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# A note on the relation between Lindelöf and $\aleph_1$-compact spaces

Creede [1, 2] demonstrates that if $S$ is a semistratifiable space, then the statement that $S$ is Lindelöf is equivalent to the statement that $S$ is $\aleph_1$-compact. It will be shown in this paper that this holds for a much weaker space.

All theorems which hold for semistratifiable spaces hold for both semimetric and stratifiable spaces since the concept of semistratifiable spaces is a generalization of both semimetric and stratifiable spaces. This generalization is due to E. A. Michael.

DEFINITION 1. The statement that a topological space $S$ is *semistratifiable* means that there is a function $G$ from the product of the collection of closed subsets of $S$ with the natural numbers into the collection of open sets in $S$ such that

(i) $$\bigcap_{n=1}^{\infty} G(A, n) = A \quad \text{for each closed set } A,$$

(ii) $$G(A_1, n) \subset G(A_2, n) \text{ whenever } A_1 \subset A_2.$$

DEFINITION 2. The statement that $K$ is a *minimal cover* of the point set $M$ means that $K$ is an open cover of $M$ and if $g$ is an element of $K$, then $\{h\colon h \in K \text{ and } h \neq g\}$ is not a cover of $M$.

DEFINITION 3. The statement that a point set $M$ is *minimal cover refineable* means that if $K$ is an open cover of $M$, then there is an open refinement of $K$ which is a minimal cover of $M$.

DEFINITION 4. The statement that a topological space $S$ is *$F_\sigma$-screenable* means that every open cover of $S$ has a $\sigma$-discrete closed refinement which covers $S$.

DEFINITION 5. The statement that a space $S$ is *$\aleph_1$-compact* means that no collection of uncountably many subsets of $S$ is discrete.

DEFINITION 6. The statement that a topological space $S$ is *semimetric* means that there is a distance function $d$ defined on $S \times S$ such that

(i) for all points $x$ and $y$ in $S$, $d(x, y) = d(y, x) \geqslant 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$, and

(iii) if $x$ is a point of $S$ and $M$ is a subset of $S$, $x \in \bar{M}$ if and only if $\inf\{d(x, y) : y \in M\} = 0$.

All other terms are as used in [4].

First, it will be shown that every subset of a semistratifiable space is minimal cover refineable. Then, it will be shown that in a minimal cover refineable space $S$, the statement that $S$ is Lindelöf is equivalent to the statement that $S$ is $\aleph_1$-compact.

THEOREM 1. *Every $F_\sigma$-screenable space is minimal cover refineable.*

Proof. Let $S$ be an $F_\sigma$-screenable space and $O$ any open cover of $S$. Let $\{F_i\}$ be a countable collection of discrete closed refinements of $O$ such that $\{F_i\}$ covers $S$. For each integer $i > 0$, associate with each element $f$ of $F_i$ one and only one open set $o = g \setminus (F_i^* \setminus f)$, where $g$ is an element of $O$ and $f \subset g$. Let $O'_i$ be the collection of elements of $O$ associated with $F_i$. Let $O_1 = O'_1$. For each integer $i > 1$, let

$$O_i = \{o:\ o' \in O'_i,\ o = o' \setminus \bigcup_{j=1}^{i-1} F_j^*,\ \text{and}\ o \cap F_i^* \neq \emptyset\}.$$

Then the collection of open sets $\{o$: for some integer $i > 0$, $o \in O_i\}$ is a minimal cover of $S$ which refines $O$ and Theorem 1 is established.

THEOREM 2. *Every subset of a semistratifiable space is minimal cover refineable.*

Proof. Creede [1, 2] demonstrates that a semistratifiable space is both $F_\sigma$-screenable and hereditarily semistratifiable. By Theorem 1, this is sufficient to show that a semistratifiable space and every subset of it is minimal cover refineable.

THEOREM 3. *A minimal cover refineable space is Lindelöf if and only if it is $\aleph_1$-compact.*

Proof. Since Lindelöf implies no collection of uncountably many subsets is discrete, only sufficiency must be shown. Let $S$ be a minimal cover refineable space with the property that no collection of uncountably many subsets of the space is discrete and $O$ any open cover of $S$. Let $O'$ be a refinement of $O$ which is a minimal cover of $S$. Let $H = \{h$: $h$ is the set of all the points in some $g \in O'$ which are in no other element of $O'\}$. The collection $H$ is discrete, so both $H$ and $O'$ have at most countably many members. Therefore, $S$ is Lindelöf and Theorem 3 is established.

The following theorem is an application of Theorem 3:

THEOREM 4. *A hereditarily separable minimal cover refineable space is Lindelöf.*

Proof. Let $D$ be a discrete collection. Let $C$ be a countable point set such that $C \subset D^*$ and $\overline{D^*} \subset \overline{C}$. Each element of $D$ contains a point of $C$, so $D$ must be a countable collection. Then, by Theorem 3, Theorem 4 is established.

In [3], R. W. Heath points out that there is a separable semimetric space which is not Lindelöf. One such example is given by McAuley [5].

Creede [1, 2] demonstrates that in a semistratifiable space hereditary separability is equivalent to Lindelöf. This is not the case in minimal cover refineable spaces.

THEOREM 5. *There is a minimal cover refineable Lindelöf $T_4$ space which is not separable.*

Proof. Let $\Omega$ denote the first uncountable ordinal. Consider the collection $[0, \Omega]$ of ordinals with the order topology. This space is compact and $T_4$. Hence, it is minimal cover refineable and Lindelöf, but it is not separable.

## Bibliography

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