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Theorems of the Romanovski type for the Denjoy–Perron integrals

1. Let $D^*_{\langle a,b\rangle}$ be the class of all functions f integrable in the Denjoy–Perron sense on the interval $\langle a, b\rangle$. In this case the function f is said to be *D^*-integrable on* $\langle a, b\rangle$. Denote by $\Phi(t, x, \xi)$ a function bounded in $t \in \langle a, b\rangle$ and $x \in \langle a', b'\rangle$ $(a < a' \leqslant b' < b)$ for every fixed $\xi \in E$, where E is a given set of numbers with an accumulation point ξ_0. Suppose that

$$\lim_{\xi\to\xi_0} \int_{\alpha}^{\beta} \Phi(t, x, \xi)\,dt = 1 \tag{1}$$

for each α, β such that $a \leqslant \alpha < x < \beta \leqslant b$. Further restrictions on $\Phi(t, x, \xi)$ will be specified below.

In the present paper we generalize Theorem 3 of [1] concerning the convergence of integrals

$$U(x, \xi; f) = (D^*) \int_{a}^{b} f(t)\Phi(t, x, \xi)\,dt. \tag{2}$$

Following Krein–Levin [2], we shall weaken slightly the hypotheses of this theorem.

Given any 2π-periodic function $f \in D^*_{\langle -\pi,\pi\rangle}$, we consider also the operator

$$W(x, \xi; f) = (D^*) \int_{-\pi}^{\pi} f(t) K(t-x, \xi)\,dt, \tag{3}$$

where the kernel $K(t, \xi)$ is 2π-periodic, even, bounded, non-negative and non-increasing in t on $\langle 0, \pi\rangle$ for every $\xi \in E$, and such that

$$\lim_{\xi\to\xi_0} \int_{-\pi}^{\pi} K(t, \xi)\,dt = 1.$$

We complete Theorem 3.1 of [5] by establishing a similar fact for the Denjoy–Perron integrals.

2. We shall begin with some auxiliary results.

LEMMA 1. *Let $g(t)$ be a function of bounded variation in every interval $\langle a+\eta, b\rangle$ $(0<\eta<b-a)$ and such that $\int_a^b \operatorname{var}_{s\leqslant t\leqslant b} g(t)\,ds<\infty$. Consider $f\in D^*_{\langle a,b\rangle}$ for which*

$$M=\sup_{0<h\leqslant b-a}\left|\frac{1}{h}(D^*)\int_a^{a+h} f(t)\,dt\right|<\infty.$$

Then the function $f(t)g(t)$ is D^*-integrable on $\langle a, b\rangle$ and

$$\left|(D^*)\int_a^b f(t)g(t)\,dt\right|\leqslant M\int_a^b\{\operatorname{var}_{s\leqslant t\leqslant b} g(t)+|g(b)|\}\,ds.$$

The proof runs as in 2.1 of [5] (see also [4], p. 246). Clearly, the related estimate as in Remark of [5], p. 175, remains also true.

Given any $f\in D^*_{\langle a,b\rangle}$, we set

$$F(u)=(D^*)\int_a^u f(t)\,dt,\qquad u\in\langle a, b\rangle.$$

Consider the function $g(t)$ of bounded variation in $\langle a,\beta\rangle\subset\langle a, b\rangle$. By the Jordan decomposition, $g(t)=g_1(t)-g_2(t)$, where $g_1(t)$, $g_2(t)$ are non-negative and non-decreasing, either $g_1(a)=0$ or $g_2(a)=0$, and $\operatorname{var}_{a\leqslant t\leqslant\beta} g(t)=\operatorname{var}_{a\leqslant t\leqslant\beta} g_1(t)+\operatorname{var}_{a\leqslant t\leqslant\beta} g_2(t)$. In view of the second mean value theorem ([4], p. 246),

$$(D^*)\int_a^\beta f(t)g(t)\,dt=g_1(\beta)(D^*)\int_{\xi_1}^\beta f(t)\,dt-g_2(\beta)(D^*)\int_{\xi_2}^\beta f(t)\,dt,$$

where $a<\xi_1, \xi_2<\beta$. Without loss of generality it can be assumed $g_1(a)=0$, $\xi_1\leqslant\xi_2$. Hence,

$$\left|(D^*)\int_a^\beta f(t)g(t)\,dt\right|\leqslant g_1(\beta)|F(\xi_2)-F(\xi_1)|+|g(\beta)|\,|F(\beta)-F(\xi_2)|,$$

and

$$g_1(\beta)=\operatorname{var}_{a\leqslant t\leqslant\beta} g_1(t)\leqslant\operatorname{var}_{a\leqslant t\leqslant\beta} g(t).$$

Denote by $\omega(F)=\omega(F;\langle a,\beta\rangle)$ the oscillation of F over the interval $\langle a,\beta\rangle$. Then, it is easy to see that inequality (1) of [1], given without proof, may be stated in a slightly more precise form:

$$(4)\qquad\left|(D^*)\int_a^\beta f(t)g(t)\,dt\right|\leqslant\{\operatorname{var}_{a\leqslant t\leqslant\beta} g(t)+\sup_{a\leqslant t\leqslant\beta}|g(t)|\}\,\omega(F).$$

Now let $g(t, x, \xi)$ be the function of bounded variation with respect to t in $\langle a, b\rangle$ for every $x \in \langle a', b'\rangle$ $(a < a' \leqslant b' < b)$, $\xi \in E$. Consider a fixed $x_0 \in \langle a, b\rangle$.

LEMMA 2. *Suppose that for every* c $(a \leqslant c \leqslant b)$

$$\lim_{\xi \to \xi_0} \int_a^c g(t, x, \xi)\,dt = 0$$

uniformly in $x \in \langle a', b'\rangle$. *If there exist a constant* $L > 0$ *such that*

$$|g(t, x, \xi)| \leqslant L, \qquad \operatorname*{var}_{a \leqslant t \leqslant b} g(t, x, \xi) \leqslant L$$

whenever $x \in \langle a', b'\rangle$, $\xi \in E$, *then*

$$\lim_{(x, \xi) \to (x_0, \xi_0)} (D^*) \int_a^b f(t) g(t, x, \xi)\,dt = 0$$

for any $f \in D^*_{\langle a, b\rangle}$.

The idea of the proof, which is directly based on the fundamental Romanovski's lemma ([3], §2), is similar to that of Theorem 2 in [1]. We must only notice that inequality (4) leads to

$$\omega(F^x_\xi;\ \langle \alpha, \beta\rangle) \leqslant 2\omega(F;\ \langle \alpha, \beta\rangle),$$

where

$$F^x_\xi(u) = (D^*) \int_a^u f(t) g(t, x, \xi)\,dt.$$

Therefore, the series

$$\sum_{k=1}^{\infty} \omega(F^x_\xi;\ \langle \alpha_k, \beta_k\rangle)$$

converges uniformly in $x \in \langle a', b'\rangle$ and $\xi \in E$.

3. Now, we shall present two theorems concerning the convergence of operators (2) and (3) at the points $x_0 \in (a, b)$ for which

$$\lim_{h \to 0} \frac{1}{h} (D^*) \int_{x_0}^{x_0+h} f(t)\,dt = f(x_0). \tag{5}$$

THEOREM 1. *Let the function* $\Phi(t, x_0, \xi)$, *introduced in par.* 1, *be of bounded variation in* t *outside every subinterval* $(x_0 - \varepsilon, x_0 + \varepsilon)$ *of* $\langle a, b\rangle$, *and such that*

$$\int_a^{x_0} \operatorname*{var}_{a \leqslant t \leqslant s} \Phi(t, x_0, \xi)\,ds + \int_{x_0}^{b} \operatorname*{var}_{s \leqslant t \leqslant b} \Phi(t, x_0, \xi)\,ds \leqslant C(x_0), \tag{6}$$

where $C(x_0)$ *is a constant depending only on* x_0. *Then, for any* $f \in D^*_{\langle a, b\rangle}$,

$$\lim_{\xi \to \xi_0} U(x_0, \xi;\ f) = f(x_0).$$

Proof. Of course, it is enough to show that

$$I(x_0, \xi) = (D^*)\int_a^b \{f(t)-f(x_0)\}\Phi(t, x_0, \xi)\,dt \to 0 \quad \text{as } \xi \to \xi_0.$$

In view of (5), given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup_{0<h\leqslant\delta}\left|\frac{1}{h}(D^*)\int_{x_0}^{x_0\pm h}\{f(t)-f(x_0)\}\,dt\right| \leqslant \varepsilon.$$

Let us consider first the interval $\langle x_0+\delta, b\rangle$. Since the function $v(s) = \operatorname*{var}_{s\leqslant t\leqslant b}\Phi(t, x_0, \xi)$ is non-increasing, inequality (6) leads to

$$C(x_0) \geqslant \int_{x_0+\frac{1}{2}\delta}^{x_0+\delta} v(s)\,ds \geqslant \tfrac{1}{2}\delta\cdot v(x_0+\delta) \tag{7}$$

for $t \in \langle x_0+\delta, b\rangle$. Consequently,

$$\operatorname*{var}_{x_0+\delta\leqslant t\leqslant b}\Phi(t, x_0, \xi) \leqslant \frac{2}{\delta}C(x_0) \qquad (\xi \in E).$$

The family $\Phi(t, x_0, \xi)$ is uniformly bounded in $t \in \langle x_0+\delta, b\rangle$ and ξ near to ξ_0. To establish this, suppose that $\limsup_{\xi\to\xi_0}|\Phi(b, x_0, \xi)| = \infty$. Then $\lim_{k\to\infty}\Phi(b, x_0, n_k) = \pm\infty$ for a certain sequence n_k. For example, when the above limit is equal to $+\infty$ we should have, by (7),

$$C(x_0) \geqslant \tfrac{1}{2}\delta\{\Phi(b, x_0, n_k) - \Phi(t, x_0, n_k)\}.$$

Hence, $\lim_{k\to\infty}\Phi(t, x_0, n_k) = \infty$ uniformly in $t \in \langle x_0+\delta, b\rangle$. Then, for any $N > 0$, there would be an integer $K(N)$, that $\Phi(t, x_0, n_k) \geqslant N$ for $k > K(N)$, $t \in \langle x_0+\delta, b\rangle$. Consequently

$$\int_{x_0+\delta}^{b}\Phi(t, x_0, n_k)\,dt \geqslant (b-x_0-\delta)\cdot N \qquad \text{for } k > K(N),$$

and

$$\limsup_{\xi\to\xi_0}(D^*)\int_{x_0+\delta}^{b}\Phi(t, x_0, \xi)\,dt = \infty.$$

But this clearly contradicts (1). Thus, $\Phi(b, x_0, \xi)$ remains bounded in a neighbourhood of ξ_0. By (7),

$$|\Phi(t, x_0, \xi) - \Phi(b, x_0, \xi)| \leqslant \frac{2}{\delta}C(x_0) \qquad \text{when } t \in \langle x_0+\delta, b\rangle,$$

and the above statement follows.

By symmetry, $\operatorname{var}\Phi(t, x_0, \xi)$ and $\Phi(t, x_0, \xi)$ are uniformly bounded in $t \in \langle a, x_0-\delta\rangle$ and ξ near to ξ_0.

Let us pass on now to the integral $I(x_0, \xi)$. Write

$$I(x_0, \xi) = (D^*)\Big(\int_a^{x_0+\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^{b}\Big)\{f(t)-f(x_0)\}\Phi(t, x_0, \xi)\,dt = I_1+I_2+I_3.$$

Applying Lemma 2 to the interval $\langle x_0+\delta, b\rangle$ with $x = x_0$, we get $\lim_{\xi\to\xi_0} I_3 = 0$. Analogously, $\lim_{\xi\to\xi_0} I_1 = 0$. Finally, by Lemma 1,

$$|I_2| \leqslant \varepsilon\{C(x_0)+\delta\cdot|\Phi(x_0-\delta, x_0, \xi)|+\delta\cdot|\Phi(x_0+\delta, x_0, \xi)|\}.$$

Hence $\lim_{\xi\to\xi_0} I_2 = 0$, and this completes the proof.

Remark. If x_0 is an end-point of $\langle a, b\rangle$, it is sufficient to restrict the assumptions of Theorem 1 to the corresponding intervals on one side of x_0.

THEOREM 2. *Suppose that the kernel $K(t, \xi)$ satisfies the conditions listed in par.* 1, *and that*

(8) $$\lim_{\xi\to\xi_0} K(t, \xi) = 0 \quad \textit{if } 0 < t \leqslant \pi.$$

Then

$$\lim W(x, \xi; f) = f(x_0)$$

as $(x, \xi)\to(x_0, \xi_0)$ on any plane set Z in which the function $\lambda(x, \xi) = (x-x_0)K(0, \xi)$ is bounded.

Proof. Write

$$(D^*)\int_{-\pi}^{\pi}\{f(t)-f(x_0)\}K(t-x, \xi)\,dt$$

$$= (D^*)\Big(\int_{-\pi}^{x_0-\delta} + \int_{x_0-\delta}^{x_0+\delta} + \int_{x_0+\delta}^{\pi}\Big)\{f(t)-f(x_0)\}K(t-x, \xi)\,dt = J_1+J_2+J_3.$$

We may clearly assume that $-\pi < x_0 \leqslant 0$ and $0 < \delta < \pi+x_0$, $0 < x_0-x < \frac{1}{2}\delta$.

Applying Lemma 1 and arguing as in [5], p. 176, we obtain $\lim J_2 = 0$ as $(x, \xi)\to(x_0, \xi_0)$ on Z.

By the hypotheses,

$$K(t-x, \xi) \leqslant K(\tfrac{1}{2}\delta, \xi) \quad \text{when } t\in\langle-\pi, x_0-\delta\rangle$$

and

$$\operatorname*{var}_{-\pi\leqslant t\leqslant x_0-\delta} K(t-x, \xi) = K(x_0-x-\delta, \xi) \leqslant K(\tfrac{1}{2}\delta, \xi);$$

moreover, for any $\eta\in\langle-\pi, x_0-\delta\rangle$,

$$\int_{-\pi}^{\eta} K(t-x, \xi)\,dt \leqslant 2\pi\cdot K(x_0-\delta-x, \xi) \leqslant 2\pi\cdot K(\tfrac{1}{2}\delta, \xi).$$

It follows at once from (8) that

$$\lim_{\xi\to\xi_0}\int_{-\pi}^{\eta} K(t-x,\,\xi)\,dt = 0$$

uniformly in x ($|x-x_0|<\frac{1}{2}\delta$), and that the function $K(t-x,\,\xi)$ and its variation in t in the interval $\langle -\pi, x_0-\delta\rangle$ are bounded uniformly in $x\in(x_0-\frac{1}{2}\delta,\, x_0+\frac{1}{2}\delta)$ and ξ near to ξ_0. Hence, by Lemma 2, $\lim J_1=0$ as $(x,\,\xi)\to(x_0,\,\xi_0)$. Analogously $\lim J_3=0$. Thus the proof is finished.

4. As an application of our main results, let us consider the Jackson integral of 2π-periodic function $f\in D^*_{\langle-\pi,\pi\rangle}$, i.e.

$$U_n(x;\,f) = (D^*)\int_0^{\pi} f_x(t)\,\Phi_n(t,\,0)\,dt,$$

where

$$f_x(t) = \frac{f(x+t)+f(x-t)}{2},\qquad \Phi_n(t,\,x) = \frac{3}{\pi n(2n^2+1)}\left\{\frac{\sin\frac{1}{2}n(t-x)}{\sin\frac{1}{2}(t-x)}\right\}^4.$$

The kernel $\Phi_n(t)=\Phi_n(t,\,0)$ is non-increasing in $t\in\langle 0,\,\pi/n\rangle$, and

$$\operatorname*{var}_{s\leqslant t\leqslant\pi}\Phi_n(t) = \int_s^{\pi}\left|\frac{d}{dt}\,\Phi_n(t)\right|dt \leqslant \pi^3 n^{-2}s^{-3}\qquad (0<s\leqslant\pi).$$

Therefore, $\Phi_n(t)$ is of bounded variation with respect to t in every interval $\langle\varepsilon,\,\pi\rangle$ $(0<\varepsilon<\pi)$, and

$$\int_0^{\pi}\operatorname*{var}_{s\leqslant t\leqslant\pi}\Phi_n(t)\,ds = \int_0^{\pi/n}\operatorname*{var}_{s\leqslant t\leqslant\pi/n}\Phi_n(t)\,ds+\frac{\pi}{n}\cdot\operatorname*{var}_{\pi/n\leqslant t\leqslant\pi}\Phi_n(t)+\int_{\pi/n}^{\pi}\operatorname*{var}_{s\leqslant t\leqslant\pi}\Phi_n(t)\,ds$$

$$\leqslant\int_0^{\pi/n}\{\Phi_n(s)-\Phi_n(\pi/n)\}\,ds+\pi+\frac{\pi^3}{n^2}\int_{\pi/n}^{\pi}s^{-3}\,ds\leqslant\tfrac{3}{2}(1+\pi).$$

Hence, in view of Theorem 1, the relation

$$\lim_{h\to0}\frac{1}{h}\,(D^*)\int_0^{h}\{f(x+t)-2f(x)+f(x-t)\}\,dt = 0$$

which is equivalent to

$$\lim_{h\to0}\frac{1}{h}\,(D^*)\int_0^{h} f_x(t)\,dt = f_x(0)$$

implies

$$\lim_{n\to\infty}U_n(x;\,f) = f(x).$$

Now, let us observe that a biharmonic function $W(x, r; f)$ in the unit disc $|r\cdot e^{ix}| < 1$ may be represented in the form

$$W(x, r; f) = (D^*) \int_{-\pi}^{\pi} f(t) K(t-x, r)\, dt \qquad (f \in D^*_{\langle -\pi, \pi\rangle}),$$

where

$$K(t, r) = \frac{(1-r^2)^2(1-r\cos t)}{2\pi(1+r^2-2r\cos t)^2}.$$

Applying Theorem 2 we obtain

$$\lim W(x, r; f) = f(x_0)$$

as (x, r) tends to $(x_0, 1)$ in such a manner so that $|x-x_0|/(1-r) \leqslant C$ ($C = \text{const}$).

Finally, we note that (7.9), (ii) of [6], p. 101, extended to $f \in D^*_{\langle -\pi, \pi\rangle}$, is also a special case of Theorem 2.

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