ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVI (1972)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XVI (1972)

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A new characterization of Peano continua *

For a metric space (X, d), we set $S(x, \delta) = \{y \in X \mid d(x, y) < \delta\}$ and $S[x, \delta] = \{y \in X \mid d(x, y) \leqslant \delta\}$.

THEOREM 1. *Let (X, d) be compact and satisfy* $\mathrm{Cl}\, S(x, \delta) = S[x; \delta]$ *for all $x \in X$ and $\delta > 0$. Then X is a Peano continuum.*

Proof. We show that each ball $S[x, \delta]$ is connected. If not, it can be written as $A \cup B$, $x \in A$, $B \neq \emptyset$, $A \cap B = \emptyset$ and A, B closed. Since $d(x, B) > 0$, there exists (by compactness) an element $b \in B$ such that $d(x, b) = d(x, B)$. Then $b \in S[x, d(x, B)]$, but $b \notin \mathrm{Cl}\, S(x, d(x, B))$, contrary to assumption. Thus X is connected and locally connected. We now apply the theorem of Hahn and Mazurkiewicz characterizing Peano continua.

THEOREM 2. *A space X is a Peano continuum iff it is compact and admits a metric d such that $S[x, \delta] = \mathrm{Cl}\, S(x, \delta)$ for all $x \in X$ and $\delta > 0$.*

Proof. The sufficiency follows from Theorem 1. The necessity follows from the fact that every Peano continuum admits a convex metric [2] which then satisfies the required condition [1].

One might conjecture that some type of connectivity remains if some conditions are slightly weakened. However, consider the following

EXAMPLE. Let

$$A = \left\{(r, \theta) \in E^2 \mid r = 1 + \frac{1}{\theta}, \pi \leqslant \theta < \infty\right\}$$

and

$$B = \left\{(s, \sigma) \in E^2 \mid s = 2 + \frac{1}{\sigma}, \pi \leqslant \sigma < \infty\right\}.$$

Let $\mathrm{al}(x, y)$ denote arclength. Define d on $A+B$ by

$$d(a_1, a_2) = \frac{\mathrm{al}(a_1, a_2)}{1 + \mathrm{al}(a_1 a_2)} \quad \text{for} \quad a_1, a_2 \in A,$$

$$d(b_1, b_2) = \frac{\mathrm{al}(b_1, b_2)}{1 + \mathrm{al}(b_1, b_2)} \quad \text{for} \quad b_1, b_2 \in B$$

* This research was supported by a Research Faculty Fellowship from L.S.U.

and $d(a, b) = r+s$ for $a = (r, \theta)\in A$ and $b = (s, \sigma)\in B$. It is easy to see that (X, δ) is uniformly locally compact and satisfies $S[x, \delta] = \mathrm{Cl}\,S(x, \delta)$ for all $x\in X$, $\delta > 0$. But X is not even connected.

References

[1] N. Artemiadis, *A remark on metric spaces*, Proc. Kon. Ned. Akad. van Wetensch. A68 (1965), p. 315–318.

[2] R. H. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. 58 (1952), p. 536–556.