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On some classes of sets related to the symmetry of the tangency relation in a metric space

Introduction. Waliszewski in [5] has given the definition of tangency of sets in metric spaces which is a generalization of the definition of tangency of simple arcs given by S. Gołąb and Z. Moszner in the paper [2]. Tangency is not, in general, a symmetric relation. In papers [5] and [6] the classes H_p, A_p^* and M_p of sets of a metric space assuring of the symmetry of the tangency relation have been distinguished. Relationships between these classes have been studied in paper [6].

This paper contains the further investigations of these classes and also of some other classes related to them. It has been proved that the class H_p (for the definition see also [1] and [3]) is essentially included in A_p^*. It follows from example 2 that M_p is essentially wider than A_p^*. In the second part I shall give other definitions of the class M_p.

1. Comparison of the classes H_p and A_p^*. In papers [5] and [6] the classes of sets A_p^* and H_p defined as follows have been investigated: A_p^* is the family of all sets B of points of a metric space E such that

(1) p is a cluster point of the set B,

(2) there exists a number $k > 0$ such that

$$\limsup_{\substack{(x,y)\to(p,p)\\(x,y)\in[B;p]}} \frac{\varrho(x,y)-k\varrho(x;\ B)}{\varrho(x,p)} \leqslant 0,$$

where $[B;p]$ is the set of all ordered pairs (x, y) of points x, y of the space E such that $y \in B$ and

$$\varrho(x;\ B) < \varrho(x,p) = \varrho(y,p), \tag{3}$$

H_p is the family of all sets B of points of the metric space E satisfying condition (1) and such that

$$\lim_{\substack{x,y\to p\\x,y\in B}} \frac{\varrho^2(x,p)+\varrho^2(y,p)-\varrho^2(x,y)}{2\varrho(x,p)\varrho(y,p)} = 1. \tag{4}$$

1.1. *For an arbitrary point p of the metric space E the class H_p is included in the class A_p^*.*

The proof of this theorem is a modification of the proof of theorem 2 from the paper [6].

Proof. Let us assume for $y, u \in E$, $y \neq p \neq u$,

$$r(y, u) = 2 - \frac{\varrho^2(u, p) + \varrho^2(y, p) - \varrho^2(u, y)}{\varrho(u, p)\varrho(y, p)}. \tag{5}$$

Let $B \in H_p$. Take arbitrary $(x, y) \in [B; p]$. Hence it follows that $y \in B$ and condition (3) is fulfilled. Thus there exists a $b(x) \in B$ for which

$$\varrho(x, b(x)) < \varrho(x; B) + \varrho^2(p, x). \tag{6}$$

It follows from the triangle inequality that

$$|\varrho(p, x) - \varrho(p, b(x))| \leqslant \varrho(x, b(x)). \tag{7}$$

By (6) and (7) we have

$$\left|\frac{\varrho(p, b(x))}{\varrho(p, x)} - 1\right| < \frac{\varrho(x; B)}{\varrho(x, p)} + \varrho(p, x). \tag{8}$$

Set $u = b(x)$. From (5) we find $\varrho(b(x), y)/\varrho(p, y)$ and we obtain

$$\frac{\varrho^2(b(x), y)}{\varrho^2(p, y)} = \left(1 - \frac{\varrho(p, b(x))}{\varrho(p, y)}\right)^2 + r(b(x), y)\frac{\varrho(p, b(x))}{\varrho(p, y)}. \tag{9}$$

Taking into account (8) we obtain

(10)
$$\left(\frac{\varrho(b(x), y)}{\varrho(p, y)}\right)^2 \leqslant \left(\frac{\varrho(x; B)}{\varrho(p, x)} + \varrho(p, x)\right)^2 + |r(b(x), y)| \cdot \left(\frac{\varrho(x; B)}{\varrho(p, x)} + \varrho(p, x) + 1\right).$$

By the triangle inequality, (6) and (10) we have

$$\frac{\varrho(x, y)}{\varrho(x, p)} \leqslant \frac{\varrho(x; B)}{\varrho(x, p)} + \varrho(x, p) + \sqrt{\left(\frac{\varrho(x; B)}{\varrho(p, x)} + \varrho(p, x)\right)^2 + |r(b(x), y)|\left(\frac{\varrho(x; B)}{\varrho(p, x)} + \varrho(p, x) + 1\right)}.$$

Whence

$$\frac{\varrho(x, y)}{\varrho(p, x)} \leqslant 2\frac{\varrho(x; B)}{\varrho(p, x)} + 2\varrho(p, x) + \sqrt{|r(b(x), y)|\left(\frac{\varrho(x; B)}{\varrho(p, x)} + \varrho(p, x) + 1\right)}.$$

It follows from the definition of the class H_p that

$$r\big(b(x), y\big) \xrightarrow[\substack{x,y\to p\\ y\in B}]{} 0$$

because, by (3) and (6) we have

$$\varrho\big(p, b(x)\big) \leqslant \varrho(p, x)+\varrho\big(x, b(x)\big) < \varrho(p, x)\big(2+\varrho(p, x)\big) \xrightarrow[x\to p]{} 0.$$

In view of

$$(11)\qquad 2\varrho(p, x)+\sqrt{\big|r\big(b(x), y\big)\big|\big(\varrho(p, x)+2\big)} \xrightarrow[\substack{(x,y)\to(p,p)\\ x,y\in[B;p]}]{} 0$$

we obtain

$$\limsup_{\substack{(x,y)\to(p,p)\\ (x,y)\in[B;p]}} \frac{\varrho(x, y)-2\varrho(x;\ B)}{\varrho(p, x)} \leqslant 0.$$

By which we have proved $H_p \subset A_p^*$. The class A_p^* may be essentially wider than the class H_p which is proved by the following.

Example 1. Let E^1 be the space of all real numbers with the ordinary metric, B the set of all points of the form

$$a_n = \frac{1}{4^n} \quad \text{or} \quad b_n = \frac{-1}{2^{2n+1}},$$

where $n = 1, 2, \ldots$ To prove that B does not belong to H_p it suffices to observe that

$$\lim_{n\to\infty} \frac{|a_n|^2+|b_n|^2-|b_n-a_n|^2}{2\,|a_n|\,|b_n|} = -1.$$

To verify that B belongs to A_p^* we observe that $[B;\ p]$ is the set of all points of the form: (a_n, a_n), $(-a_n, a_n)$, (b_n, b_n), $(-b_n, b_n)$, where $n = 1, 2, \ldots$, because for $(x, y)\in[B;\ p]$ if $x\in B$, $y\in B$, we have

$$\varrho(x;\ B) = 0 < |x| = |y|$$

and

$$|x-y| = 0.$$

If, on the other hand, $x\notin B$, $y\in B$, we have

$$\varrho(x;\ B) = \tfrac{1}{2}|x| < |x| = |y|$$

and

$$|x-y| = 2\,|x|.$$

Thus we have for $k = 4$

$$\lim_{\substack{(x,y)\to(p,p)\\ (x,y)\in[B;p]}} \frac{\varrho(x, y)-k\varrho(x;\ B)}{|x|} = 0$$

because $\varrho(x, y) = 4\varrho(x;\ B)$. Which proves that $B\in A_p^*$.

2. The properties of the class M_p. Let E be a metric space and $\mu > 0$. Consider the set $[B; p, \mu]$ of all pairs (x, y) of points of the space E such that $y \in B$ and

$$\mu\varrho(x; B) < \varrho(x, p) = \varrho(y, p). \tag{12}$$

In particular, $[B; p] = [B; p, 1]$.

Denote by $M_p(\mu)$ the *class* of all sets $B \subset E$ satisfying the following conditions:

(i) p is a cluster point of the set B.

(ii) There exists a function f: $[B; p, \mu] \times \langle 0, \infty) \to \langle 0, \infty)$ such that for arbitrary $(x, y) \in [B; p, \mu]$ and $a > 0$ the conditions

$$f(x, y, 0) \leqslant f(x, y, a), \tag{13}$$

$$\frac{\varrho(x, y)}{\varrho(x, p)} \leqslant f\left(x, y, \frac{\varrho(x; B)}{\varrho(x, p)}\right), \tag{14}$$

$$\lim_{\substack{(x,y)\to(p,p)\\ (x,y)\in[B;p,\mu]\\ a\to 0}} f(x, y, a) = 0 \tag{15}$$

are fulfilled.

2.1. *For arbitrary positive numbers μ and ν class $M_p(\mu)$ is equal to $M_p(\nu)$.*

Proof. Let $B \in M_p(\mu)$; then for an arbitrary point $(x, y) \in [B; p, \mu]$ inequality (12) holds. Consider an arbitrary number $\nu > 0$. If $\mu < \nu$, we have $[B; p, \nu] \subset [B; p, \mu]$ and it suffices to consider a function f defined in the definition of $M_p(\mu)$ restricted to the set $[B; p, \nu]$. Now let $\nu < \mu$. Similarly as before $[B; p, \mu] \subset [B; p, \nu]$. If $(x, y) \in [B; p, \nu] \setminus [B; p, \mu]$, we have

$$\frac{1}{\mu} \leqslant \frac{\varrho(x; B)}{\varrho(x, p)} < \frac{1}{\nu}. \tag{16}$$

For a function f satisfying conditions (13), (14), (15) we define the function g as follows:

$$g(x, y, a) = \begin{cases} f(x, y, a) & \text{if } (x, y) \in [B; p, \mu],\ a > 0, \\ \mu \dfrac{\varrho(x, y)a}{\varrho(x, p)} & \text{if } (x, y) \in [B; p, \nu] \setminus [B; p, \mu],\ a > 0. \end{cases} \tag{17}$$

We will verify that the above defined function g, possesses the required properties. The inequality $g(x, y, 0) \leqslant g(x, y, a)$ for $(x, y) \in [B; p, \nu]$, $a > 0$ is obvious. For an arbitrary $(x, y) \in [B; p, \nu]$

$$\frac{\varrho(x, y)}{\varrho(x, p)} \leqslant g\left(x, y, \frac{\varrho(x; B)}{\varrho(x, p)}\right). \tag{18}$$

In fact, if $(x, y) \in [B;\ p, \mu]$, then inequality (18) follows from the properties of the function f and for $(x, y) \in [B;\ p, \nu] \setminus [B;\ p, \mu]$ in view of (16) and (17) we obtain

$$\frac{\varrho(x, y)}{\varrho(x, p)} \leqslant \mu \frac{\varrho(x, y)}{\varrho(x, p)} \cdot \frac{\varrho(x;\ B)}{\varrho(x, p)} = g\left(x, y, \frac{\varrho(x;\ B)}{\varrho(x, p)}\right).$$

Since for $(x, y) \in [B;\ p, \mu]$, $g(x, y, a) = f(x, y, a)$ thus (15) and (17) implies

$$\lim_{\substack{(x,y)\to(p,p)\\ a\to 0\\ (x,y)\in[B;p,\nu]}} g(x, y, a) = 0.$$

Thus we have proved that for arbitrary positive μ and ν $M_p(\mu) \subset \subset M_p(\nu)$. Changing the places of ν and μ we obtain the required property. It follows from 2.1 that the class M_p defined in [6] is identical with $M_p(\mu)$ for an arbitrary $\mu > 0$. Now, we will give two other definitions of the class M_p. A set $B \subset E$ is said to be *of the class* $\hat{M}_p$ if p is a cluster point of the set B and if there exists a number $\mu > 0$, such that for an arbitrary sequence of points (x_n, y_n) of the set $[B;\ p, \mu]$ satisfying the conditions

$$\lim_{n\to\infty} \varrho(x_n, p) = 0 \tag{19}$$

and

$$\lim_{n\to\infty} \frac{\varrho(x_n;\ B)}{\varrho(x_n, p)} = 0 \tag{20}$$

holds the equality

$$\lim_{n\to\infty} \frac{\varrho(x_n, y_n)}{\varrho(x_n, p)} = 0. \tag{21}$$

A set $B \subset E$ is said to belong *to the class* $\tilde{M}_p$, if p is a cluster point of the set B and there exists $\mu > 0$ such that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every $(x, y) \in [B;\ p, \mu]$, if

$$\varrho(x, p) < \delta \tag{22}$$

and

$$\frac{\varrho(x;\ B)}{\varrho(x.\ p)} < \delta, \tag{23}$$

then

$$\frac{\varrho(x, y)}{\varrho(x, p)} < \varepsilon. \tag{24}$$

2.2. *The class* $\hat{M}_p$ *includes the class* M_p.

Proof. Let $B \in M_p$ and the sequence of the elements of the set $[B; p, \mu]$ satisfy conditions (19) and (20), f be a function appearing in the definition of the function of the class $M_p(\mu)$, for a positive μ. From (19), (20), the inequality

$$\frac{\varrho(x_n, y_n)}{\varrho(x_n, p)} \leqslant f\left(x_n, y_n, \frac{\varrho(x_n; B)}{\varrho(x_n, p)}\right), \qquad n = 1, 2, \ldots,$$

and (15) if follows that

$$\lim_{n\to\infty} f\left(x_n, y_n, \frac{\varrho(x_n; B)}{\varrho(x_n, p)}\right) = 0.$$

Thus

$$\lim_{n\to\infty} \frac{\varrho(x_n, y_n)}{\varrho(x_n, p)} = 0.$$

2.3. *The class $\hat{M}_p$ includes the class $\tilde{M}_p$.*

Proof. Suppose that the set B belongs to $\hat{M}_p$ but it does not belong to $\tilde{M}_p$. Consider an arbitrary $\mu > 0$. There exists an $\varepsilon > 0$ such that for an arbitrary $\delta > 0$ there exists a point (x, y) of the set $[B; p, \mu]$ for which equalities (22) and (23) are fulfilled but (24) does not hold. Taking $\delta = 1/n$ we have $(x_n, y_n) \epsilon [B; p, \mu]$, $\varrho(x_n, p) < 1/n$,

$$\frac{\varrho(x_n; B)}{\varrho(x_n, p)} < \frac{1}{n}$$

and

$$\frac{\varrho(x_n, y_n)}{\varrho(x_n, p)} \geqslant \varepsilon.$$

In the other words, $B \notin \hat{M}_p$.

2.4. *The class $\tilde{M}_p$ is contained in M_p.*

Proof. Let B be the set of class $\tilde{M}_p$. For any $\mu > 0$, we consider the function g_μ defined, for any $a \geqslant 0$ and $y \epsilon B$ as a supremum of all η such that there exist x and y satisfying the conditions $(x, y) \epsilon [B; p, \mu]$,

$$\varrho(x; B)/\varrho(x, p) \leqslant a \quad \text{and} \quad \varrho(x, y)/\varrho(x, p) = \eta.$$

The function g_μ is non-negative, satisfies the inequality

$$0 \leqslant g_\mu(y, 0) \leqslant g_\mu(y, a) \leqslant 2$$

and

$$\frac{\varrho(x, y)}{\varrho(x, p)} \leqslant g_\mu\left(y, \frac{\varrho(x; B)}{\varrho(x, p)}\right)$$

for an arbitrary $(x, y) \epsilon [B; p, \mu]$. It follows from $B \epsilon \tilde{M}_p$ that there exists a $\mu_0 > 0$ such for an arbitrary $\varepsilon > 0$ one may find $\delta > 0$ and such that for $(x, y) \epsilon [B; p, \mu]$ the inequalities $\varrho(x, p) < \delta$ and

$$\frac{\varrho(x; B)}{\varrho(x, p)} < \delta \quad \text{imply} \quad \frac{\varrho(x, y)}{\varrho(x, p)} < \frac{1}{2}\varepsilon.$$

Let $0 < a < \delta$ and $\varrho(y, p) < \delta$. Then $g_{\mu_0}(y, a) \leqslant \frac{1}{2}\varepsilon$. Hence $g_{\mu_0}(y, a) \to 0$ if $y \to p$, $a \to 0$, $y \in B$. Put for $(x, y) \in [B; p, \mu_0]$, $a > 0$, $f(x, y, a) = g_{\mu_0}(y, a)$. Thus the function f satisfies the conditions specified in the definition of the class $M_p(\mu_0)$. Thus $B \in M_p$.

The equality of the classes $\tilde{M}_p$, $\hat{M}_p$ and M_p follows from 2.2, 2.3, and 2.4. Lemma 2.4 implies immediately the possibility of weakening the conditions imposed upon function f, appearing in the definition of the class M_p. Now we will give an example of a set which belongs to the class M_p and which is not a set of the class A_p^*.

EXAMPLE 2. Let B be a set whose elements are all points $z_{t,n}$ of the Cartesian plane R^2 of the form

$$z_{t,n} = \left(\frac{1}{4^n}, \frac{t}{4^n}\right), \quad \text{where } t \in \left\langle 0, \frac{\sqrt{3}}{3}\right\rangle, \; n = 1, 2, \ldots, \tag{25}$$

and the point $p = (0, 0)$.

We shall prove that the set B does not belong to the class A_p^*. Consider an arbitrary number $k > 0$. We shall prove the existence of sequence (x_n, y_n) of elements $[B; p, 1]$ for which $\lim\limits_{n\to\infty} |x_n| = 0$ and

$$\limsup_{n\to\infty} \frac{|x_n - y_n| - k\varrho(x_n;\, B)}{|x_n|} > 0. \tag{26}$$

Put

$$u_n(t) = \left(\frac{1}{4^n\sqrt{1+t^2}}, \frac{t}{4^n\sqrt{1+t^2}}\right) \quad \text{for } t \in \left(0, \frac{\sqrt{3}}{3}\right).$$

Then $|z_{0,n}| = |u_u(0)|$. It is obvious that

$$\frac{\varrho(u_n(t);\, B)}{|z_{0,n}|} = \left(1 - \frac{1}{\sqrt{1+t^2}}\right) < 1 \tag{27}$$

thus $(u_n(t), z_{0,n}) \in [B;\, p, 1]$ for $t \in \langle 0, \sqrt{3}/3\rangle$. From the equalities

$$|u_n(t) - z_{0,n}| = \frac{1}{4^n}\sqrt{2\left(1 - \frac{1}{\sqrt{1+t^2}}\right)}, \qquad |z_{0,n}| = \frac{1}{4^n}$$

it follows that

$$\frac{|u_n(t) - z_{0,n}|}{|z_{0,n}|} = \sqrt{2\left(1 - \frac{1}{\sqrt{1+t^2}}\right)}.$$

Putting

$$l_t = \sqrt{2\frac{\sqrt{1+t^2}}{\sqrt{1+t^2} - 1}},$$

we have, by (27),

$$\frac{|u_n(t)-z_{0,n}|}{|z_{0,n}|}-l_t\frac{\varrho(u_n(t);\, B)}{|z_{0,n}|}=0$$

and clearly l_t is the smallest number for which

$$\limsup_{n\to\infty}\left(\frac{|u_n(t)-z_{0,n}|}{|z_{0,n}|}-l_t\frac{\varrho(u_n(t);\, B)}{|z_{0,n}|}\right)\leqslant 0.$$

In view of $l_t \underset{t\to 0}{\to} \infty$, for an arbitrary number $k>0$ there exists t such that $l_t>k$ and for a sequence of elements $(u_n(t), z_{0,n})$ belonging to $[B; p, 1]$ relation (26) is satisfied, thus B does not belong to A_p^*. The set B belongs to the class M_p. Let $\mu = 3$. Then for an arbitrary pair $(x, y)\in [B; p, \mu]$ the relation $3\varrho(x;\, B) < |x| = |y|$ is satisfied. Thus there exists $y'\in B$ such that $\varrho(x;\, B) = |x-y'|$. Hence

$$\left|\frac{|y'|}{|x|}-1\right|\leqslant\frac{\varrho(x;\, B)}{|x|}<\frac{1}{3}.$$

Since $|y| = |x| = \sqrt{1+t^2}/4^n$, $|y'| = \sqrt{1+t'^2}/4^{n'}$, where $t, t'\in\langle 0, \sqrt{3}/3\rangle$; thus

$$\left|\frac{\sqrt{1+t'^2}\cdot 4^n}{\sqrt{1+t^2}\cdot 4^{n'}}-1\right|<\frac{1}{3},$$

therefore

$$\frac{2}{3}\leqslant\frac{\sqrt{1+t'^2}\cdot 4^n}{\sqrt{1+t^2}\cdot 4^{n'}}<\frac{4}{3}. \tag{28}$$

In view of $t, t'\in\langle 0, \sqrt{3}/3\rangle$ the inequality

$$\frac{\sqrt{3}}{2}\leqslant\sqrt{\frac{1+t'^2}{1+t^2}}\leqslant\frac{2}{\sqrt{3}}$$

hold. If we have $n\neq n'$, then $4^{n-n'}\geqslant 4$ or $4^{n-n'}\leqslant\frac{1}{4}$ hold. Thus

$$\sqrt{\frac{1+t'^2}{1+t^2}}\cdot 4^{n-n'}\geqslant 2\sqrt{3}\quad\text{ or }\quad\sqrt{\frac{1+t'^2}{1+t^2}}\cdot 4^{n-n'}\leqslant\frac{1}{2\sqrt{3}}$$

which contradicts inequality (28). Then $n = n'$.

Consider a sequence (x_m, y_m), $m = 1, 2, \ldots$ elements of the set $[B; p, \mu]$ satisfying conditions (19) and (20). We will prove that then equality (21) holds. In view of $(x_m, y_m)\in [B; p, \mu]$ we have $y_m = (1/4^{n_m},\ t_m/4^{n_m})$ and $|x_m| = |y_m|$, where $t_m\in\langle 0, \sqrt{3}/3\rangle$, thus

$$x_m = \left(\frac{\sqrt{1+t_m^2}}{4^{n_m}\sqrt{1+s_m^2}},\ \frac{s_m\sqrt{1+t_m^2}}{4^{n_m}\sqrt{1+s_m^2}}\right)$$

for a certain $s_m \epsilon (-\infty, \infty)$, taking into account $\mu = 3$, we have also

$$(29) \quad \varrho(x_m; B) = |x_m - y'_m| \quad \text{where } y'_m = \left(\frac{1}{4^{n_m}}, \frac{t'_m}{4^{n_m}}\right), \; t'_m \epsilon \left\langle 0, \frac{\sqrt{3}}{3}\right\rangle.$$

Since

$$0 \leqslant \frac{\big||y_m| - |y'_m|\big|}{|y_m|} \leqslant \frac{|x_m - y'_m|}{|y_m|}.$$

Thus taking into account (20) we have

$$\lim_{m\to\infty} \frac{\big||y_m| - |y'_m|\big|}{|y_m|} = 0.$$

Moreover,

$$(30) \qquad \lim_{m\to\infty} \frac{|y_m - y'_m|}{|y_m|} = 0,$$

because

$$\left|1 - \frac{|y'_m|^2}{|y_m|^2}\right| \geqslant \frac{|y_m - y'_m|^2}{|y_m|^2}.$$

The last equality follows from

$$\left|1 - \frac{|y'_m|^2}{|y_m|^2}\right| = \frac{|t_m^2 - t'^2_m|}{(4^{n_m})^2|y_m|^2} = \frac{|t_m - t'_m|^2(t_m + t'_m)}{(4^{n_m})^2|y_m|^2|t_m - t'_m|} \geqslant \frac{|t_m - t'_m|^2}{(4^{n_m})^2|y_m|^2} = \frac{|y_m - y'_m|^2}{|y_m|^2}.$$

Conditions (20), (29) and (30) implies (21). Thus we have proved, by 2.3 and 2.4 that $B \epsilon M_p$.

The given example proves that it is imposible to convert the following implication being the Theorem 1 of the paper [6]: if $B \epsilon M_p$, then for every $A \epsilon C_p$ from $\langle A, B\rangle \epsilon T_p$ it follows that $\langle B, A\rangle \epsilon T_p$.

EXAMPLE 3. Let E be the space of all real numbers with the ordinary metric and B the set of real numbers of the form $1/2^n$ or $-1/2^n$, $n = 1, 2, \ldots$ It can easily be verified that, there does not exist any set A in C_p (C_p is a class defined in [6], such that $\langle A, B\rangle \epsilon T_p$).

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