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JAN BOCHENEK (Kraków)

Some remarks on the convergence of Ritz and Galerkin methods

Introduction. This paper is a continuation of paper [1], where a manner of choice of "coordinate functions" in the method of Bubnov–Galerkin has been presented. In this paper in Section 1 we shall prove Theorem 1, which is a generalization of Theorem 1 of paper [1]. In Section 2 we shall prove a theorem about rate of convergence of Galerkin method according to a manner of choice of "coordinate functions". In Section 3 we shall transfer those results to somewhat more general equation.

Let H be a separable Hilbert space and let A be a linear operator with domain $D(A) \subset H$. We assume that $D(A)$ and $D(A^*)$ are the dense subspaces of H and that $D(A) \subset D(A^*)$, where $D(A^*)$ is the domain of adjoint operator A. Moreover, we assume that the operator A satisfies the following condition:

(W) $$|(Au, u)| \geqslant p\|u\|^2, \quad u \in D(A),\ p > 0.$$

Since $D(A^*)$ is a dense subspace in H, the operator A can be extended to a closed operator in H (cf. [7], p. 557). In the sequel by A we shall denote the least closed extension of A.

1. Convergence of Ritz–Galerkin method. First we shall prove some lemmas.

LEMMA 1. *If operator A satisfies condition* (W) *and if $D(A) \subset D(A^*)$, then range $R(A)$ of A is a dense subspace of H.*

Proof. We see that it is sufficient to prove that $R(A)$ is dense in $D(A)$. Let $z \in D(A)$ and $y \in R(A)$. We shall prove that if $(z, y) = 0$ for any $y \in R(A)$, then $z = 0$. Indeed, if $y \in R(A)$, then $y = Ax$, where $x \in D(A)$, whence $0 = (y, z) = (Ax, z) = (x, 0) = (x, A^*z)$, i.e., $A^*z = 0$. On the other hand, by (W), $p\|z\|^2 \leqslant |(Az, z)| = |(z, A^*z)| = 0$, hence $z = 0$.

COROLLARY 1. *The range $R(A)$ of A is the whole space H.*

Apart from the operator A we now consider a linear, self-adjoint, positive definite operator B such that $D(A) = D(B)$.

LEMMA 2. *If the operators A and B satisfy the above conditions, then the operators AB^{-1}, BA^{-1}, $A^{-1}B$ and $B^{-1}A$ are bounded operators in H.*

Proof. Since the operators $AB^{-1}, BA^{-1}, B(A^*)^{-1}$ and A^*B^{-1} are closed and defined in whole space H, it is known that they are bounded. It follows that the operators $A^{-1}B$ and $B^{-1}A$ are bounded as the adjoint operators to $B(A^*)^{-1}$ and A^*B^{-1}, respectively.

We still asume that there exist denumerable sequence of eigenvalues $\{\lambda_n\}$ of operator B, and corresponding sequence of eigenvectors $\{\varphi_n\}$ which form a complete system in H. We assume also that $\{\varphi_n\}$ is an orthonormal sequence and that $\lim_{n\to\infty}\lambda_n = +\infty$.

Let us denote by M_n and L_n closed linear subspaces spanned by $\{\varphi_1, \ldots, \varphi_n\}$ and $\{\psi_1, \ldots, \psi_n\}$, respectively, and by P_n and π_n the projectors of H onto M_n and L_n, respectively, where $\psi_k = AB^{-1}\varphi_k$, $k = 1, \ldots, n$.

We shall prove the following

LEMMA 3. *If the operators B and P_n are defined as above, then P_n and B^{-a} $(a > 0)$ are commutative operators, i.e.,*

$$B^{-a}P_n u = P_n B^{-a} u \quad \text{for any } u \in H. \tag{1}$$

Proof. Since $\{\varphi_n\}$ is an orthonormal and complete system in the space H, then

$$u = \sum_{k=1}^{\infty} \gamma_k \varphi_k \quad \text{for any } u \in H, \quad \text{and} \quad P_n u = \sum_{k=1}^{n} \gamma_k \varphi_k.$$

Hence

$$B^{-a}P_n u = \sum_{k=1}^{n} \gamma_k \lambda_k^{-a} \varphi_k. \tag{2}$$

On the other hand

$$B^{-a} u = \sum_{k=1}^{\infty} \gamma_k \lambda_k^{-a} \varphi_k.$$

therefore

$$P_n B^{-a} u = \sum_{k=1}^{n} \gamma_k \lambda_k^{-a} \varphi_k.$$

From this by (2) we have (1).

Now we shall consider the following equation

$$Au = f, \tag{3}$$

where A is the operator defined above and $f \in H$. Let us denote

$$u_n = \sum_{k=1}^{n} a_k \varphi_k \tag{4}$$

the n-th approximation of solution of equation (3) in the sense of Galerkin. Let u_0 denotes the solution of equation (3). We assume that u_0 exists.

We shall prove the following

THEOREM 1. *If $\varphi_1, \ldots, \varphi_n$ in formula* (4) *are eigenvectors of the operator B which is defined above, and if*

$$(5)\qquad (Bu, u) \leqslant C|(Au, u)|, \qquad u \in D(A),\ C > 0,$$

then

$$(6)\qquad \|Au_n - f\| \to 0 \quad \text{for } n \to +\infty.$$

Proof. It is known that the coefficients a_k $(k = 1, \ldots, n)$ in formula (4) satisfy the following so-called Ritz–Galerkin system

$$(7)\qquad \sum_{k=1}^{n} a_k(A\varphi_k, \varphi_j) = (f, \varphi_j), \qquad j = 1, \ldots, n.$$

Let us denote

$$(8)\qquad z_n = Au_n,$$

and let us observe that

$$z_n = Au_n = \sum_{k=1}^{n} a_k A\varphi_k = \sum_{k=1}^{n} a_k \lambda_k AB^{-1}\varphi_k = \sum_{k=1}^{n} c_k \psi_k,$$

where $c_k = \lambda_k a_k$ and $\psi_k = AB^{-1}\varphi_k$, $k = 1, \ldots, n$.

Now system (7) may be written in the form

$$(9)\qquad \sum_{k=1}^{n} c_k(\psi_k, \varphi_j) = (f, \varphi_j), \qquad j = 1, \ldots, n,$$

or in the form

$$(10)\qquad P_n z_n = P_n f.$$

Let $v_n \in L_n$ hence $v_n = \sum_{k=1}^{n} \beta_k \psi_k$ and $P_n v_n = \sum_{k=1}^{n} \alpha_k \varphi_k$, where the coefficients $\alpha_1, \ldots, \alpha_n$ are defined by the condition

$$(11)\qquad \|v_n - P_n v_n\| = \min \qquad (\text{cf. } [2]).$$

It follows from (11) that the coefficients $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ satisfy the following system:

$$(12)\qquad \sum_{k=1}^{n} \beta_k(\psi_k, \varphi_i) = \alpha_i, \qquad i = 1, \ldots, n.$$

Let us observe that

$$(13)\quad \|v_n\|^2 = \|AB^{-1}\sum_{k=1}^{n} \beta_k \varphi_k\|^2 \leqslant \|AB^{-1}\|^2 \Big\|\sum_{k=1}^{n} \beta_k \varphi_k\Big\|^2 \leqslant \lambda_n \|AB^{-1}\|^2 \sum_{k=1}^{n} \frac{\beta_k^2}{\lambda_k}.$$

On the other hand

$$\sum_{k=1}^{n}\frac{\beta_k^2}{\lambda_k}=\Big(\sum_{k=1}^{n}\beta_k\varphi_k,\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}\varphi_k\Big)=\Big(B\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}\varphi_k,\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}\varphi_k\Big)$$
$$\leqslant C\Big|\Big(A\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}\varphi_k,\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}\varphi_k\Big)\Big|\leqslant C\sum_{k=1}^{n}\Big|\frac{\beta_i}{\lambda_i}\Big|\Big|\Big(A\sum_{k=1}^{n}\frac{\beta_k}{\lambda_k}\varphi_k,\varphi_i\Big)\Big|$$
$$=C\sum_{i=1}^{n}\Big|\frac{\beta_i}{\lambda_i}\Big|\Big|\sum_{k=1}^{n}\beta_k(\psi_k,\varphi_i)\Big|=C\sum_{i=1}^{n}\Big|\frac{\alpha_i\beta_i}{\lambda_i}\Big|$$
$$\leqslant C\Big(\sum_{i=1}^{n}\frac{\alpha_i^2}{\lambda_i}\Big)^{\frac12}\Big(\sum_{i=1}^{n}\frac{\beta_i^2}{\lambda_i}\Big)^{\frac12}.$$

From this we get

$$\Big(\sum_{i=1}^{n}\frac{\beta_i^2}{\lambda_i}\Big)^{\frac12}\leqslant C\Big(\sum_{i=1}^{n}\frac{\alpha_i^2}{\lambda_i}\Big)^{\frac12}. \tag{14}$$

By (13) and (14) we have

$$\|v_n\|\leqslant\lambda_n^{\frac12}C_1\|B^{-\frac12}P_nv_n\|,\qquad \text{where } C_1=C\|AB^{-1}\|. \tag{15}$$

Since $z_n-\pi_nf\in L_n$, hence by (15) we get

$$\|z_n-\pi_nf\|\leqslant C_1\lambda_n^{\frac12}\|B^{-\frac12}P_n(z_n-\pi_nf)\|=C_1\lambda_n^{\frac12}\|B^{-\frac12}P_n(f-\pi_nf)\|. \tag{16}$$

From (1) and (16) follows

$$\|z_n-\pi_nf\|\leqslant C_1\lambda_n^{\frac12}\|B^{-\frac12}(f-\pi_nf)\|. \tag{17}$$

Since

$$f-\pi_nf=\sum_{k=n+1}^{\infty}\gamma_k\psi_k=AB^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k,$$

therefore

$$\|B^{-\frac12}(f-\pi_nf)\|^2=\Big\|B^{-\frac12}AB^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k\Big\|^2 \tag{17'}$$
$$=\Big(B^{-\frac12}AB^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k,B^{-\frac12}AB^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k\Big)$$
$$=\Big(B^{-1}AB^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k,AB^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k\Big)$$

$$\leqslant \|B^{-1}A\| \Big\|B^{-1}\sum_{k=n+1}^{\infty}\gamma_k\varphi_k\Big\| \|AB^{-1}\| \Big\|\sum_{k=n+1}^{\infty}\gamma_k\varphi_k\Big\|$$

$$= \|B^{-1}A\|\,\|AB^{-1}\|\Big[\sum_{k=n+1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^2\Big]^{\frac{1}{2}}\Big[\sum_{k=n+1}^{\infty}\gamma_k^2\Big]^{\frac{1}{2}}$$

$$\leqslant \|B^{-1}A\|\,\|AB^{-1}\|\,\lambda_{n+1}^{-1}\sum_{k=n+1}^{\infty}\gamma_k^2.$$

In virtue of Lemma 2 the operators AB^{-1} and $B^{-1}A$ are bounded and from this by (17) we have

(18) $$\|z_n-\pi_n f\| \leqslant C_2\Big[\sum_{k=n+1}^{\infty}\gamma_k^2\Big]^{\frac{1}{2}},$$

where $C_2 = C_1\|B^{-1}A\|\,\|AB^{-1}\|$.

Let us observe that

(19) $$\lim_{n\to\infty} r_n = \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\gamma_k^2 = 0.$$

Indeed, let us denote $y_n = \sum_{k=1}^{n}\gamma_k\varphi_k$, whence $AB^{-1}y_n = \pi_n f$, therefore $y_n = BA^{-1}\pi_n f$ and $\|y_n\| = \|BA^{-1}\pi_n f\| \leqslant \|BA^{-1}\|\,\|\pi_n f\|$. Therefore

(20) $$\|y_n\|^2 = \sum_{k=1}^{n}\gamma_k^2 \leqslant \|BA^{-1}\|^2\|\pi_n f\|^2.$$

Since $\{\psi_n\}$ is a complete system in H, hence

(21) $$\lim_{n\to\infty}\|\pi_n f\|^2 = \|f\|^2.$$

From (20) and (21) follows (19).

In virtue (17), (17′), (18) and (19) we get

(22) $$\|z_n-\pi_n f\| \to 0 \quad \text{when } n\to+\infty.$$

Since $\|f-z_n\| = \|(f-\pi_n f)-(z_n-\pi_n f)\| \leqslant \|f-\pi_n f\|+\|z_n-\pi_n f\|$ whence from this by (8), (22) and by completeness of $\{\psi_n\}$ follows (6). This yields Theorem 1.

Remark 1. If the operator A in equation (3) is a linear, self-adjoint, positive definite operator such that $D(A) = D(B)$, then as it is that known the method of Galerkin is the classical method of Ritz (cf. [6]). In this case Theorem 1 is a generalization of Theorem 23.1 of monograph [4], in this sense that the assumption

$$|(Au, Bu)| \geqslant c\|Bu\|^2, \quad u\in D(A),\ c>0$$

is omitted.

2. The rate of convergence of approximation in the sense of Ritz–Galerkin. In paper [1] we have mentioned that the rate of convergence of sequence $\{u_n\}$ depends on the choice of "coordiante functions" $\{\varphi_n\}$.

Now we shall prove the following

THEOREM 2. *If A and B are the operators defined in Section 1 and satisfying condition* (5), *then*

$$\|u_n - u_0\| = o(\lambda_n^{-1}), \tag{23}$$

where $\{u_n\}$ is defined in (4) *and u_0 is the solution of equation* (3), *but $\{\varphi_n\}$ is the sequence of eigenvalues of operator B.*

Proof. By Theorem 1 $\|\delta_n\| = \|Au_n - f\| \to 0$, as $n \to +\infty$. Let us observe, that

$$\delta_n = \sum_{k=1}^{\infty} (\delta_n, \varphi_k)\varphi_k. \tag{24}$$

By orthonormalization of $\{\varphi_n\}$ and by (24) we get

$$\|\delta_n\| = \Big\{\sum_{k=1}^{\infty} |(\delta_n, \varphi_k)|^2\Big\}^{\frac{1}{2}}.$$

On the other hand by (7) it is obvious, that

$$(\delta_n, \varphi_k) = (Au_n - f, \varphi_k) = 0 \quad \text{for } k = 1, \ldots, n.$$

It means that

$$\|\delta_n\| = \Big\{\sum_{k=n+1}^{\infty} |(\delta_n, \varphi_k)|^2\Big\}^{\frac{1}{2}}. \tag{25}$$

Now we shall estimate $\|B^{-1}\delta_n\|$. We get

$$\|B^{-1}\delta_n\|^2 = \sum_{k=1}^{\infty} |(B^{-1}\delta_n, \varphi_k)|^2 = \sum_{k=1}^{\infty} |(\delta_n, B^{-1}\varphi_k)|^2$$

$$= \sum_{k=1}^{\infty} \frac{|(\delta_n, \varphi_k)|^2}{\lambda_k^2} = \sum_{k=n+1}^{\infty} \frac{|(\delta_n, \varphi_k)|^2}{\lambda_k^2} \leqslant \frac{\|\delta_n\|^2}{\lambda_{n+1}^2}.$$

Therefore

$$\|B^{-1}\delta_n\| \leqslant \frac{\|\delta_n\|}{\lambda_{n+1}}. \tag{26}$$

By (26) we have

$$\|u_n - u_0\| = \|A^{-1}BB^{-1}\delta_n\| \leqslant \|A^{-1}B\|\,\|B^{-1}\delta_n\| \leqslant \frac{\|A^{-1}B\|\,\|\delta_n\|}{\lambda_{n+1}}. \tag{27}$$

Since $A^{-1}B$ by Lemma 2 is a bounded operator, whence by (27) we get (23). The proof is completed.

3. A generalization of previous results. In this section we deal with equation of the form

$$A w + K w = f, \tag{28}$$

where A is a operator satisfying all conditions of Section 1, and K is a operator such that $D(A) \subset D(K) \subset D(K^*)$ and $A^{-1}K$, KA^{-1} are completely continuous operators. We assume that w_0 is the unique solution of equation (28).

At first we shall prove the following lemmas:

LEMMA 4. *If*

$$x_n = \sum_{k=1}^{n} a_k \varphi_k \tag{29}$$

is the n-th approximation of solution (in the sense of Galerkin) of the following equation

$$x + Tx = f, \tag{30}$$

where T is a completely continuous operator, such that $B^{-\frac{1}{2}}(I+T)^{-1}B^{\frac{1}{2}}$ is a bounded operator, then

$$\lambda_n^{\frac{1}{2}} \|B^{-\frac{1}{2}}(x_n - x)\| \to 0, \tag{31}$$

where $\{\lambda_n\}$ and $\{\varphi_n\}$ are the sequences of eigenvalues and eigenvectors of the operator B, respectively, B is defined in Section 1 *and I is the identity operator.*

Proof. Let us denote by $\delta_n = x_n + Tx_n - f$, and observe that by definition of $\{x_n\}$ we have

$$(\delta_n, \varphi_k) = (x_n + Tx_n - f, \varphi_k) = 0 \quad \text{for } k = 1, \ldots, n. \tag{32}$$

On the other hand, because T is a completely continuous operator, then

$$\|\delta_n\| \to 0, \quad \text{as } n \to +\infty \quad \text{(cf. [5])}. \tag{33}$$

Let us remark that

$$\begin{aligned} \|B^{-\frac{1}{2}}(x_n - x)\| &= \|B^{-\frac{1}{2}}(I+T)^{-1}\delta_n\| = B^{-\frac{1}{2}}(I+T)^{-1}B^{\frac{1}{2}}B^{-\frac{1}{2}}\delta_n\| \\ &\leqslant \|B^{-\frac{1}{2}}(I+T)^{-1}B^{\frac{1}{2}}\| \, \|B^{-\frac{1}{2}}\delta_n\|. \end{aligned} \tag{34}$$

By (32) we get

$$\|B^{-\frac{1}{2}}\delta_n\|^2 = \sum_{k=1}^{\infty} \frac{|(\delta_n, \varphi_k)|^2}{\lambda_k} = \sum_{k=n+1}^{\infty} \frac{|(\delta_n, \varphi_k)|^2}{\lambda_k} \leqslant \frac{\|\delta_n\|^2}{\lambda_{n+1}}. \tag{35}$$

Since $\lambda_{n+1} \geqslant \lambda_n$ for any n, whence by (33), (34) and (35) we have (31).

LEMMA 5. *If $\{x_n\}$ satisfies condition* (31) *and P_n is the projector defined in section* 1 *and satisfying condition* (15), *then*

$$\|y_n - x\| \to 0, \tag{36}$$

where $x_n = P_n y_n$.

Proof. By (17′) and (18) we get

$$\lambda_n^{\frac{1}{2}}\|B^{-\frac{1}{2}}(x-\pi_n x)\| \to 0. \tag{37}$$

But by inequality

$$\|B^{-\frac{1}{2}}(x_n-P_n x)\| \leqslant \|B^{-\frac{1}{2}}(x_n-x)\| + \|B^{-\frac{1}{2}}(x-P_n x)\|$$

and by (31) we have

$$\lambda_n^{\frac{1}{2}}\|B^{-\frac{1}{2}}(x_n-P_n x)\| \to 0, \tag{38}$$

because it is easy to see that $\lambda_n^{\frac{1}{2}}\|B^{-\frac{1}{2}}(x-P_n x)\| \to 0$.

Since $y_n-\pi_n x \in L_n$, whence it follows from (15)

$$\begin{aligned}\|y_n-\pi_n x\| &\leqslant C\lambda_n^{\frac{1}{2}}\|B^{-\frac{1}{2}}P_n(y_n-\pi_n x)\| \\ &\leqslant C\lambda_n^{\frac{1}{2}}\{\|B^{-\frac{1}{2}}(x_n-P_n x)\| + \|B^{-\frac{1}{2}}P_n(x_n-\pi_n x\|\} \\ &= C\lambda_n^{\frac{1}{2}}\{\|B^{-\frac{1}{2}}(x_n-P_n x)\| + \|P_n B^{-\frac{1}{2}}(x-\pi_n x)\|\} \\ &\leqslant C\lambda_n^{\frac{1}{2}}\{\|B^{-\frac{1}{2}}(x_n-P_n x)\| + \|B^{-\frac{1}{2}}(x-\pi_n x)\|\}.\end{aligned}$$

By the last inequality and by (37) and (38) we have

$$\|y_n-\pi_n x\| \to 0. \tag{39}$$

Further we have

$$\|y_n-x\| \leqslant \|y_n-\pi_n x\| + \|x-\pi_n x\|. \tag{40}$$

From (40) follows (36), Q. E. D.

LEMMA 6. *If the operators T and P_n satisfy the assumptions of Lemma 5, then*

$$\|T-P_n T P_n^{-1}\| \to 0 \quad \text{as } n \to +\infty. \tag{41}$$

Proof. Let us put in inequality (15) $w = B^{-\frac{1}{2}}P_n v$, we get

$$\|P_n^{-1}B^{\frac{1}{2}}w\| \leqslant C_1\lambda_n^{\frac{1}{2}}\|w\|, \quad w \in M_n, \tag{42}$$

where P_n^{-1} denotes the inverse operator of P_n. The existence of P_n^{-1} follows from inequality (15). Let T^* denote the adjoint operator to T, obviously T^* is also the completely continuous operator. As we know

$$T^* u = \sum_{k=1}^{\infty}(T^* u, \varphi_k)\varphi_k = \sum_{k=1}^{\infty}(u, T\varphi_k)\varphi_k \tag{43}$$

and

$$P_n T^* u = \sum_{k=1}^{n}(u, T\varphi_k)\varphi_k. \tag{43′}$$

We see that (41) is equivalent to

$$\|T^*-P_n^{-1}T^*P_n\| \to 0. \tag{44}$$

Taking the identity

$$T^* - P_n^{-1}T^*P_n = (T^* - T^*P_n) + P_n^{-1}(P_nT^* - T^*)P_n,$$

we have

(45) $\|T^* - P_n^{-1}T^*P_n\| \leqslant \|T^* - T^*P_n\| + \|P_n^{-1}B^{\frac{1}{2}}\|\,\|B^{-\frac{1}{2}}(P_nT^* - T^*)\|\,\|P_n\|.$

It is known, that (cf. [5])

(46) $\|T^* - T^*P_n\| \to 0$ and $\|P_nT^* - T^*\| \to 0$ as $n \to +\infty$.

By (42) we have

(47) $\|P_n^{-1}B^{\frac{1}{2}}\| \leqslant C_1\lambda_n^{\frac{1}{2}}$ and $\|P_n\| \leqslant 1.$

On the other hand in virtue (43) and (43′) we get

$$(T^* - P_nT^*)u = \sum_{k=n+1}^{\infty} (u, T\varphi_k)\varphi_k,$$

whence

$$B^{-\frac{1}{2}}(T^* - P_nT^*)u = \sum_{k=n+1}^{\infty} (u, T\varphi_k)B^{-\frac{1}{2}}\varphi_k = \sum_{k=n+1}^{\infty} (u, T\varphi_k)\lambda_k^{-\frac{1}{2}}\varphi_k.$$

From the last inequality follows

(48) $\|B^{-\frac{1}{2}}(P_nT^* - T^*)\| \leqslant \lambda_{n+1}^{-\frac{1}{2}}\|P_nT^* - T^*\|.$

In virtue of (45), (46) and (47) we have (44), which is equivalent to (41).

Using Lemmas 4, 5 and 6 we shall prove the following

THEOREM 4. *If*

(49) $$w_n = \sum_{k=1}^{n} b_k\varphi_k$$

is the n-th approximation of the solution of equation (28) *in the sense of Galerkin, then*

(50) $\|Aw_n + Kw_n - f\| \to 0$ *as* $n \to +\infty$.

Proof. Let us denote by $y_n = Aw_n$ and observe that the coefficients $b_1, \ldots, b_n$ satisfy the following system:

(51) $$P_ny_n + P_nKA^{-1}y_n = P_nf.$$

It may be written in the form

(52) $$x_n + P_nTP_n^{-1}x_n = P_nf,$$

where $x_n = P_ny_n$ and $T = KA^{-1}$.

By the assumptions on the operators A, B and K it is easy to prove that $B^{-\frac{1}{2}}(I+T)^{-1}B^{\frac{1}{2}}$ is a bounded operator, as the adjoint operator to $B^{\frac{1}{2}}[(I+T)^{-1}]^*B^{-\frac{1}{2}}$, which is defined in the whole space H and closed.

In virtue of (41) as we know (cf. [5]), x_n is the n-th approximation of the solution of equation (30) in the sense of Galerkin. Hence by Lemmas 4 and 5 we have

$$\|y_n - x\| \to 0 \quad \text{as } n \to +\infty, \tag{53}$$

where x is the unique solution of equation (30). Since $y_n = Aw_n$ and $x = Aw_0$, we get $\|Aw_n - Aw_0\| \to 0$ as $n \to +\infty$.

By the inequality

$$\|Kw_n - Kw_0\| \leqslant \|KA^{-1}\| \, \|Aw_n - Aw_0\|,$$

we have $\|Kw_n - Kw_0\| \to 0$ as $n \to +\infty$. Whence

$$\|\delta_n\| = \|Aw_n + Kw_n - f\| \to 0 \quad \text{as } n \to +\infty. \tag{54}$$

The proof of Theorem 4 is completed.

THEOREM 5. *If the operators A, B and K satisfy the all previous assumptions, then*

$$\|w_n - w_0\| = o(\lambda_n^{-1}), \tag{55}$$

where $\{w_n\}$, $\{\lambda_n\}$ and w_0 are defined in Theorem 4.

The proof of Theorem 5 as quite analogous to the proof of Theorem 3 is omitted.

Remark 2. That KA^{-1} and $A^{-1}K$ be the completely continuous operators it suffices that KB^{-1} and $B^{-1}K$ are the completely continuous operators, since $KA^{-1} = KB^{-1}(BA^{-1})$ and $A^{-1}K = (A^{-1}B)B^{-1}K$, where BA^{-1} and $A^{-1}B$ are the bounded operators.

Remark 3. Theorems 4 and 5 the generalizations of the Theorems 1′ and 2′ from paper [2], respectively. In paper [2] author has assumed that the operators A and B satisfy the following conditions:

1° operator A has a discrete spectrum,

2° $B = UAU^{-1}$, where U is a unitary operator,

3° $\{A\varphi_n\}$ form a complete system in H, where $\{\varphi_n\}$ is a sequence of eigenvectors of operator B,

4° $|(Au, Bu)| \geqslant \gamma_0 \|Au\| \, \|Bu\|$ or $|(Au, u)| \geqslant \gamma_1 \|Au\| \, \|u\|$, where $u \in D(A) \cap \cap D(B)$, $\gamma_0 > 0$ and $\gamma_1 > 0$.

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