ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XVI (1972)

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Seria I: PRACE MATEMATYCZNE XVI (1972)

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Nörlund summability of the derived series of Fourier series

1. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of real constants and

$$P_n = p_0+p_1+\ldots+p_n, \quad P_{-1} = p_{-1} = 0.$$

In the sequel it is assumed that for $n \geqslant 0$, $P_n \neq 0$ and $P_n \to \infty$ as $n \to \infty$.

Let $p(t)$ be continuous in $(0, \infty)$, linear in each interval $(n, n+1)$ $(n = 0, 1, 2, \ldots)$ and such that $p(n) = p_n$ for $n = 0, 1, 2, \ldots$ Also we write $P(u) = \int_0^u p(x)dx$ so that $P(n) \simeq P_n$ as $n \to \infty$.

The transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} S_\nu$$

defines the n-th (N, p_n) mean or the n-th Nörlund mean of the sequence S_n. If

$$\lim_{n\to\infty} t_n = S,$$

the series $\sum a_n$ is said to be *summable* (N, p_n) *to the sum* S.

2. Let $f(t)$ be a continuous function of bounded variation, periodic with period 2π and integrable in $(-\pi, \pi)$. Let the function $f(t)$ have a derivative $f'(x)$ at the point $t = x$ and let the Fourier Series associated with $f(t)$ be

$$\tag{2.1} \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}{}' (a_n \cos nt + b_n \sin nt),$$

where the constants a_n and b_n are given by the usual Euler–Fourier formulae.

The derived series of Fourier series is

$$(2.2)\qquad \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt)$$

and its allied series is

$$(2.3)\qquad \sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt).$$

In this paper we adopt the following notations:

$$g(t) = f(x+t) - f(x-t) - 2tf'(x),$$

$$h(t) = f(x+t) + f(x-t) - 2f(x),$$

$$G(t) = \int_0^t |dg(u)|,$$

$$X(t) = \int_0^t |dh(u)|,$$

$$H_n(x) = -\frac{1}{4\pi} \int_{1/n}^{\pi} h(t) \operatorname{cosec}^2 \tfrac{1}{2}t \, dt.$$

3. The Nörlund summability of a Fourier series and its allied series has been studied by a number of workers such as Iyengar [4], Siddiqi [9], Pati [7], Singh [10], Rajagopal [8], Hirokawa and Kayashima [2], Hirokawa [1], M. Izumi and S. Izumi [5] and others. In this paper we establish the following two theorems on the Nörlund Summability of the derived series of Fourier Series and its allied series.

THEOREM 1. *If*

$$(3.1)\qquad G(t) \equiv \int_0^t |dg(u)| = o\left[\frac{p\left(\frac{1}{t}\right)}{P\left(\frac{1}{t}\right)}\right]$$

as $t \to 0$ *and* p_n *is a positive sequence such that*

$$(3.2)\qquad \int_1^n u\,|p'(u)|\,du = O(P_n)$$

as $n \to \infty$, *then the derived series of Fourier series of* $f(x)$ *is summable* (N, p_n) *to the sum* $f'(x)$ *at the point* x.

THEOREM 2. *If* p_n *is a positive sequence satisfying condition* (3.1) *and*

$$X(t) = \int_0^t |dh(u)| = o\left[\frac{p\left(\dfrac{1}{t}\right)}{P\left(\dfrac{1}{t}\right)}\right]$$

as $t \to 0$, *then series* (2.3) *is summable* (N, p_n) *to the sum*

$$-\frac{1}{4\pi}\int_0^\pi h(t)\operatorname{cosec}^2 \tfrac{1}{2}t\, dt$$

at every point x *at which this integral exists.*

It is interesting to note that these theorems generalize earlier results due to Tripathi ([11], [12]) and Joshi [6].

The following lemma is required in the proof of Theorem 1:

LEMMA [5]. *For* $t \geqslant 1/n$,

$$\sum_{k=0}^{n} p_k \sin(n-k+\tfrac{1}{2})t = O\big(P(1/t)\big) + O(1/t)\Big\{p(1/t) + p_n + \int_{1/t}^{n} |p'(u)|\, du\Big\}.$$

4. Proof of Theorem 1. Before proceeding with the proof of the theorem we note that (see [5]) hypothesis (3.2) of the theorem implies that

(4.1) $$np_n = O(P_n)$$

as $n \to \infty$. Clearly if (4.1) is satisfied, then (3.1) gives

(4.2) $$G(t) = o(t)$$

as $t \to 0$.

We have

$$\begin{aligned}
\sigma_n(x) &\equiv \sum_{\nu=1}^{n} \nu(b_\nu \cos \nu x - a_\nu \sin \nu x) \\
&= \frac{d}{dx}\Big[\sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\Big] \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(u)\left\{\frac{d}{dx}\,\frac{\sin(n+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)}\right\} du \\
&= -\frac{1}{2\pi}\int_0^{2\pi} f(u)\,\frac{d}{du}\left\{\frac{\sin(n+\frac{1}{2})(x-u)}{\sin\frac{1}{2}(x-u)}\right\} du \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \{f(x+u) - f(x-u)\}\,\frac{d}{du}\left\{\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\right\} du,
\end{aligned}$$

so that on integration by parts

$$\sigma_n(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\, d\{f(x+u)-f(x-u)\}.$$

It is known that

$$1 = \frac{1}{2\pi} \int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\, 2\,du,$$

and therefore

$$\sigma_n(x)-f'(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} [d\{f(x+u)-f(x-u)\}-2f'(x)]\,du$$

$$= \frac{1}{2\pi} \int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\, dg(u).$$

In order to establish the theorem we have to show that

$$\lim_{n\to\infty}\left[\frac{1}{P_n}\sum_{k=0}^{n} p_k\big(\sigma_{n-k}-f'(x)\big)\right] = 0. \tag{4.3}$$

Now

$$\frac{1}{P_n}\sum_{k=0}^{n} p_k\big(\sigma_{n-k}-f'(x)\big) \tag{4.4}$$

$$= \frac{1}{2\pi P_n}\int_0^\pi \frac{dg(t)}{\sin\frac{1}{2}t}\left(\sum_{k=0}^{n} p_k \sin(n-k+\tfrac{1}{2})t\right)$$

$$= \frac{1}{2\pi P_n}\int_0^{1/n} \frac{dg(t)}{\sin\frac{1}{2}t}\left(\sum_{k=0}^{n} p_k \sin(n-k+\tfrac{1}{2})t\right) +$$

$$+ \frac{1}{2\pi P_n}\int_{1/n}^{\pi} \frac{dg(t)}{\sin\frac{1}{2}t}\left(\sum_{k=0}^{n} p_k \sin(n-k+\tfrac{1}{2})t\right) = I_1 + I_2, \quad \text{say.}$$

Let us first consider I_1. We have

$$I_1 = O\left(\frac{1}{P_n}\right)\int_0^{1/n} \frac{|dg(t)|}{t} \sum_{k=0}^{n} p_k \cdot nt \tag{4.5}$$

$$= O(n)\int_0^{1/n} |dg(t)|$$

$$= O(n)\cdot o\left(\frac{1}{n}\right) = o(1),$$

as $n\to\infty$, by the application of (4.2).

Making use of the lemma, we have

$$(4.6)\qquad I_2 = O\left(\frac{1}{P_n}\right)\Bigg[\int_{1/n}^{\pi}\frac{|dg(t)|}{t}P\left(\frac{1}{t}\right)+\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}p\left(\frac{1}{t}\right)+$$

$$+p_n\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}+\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}\int_{1/t}^{n}|p'(u)|\,du\Bigg]$$

$$= O\left(\frac{1}{P_n}\right)\Bigg[\int_{1/n}^{\pi}\frac{|dg(t)|}{t}P\left(\frac{1}{t}\right)+p_n\cdot\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}+$$

$$+\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}\int_{1/t}^{n}|p'(u)|\,du\Bigg]$$

remembering that $\frac{1}{t}\,p\left(\frac{1}{t}\right)=O\left(P\left(\frac{1}{t}\right)\right)$, as $t\to 0$.

By integration by parts and hypothesis (3.1) we have

$$(4.7)\qquad \frac{1}{P_n}\int_{1/n}^{\pi}\frac{|dg(t)|}{t}P\left(\frac{1}{t}\right)$$

$$=\frac{1}{P_n}\left[\frac{P\left(\frac{1}{t}\right)}{t}G(t)\right]_{1/n}^{\pi}+\frac{1}{P_n}\int_{1/n}^{\pi}G(t)\,\frac{P\left(\frac{1}{t}\right)}{t^2}\,dt+\frac{1}{P_n}\int_{1/n}^{\pi}\frac{G(t)p\left(\frac{1}{t}\right)}{t^3}\,dt$$

$$=o(1)+o\left(\frac{1}{P_n}\right)\int_{1/n}^{\pi}\frac{p\left(\frac{1}{t}\right)}{P\left(\frac{1}{t}\right)}\,\frac{P\left(\frac{1}{t}\right)}{t^2}\,dt+o\left(\frac{1}{P_n}\right)\int_{1/n}^{\pi}\frac{p\left(\frac{1}{t}\right)}{t^2}\,dt$$

$$=o(1)+o\left(\frac{1}{P_n}\right)\int_{0}^{n}p(u)\,du=o(1),$$

as $n\to\infty$. Also

$$(4.8)\qquad \frac{p_n}{P_n}\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}$$

$$=\frac{p_n}{P_n}\left[\frac{1}{t^2}G(t)\right]_{1/n}^{\pi}+2\,\frac{p_n}{P_n}\int_{1/n}^{\pi}\frac{G(t)}{t^3}\,dt$$

$$=o(1)+o\left(\frac{p_n}{P_n}\right)\int_{1/n}^{\pi}\frac{dt}{t^2}=o(1)+o\left(\frac{np_n}{P_n}\right)=o(1),$$

as $n\to\infty$, by the application of (4.1).

Finally

$$(4.9)\qquad \frac{1}{P_n}\int_{1/n}^{\pi}\frac{|dg(t)|}{t^2}\int_{1/t}^{n}|p'(u)|\,du$$

$$=\frac{1}{P_n}\left[\frac{G(t)}{t^2}\int_{1/t}^{n}|p'(u)|\,du\right]_{1/n}^{\pi}+\frac{2}{P_n}\int_{1/n}^{\pi}\frac{G(t)}{t^3}\,dt\int_{1/t}^{n}|p'(u)|\,du-$$

$$-\frac{1}{P_n}\int_{1/n}^{\pi}\frac{G(t)}{t^2}\,\frac{|p'(1/t)|}{t^2}\,dt$$

$$=O\left(\frac{1}{P_n}\right)\int_{1/\pi}^{n}|p'(u)|\,du+o\left(\frac{1}{P_n}\right)\int_{1/n}^{\pi}\frac{dt}{t^2}\int_{1/t}^{n}|p'(u)|\,du+$$

$$+o\left(\frac{1}{P_n}\right)\int_{1/n}^{\pi}\frac{|p'(1/t)|}{t^3}\,dt$$

$$=o(1)+O\left(\frac{1}{P_n}\right)\int_{1}^{n}|p'(u)|\,du+o\left(\frac{1}{P_n}\right)\int_{1/\pi}^{n}dv\int_{v}^{n}|p'(u)|\,du+$$

$$+o\left(\frac{1}{P_n}\right)\int_{1/\pi}^{n}u\,|p'(u)|\,du$$

$$=o(1)+O\left(\frac{1}{P_n}\right)\int_{1}^{n}|p'(u)|\,du=o(1),$$

if we show that

$$(4.10)\qquad \frac{1}{P_n}\int_{1}^{n}|p'(u)|\,du=o(1),$$

as $n\to\infty$. We have

$$(4.11)\qquad \frac{1}{P_n}\int_{1}^{n}|p'(u)|\,du=\frac{1}{nP_n}\int_{1}^{n}u\,|p'(u)|\,du+\frac{1}{P_n}\int_{1}^{n}\frac{1}{u^2}\int_{1}^{u}v|p'(v)|\,dv$$

$$=O\left(\frac{1}{n}\right)+O\left(\frac{1}{P_n}\right)\int_{1}^{n}\frac{P(u)}{u^2}\,du$$

$$=o(1)+O\left(\frac{1}{P_n}\right)\sum_{k=1}^{n}\frac{P_k}{k^2},$$

and since the series $\sum_{k=1}^{\infty} 1/k^2$ is convergent by virtue of a known result (see Hobson [3], example 3 on p. 8) it follows that

$$\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^2} = o(1), \tag{4.12}$$

as $n \to \infty$. Combining the estimates in (4.11) and (4.12) we find that (4.10) is established.

Again combining the estimates in (4.6) through (4.9) we find that

$$I_2 = o(1) \tag{4.13}$$

as $n \to \infty$. Finally from (4.4), (4.5) and (4.13) we get that

$$\sum_{k=0}^{n} p_k\big(\sigma_{n-k} - f'(x)\big) = o(P_n),$$

as $n \to \infty$ and this establishes (4.3). Hence the theorem.

Proof of Theorem 2. If we denote by $\bar{\sigma}_n(x)$ the sum of the first n terms of the series (2.3) at a point $t = x$, then proceeding as in [11] we have

$$\bar{\sigma}_{n-k}(x) - H_n(x) = \frac{1}{2\pi} \int_{1/n}^{\pi} \frac{\cos(n-k+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dh(t) + o(1).$$

Hence in order to establish the theorem we have to show that

$$\frac{1}{2\pi P_n} \int_{1/n}^{\pi} \frac{dh(t)}{\sin\frac{1}{2}t} \Big(\sum_{k=0}^{n} p_k \cos(n-k+\tfrac{1}{2})t\Big) = o(1)$$

as $n \to \infty$. But this can be proved by exactly similar arguments as in the case of the proof of $I_2 = o(1)$ as $n \to \infty$, in the proof of Theorem 1. This completes the proof of the theorem.

The author is deeply indebted to Dr. S. N. Lal for his valuable guidance and suggestions.

References

[1] Hiroshi Hirokawa, *On the Nörlund summability of Fourier series and its conjugate series*, Proc. Japan Acad. 44 (1968), p. 449-451.

[2] — and Ikuko Kayashima, *On the Nörlund summability of the conjugate series of Fourier series*, ibidem 44 (1968), p. 301-306.

[3] E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, New York 1957.

[4] K. S. K. Iyengar, *A Tauberian theorem and its application to convergence of Fourier series*, Proc. Indian Acad. Sci. Sec. A, 18 (1943), p. 81-87.
[5] M. Izumi and S. Izumi, *Nörlund summability of Fourier series*, Pacific J. of Math. 26 (1968), p. 289-301.
[6] H. P. Joshi, *On the Nörlund summability of the derived series of a Fourier series*, J. of Sci. Res. 2, 16 (1965/66), p. 183-189.
[7] T. Pati, *A generalization of a theorem of Iyenger on the harmonic summability of Fourier series*, Indian J. Math. 3 (1961), p. 85-90.
[8] C. T. Rajagopal, *On the Nörlund summability of Fourier series*, Proc. Cambridge Phil. Soc. 59 (1963), p. 47-53.
[9] J. A. Siddiqi, *On the harmonic summability of Fourier series*, Proc. Indian Acad. Sci. Sec. A, 28 (1948), p. 527-531.
[10] T. Singh, *Nörlund summability of Fourier series and its conjugate series*, Ann. Mat. Pura Appl. Ser. IV, 64 (1964), p. 123-132.
[11] L. M. Tripathi, *On the harmonic summability of the derived Fourier series and its conjugate series*, Proc. Nat. Acad. Sci. India Sec. A, 33 (1963), p. 443-454.
[12] — *On the Nörlund summability of the derived Fourier series*, ibidem 30 (1964), p. 548-555.

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