end{aligned}$$

$$(3.25) \quad U(p) \leq (1+r)^{-s} \quad \text{when } (r, s) \in E_4.$$

Estimations (3.24) and (3.25) are sharp ($p^*(z) = 1 + \varepsilon z$, $|\varepsilon| = 1$).

In the limit case $b = 1$ ($M = +\infty$), it follows from definitions (3.6)-(3.8) that the sets E_3 and E_5 are empty. Thus from estimation (3.9) we find inequality (2.6) in the family \mathfrak{P} and from (3.12) results (2.7) but only for $s \leq 0$.

Analogous results to those in (3.3) and (3.9)-(3.13) may be obtained in the family \mathfrak{P}_M with $M \in (\frac{1}{2}, 1)$.

Applying the above method, however, one has to take into account that in this case $b \in (-1, 0)$.

4. The family of functions $\mathfrak{P}_{m,M}$ of form (2.1) satisfying in the circle K condition

$$(4.1) \quad |p(z) - m| < M$$

has been introduced in [6], where m, M are arbitrary fixed numbers which satisfy the relations

$$(4.1') \quad \frac{1}{2} < m \leq 1, \quad 1 - m < M \leq m$$

or

$$(4.1'') \quad 1 < m, \quad m - 1 < M \leq m.$$

It is obvious that $\mathfrak{P}_{M,M} \equiv \mathfrak{P}_M$ and $\mathfrak{P}_{m,M} \subset \mathfrak{P}$. It is also known that if $p \in \mathfrak{P}_{m,M}$, then

$$(4.2) \quad p(z) = (1 + aq(z))(1 - bq(z))^{-1} \quad z \in K$$

with $q \in Q$ and

$$(4.3) \quad a = M^{-1}(M^2 - m^2 + m), \quad b = M^{-1}(m - 1).$$

The sharp estimation [6]

$$(4.4) \quad |p_n| \leq a + b, \quad n = 1, 2, \dots$$

is known, which is a generalization of the results (2.2) and (3.2).

Now we shall prove a theorem on distortion in the class $\mathfrak{P}_{m,M}$.

THEOREM 4.1. *If $p \in \mathfrak{P}_{m,M}$, then*

$$(4.5) \quad (1 - ar)(1 + br)^{-1} \leq |p(z)| \leq (1 + ar)(1 - br)^{-1}, \quad |z| = r, \quad 0 < r < 1.$$

The extremal function is of the form

$$(4.6) \quad p^*(z) = (1 + a\varepsilon z)(1 - b\varepsilon z)^{-1}, \quad |\varepsilon| = 1.$$

Proof. Accept in formula (4.2) $q(z) = \lambda e^{it}$, $0 \leq \lambda \leq r$, $0 \leq t \leq 2\pi$. Then we get

$$|p(z)| = \Phi(\lambda, t),$$

$$\text{where } \Phi(\lambda, t) = [1 + a^2 \lambda^2 + 2a\lambda \cos t]^{1/2} [1 + b^2 \lambda^2 - 2b\lambda \cos t]^{-1/2}.$$

If, however, m, M satisfying relations (4.1') and (4.1''), then because of (4.3) we have $|b| < 1$, $-1 < a \leq 1$ and $a + b > 0$, thus

$$(1 - ar)(1 + br)^{-1} \leq \Phi(\lambda, t) \leq (1 + ar)(1 - br)^{-1}$$

whence inequality (4.5) follows. Function (4.6) belongs to family $\mathfrak{P}_{m, M}$ and its values at the points $z_1 = r\bar{\varepsilon}$, $z_2 = -r\varepsilon$ are equal to the upper or the lower bound of the functional in question, thus estimation (4.5) is sharp.

THEOREM 4.2. *If $p \in \mathfrak{P}_{m, M}$, $|z| = r$, $0 < r < 1$, then*

$$(4.7) \quad |p'(z)| \leq (a + b)(1 - |b|r)^{-2} \quad \text{when } 0 < r \leq |b|$$

and

$$(4.8) \quad |p'(z)| \leq (a + b)(1 - r^2)^{-1}(1 - b^2)^{-1} \quad \text{when } |b| < r < 1.$$

Estimation (4.7) is sharp.

Proof. By formulas (4.2), (4.3) and (1.5) we have

$$|p'(z)| \leq (a + b)(1 - r^2)^{-1}g(|q(z)|)$$

where

$$g(x) = (1 - x^2)(1 - |b|x)^{-2}, \quad 0 \leq x \leq r.$$

Since

$$\max_{0 \leq x \leq r} g(x) = g(r) \quad \text{when } 0 < r \leq |b|$$

and

$$\max_{0 \leq x \leq r} g(x) = g(|b|) \quad \text{when } |b| < r < 1$$

estimations (4.7) and (4.8) are true. The upper bound in the first of them is attained by the function (4.6) at the point $z_1 = \pm \bar{\varepsilon}r$ respectively according to the sign of b .

THEOREM 4.3. *If $p \in \mathfrak{P}_{m, M}$, ($m > 1$, $\sqrt{m^2 - 1} < M \leq m$), then for an arbitrary real number s , $s \geq 1$ and an arbitrary z , $|z| = r$, $0 < r < 1$ the following sharp estimation*

$$(4.9) \quad |p'(z)/p^s(z)| \leq (a + b)(1 + br)^{s-2}(1 - ar)^{-s}, \quad 0 < r < x_1,$$

with $p^s(0) = 1$, $x_1 = [1 - ab - ((1 - a^2)(1 - b^2))^{1/2}](a - b)^{-1}$ holds.

Proof. From the conditions assumed as to m and M and denotations (4.3) it follows that $0 < b < a \leq 1$. Thus by (4.2) and (1.5) we have

$$|p'(z)/p^s(z)| \leq \frac{a + b}{1 - |z|^2} \left(\frac{1 + b|q(z)|}{1 - a|q(z)|} \right)^{s-1} \cdot h(|q(z)|),$$

where

$$h(x) = (1 - x^2)(1 - ax)^{-1}(1 + bx)^{-1}, \quad 0 \leq x \leq r.$$

Thus if $r < x_1$, then

$$\max_{0 \leq x \leq r} h(x) = h(r)$$

whence the announced estimation (4.9) follows. Function (4.6) attains the upper bound of the functional at the point $z_1 = -\bar{\varepsilon}r$ thus this estimation is sharp.

Since $\mathfrak{P}_{M,M} \equiv \mathfrak{P}_M$ and if $m = M$, then $a = 1$, estimations (4.5) and (4.7)-(4.9) are a natural generalization of the results (3.3), (3.12), (3.13), (for $s = 0$) and (3.9).

Theorems analogous to 4.3 could be proved for other admissible values of m and M . Since the majority of the estimations which could be found by the method employed in this paper would not be sharp, they have been omitted.

5. Condition (4.1) required in the definition of the class $\mathfrak{P}_{m,M}$ can be generalized to the case when m is replaced by a complex number c . So we consider the family $\mathfrak{P}_{c,M}$ of functions of form (2.1) satisfying the condition

$$(5.1) \quad |p(z) - c| < M$$

in the circle K , where c and M are arbitrary fixed numbers satisfying inequality

$$(5.2) \quad |1 - c| < M \leq \operatorname{re} c.$$

If $p \in \mathfrak{P}_{c,M}$, then the function

$$q(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)}h(z)}$$

with $h(z) = M^{-1}(p(z) - c)$ belongs to the family Q (comp. (5.1)). Thus every function of the family $\mathfrak{P}_{c,M}$ may be represented in the form

$$(5.3) \quad p(z) = (1 + aq(z))(1 - bq(z))^{-1},$$

where $q \in Q$ and

$$(5.4) \quad a = M^{-1}(M^2 - |c|^2 + c), \quad b = M^{-1}(\bar{c} - 1).$$

Conversely, from (5.4) and (5.2) it follows that for every function $q \in Q$ function (5.3) belongs to the class $\mathfrak{P}_{c,M}$.

THEOREM 5.1. *If $p \in \mathfrak{P}_{c,M}$, then*

$$(5.5) \quad |p_n| \leq a + b = M^{-1}(M^2 - |c - 1|^2), \quad n = 1, 2, \dots,$$

$$(5.6) \quad \sum_{k=1}^{\infty} |p_k|^2 \leq M^2 - |c - 1|^2,$$

$$(5.7) \quad (1 - |a|r)(1 + |b|r)^{-1} \leq |p(z)| \leq (1 + |a|r)(1 - |b|r)^{-1},$$

where $|z| = r$, $0 < r < 1$,

and

$$(5.8) \quad |p'(z)| \leq (a+b)(1-|b|r)^{-2} \quad \text{when } 0 < r \leq |b|.$$

Estimations (5.5) and (5.8) are sharp.

Proof. Estimations (5.7) and (5.8) are obtained from formulas (5.3) and (5.4) by an analogous argument as in the proofs of Theorems 4.1 and 4.2.

To obtain inequality (5.5) we employ the known method of Clunie [3]. Thus from formula (5.3) we get

$$p(z) - 1 = [a + bp(z)]q(z)$$

whence by (1.1) and (2.1) we have

$$(5.9) \quad p_1 = (a+b)q_1,$$

$$(5.10) \quad p_n = (a+b)q_n + b(p_{n-1}q_1 + \dots + p_1q_{n-1}), \quad n = 2, 3, \dots,$$

$$(5.11) \quad \sum_{k=1}^n p_k z^k + \sum_{k=n+1}^{\infty} d_k z^k = \left[a + b + b \sum_{k=1}^{n-1} p_k z^k \right] q(z), \quad n = 2, 3, \dots,$$

where the coefficients d_k have been chosen suitably. From relations (5.9) and (1.3) we find the estimation $|p_1| \leq |a+b|$.

Accepting in the identity (5.11) $z = re^{it}$, $0 < r < 1$, $0 \leq t \leq 2\pi$ and employing condition $|q(z)| < 1$ and the integrating the obtained inequality in the interval $(0, 2\pi)$ we obtain

$$\sum_{k=1}^n |p_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} < |a+b|^2 + |b|^2 \sum_{k=1}^{n-1} |p_k|^2 r^{2k}.$$

Thus since r is arbitrary we have

$$(5.12) \quad |p_n|^2 \leq |a+b|^2 + (|b|^2 - 1) \sum_{k=1}^{n-1} |p_k|^2.$$

Since, however, $|b| < 1$, $|p_n| \leq |a+b|$ for $n = 2, 3, \dots$

By the accepted denotations (5.4) we have proved inequality (5.5) for every natural n .

By a similar argument inequality (5.6) may be deduced from (5.3) and the condition $|q(z)| < 1$.

From relationships (5.12) and (5.10) we find that equality can be the case in estimation (5.5) if and only if

$$(5.13) \quad p^*(z) = (1 + a\varepsilon z^n)(1 - b\varepsilon z^n)^{-1}, \quad |\varepsilon| = 1$$

with a and b defined by formulas (5.4). Function (5.13) with $n = 1$ is also an extremal function in estimation (5.8).

Conclusion. The simple method applied in this paper, which in fact is based on Schwarz lemma enabled us to obtain several theorems on distortion in the family \mathfrak{P} and some of its subclasses. The effectiveness of this method for finding sharp estimations of the investigated functionals, however, decreases as the considered class of functions becomes more general. Because of many applications of functions of Carathéodory family and its subclasses especially in the study of some metric properties of classes of functions generated by the above mentioned families, it seems purposeful to search special solutions of the problems, presented in this paper, on another way. The author will concern this subject in one of his next papers.

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