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# Some generalizations of the Cantor–Lebesgue Theorem, II

This paper is concerned with generalizations of the results to be found on pages 95 thru 103 of my ealier paper [3], and with certain new results along these lines. In general these generalizations have required complete restatements of the theorems and their proofs, however, whenever possible we have simply made reference to the result given in the earlier paper [3]. At first the generalization will be confined mainly to replacing Lebesgue measure by Lebesgue–Stieltjes measures and introducing a weighting function $g(x)$ in place of $x$, but in the end we also obtain results in which we introduce several weighting functions $g_m(x), \ldots, g_0(x)$ and several corresponding lambda sequences $\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\}$. An effort has also been made to improve the notation, and in conjunction with this it should be noted that the property $P(\delta, l, r)$ found in this paper is the property $Q'(r, \delta, l)$ found in [3].

Throughout this paper we will assume that $f(x)$ is a real or complex valued function defined for all real $x$, and that unless specified otherwise, $h(x)$ is a strictly increasing real valued function defined and right continuous on an open interval $D$, finite or infinite, that $\mu_h$ is the Lebesgue–Stieltjes measure function on $D$ generated by $h(x)$, and that $\mu$ is the Lebesgue measure function.

THEOREM 1. *Let $f(x)$ be any real or complex valued function, $p$ be a fixed real or complex number, and $\{t_n\}$ be any convergent sequence of real numbers. Let $D$ be an open interval, finite or infinite, and $g(x)$ be defined on $D$. If $t_0 = \lim_n t_n$ and $f(x)$ is continuous at $t_0$, then a necessary condition that the sequence $f(\lambda_n \cdot g(x) + t_n)$ converge to $p$ on a subset of $D$ having positive $\mu_h$-measure for no bounded real sequence $\{\lambda_n\}$ is*

$$f(t_0) \neq p. \tag{i}$$

Proof. Let $E$ be a subset of $D$ having positive $\mu_h$-measure and $\lambda_n = 0$ $(n = 1, 2, 3, \ldots)$, then $\lim_n f(\lambda_n \cdot g(x) + t_n) = \lim_n f(t_n) = f(t_0)$ on $E$.

Thus if $\lim_n f(\lambda_n \cdot g(x) + t_n) \neq p$ on subsets of $D$ having positive $\mu_h$-measure for all bounded real sequences $\{\lambda_n\}$, then $f(t_0) \neq p$.

Note. If $t_n = t_0$ for $n = 1, 2, 3, \ldots$ the hypothesis that $f(x)$ is continuous at $t_0$ may be deleted.

Lemma 2. *Let the function $g(x)$ be defined and strictly monotonic on an open interval $D$, finite or infinite, and $h(x)$ be strictly increasing and right continuous on an open interval containing $g(D)$. If $E$ is a Borel subset of $D$, then $g(E)$ is $\mu_h$-measurable.*

Proof. Since $g$ is strictly monotonic on the open interval $D$; then $D \cap \{x : g(x) \leqslant a\}$ is an interval or the null set for every real $a$, hence $g$ is a Baire function. Thus $g$ is a one-to-one Baire function whose domain is a Borel set $D$, hence it maps Borel subsets of $D$ onto Borel sets (see p. 489 of [2]).

Lemma 3. *Let the functions $g(x)$ and $h(x)$ be continuous on the interval $D$, finite or infinite, differentiable on $D^0$, and the function $g'(x)/h'(x)$ be bounded on $D^0$. If $M = \operatorname{Sup}\{|g'(x)/h'(x)| : x \in D^0\}$, then $|g(x_1) - g(x_2)| \leqslant M|h(x_1) - h(x_2)|$ for every $x_1, x_2 \in D$.*

Proof. Let $x_1, x_2$ be any two points in $D$, then it follows from the Generalized Mean Value Theorem that

$$[g(x_1) - g(x_2)] \cdot h'(\xi) = [h(x_1) - h(x_2)] \cdot g'(\xi) \quad \text{for some } \xi \in (x_1, x_2),$$

and consequently

$$|g(x_1) - g(x_2)| = \left|\frac{g'(\xi)}{h'(\xi)}\right| \cdot |h(x_1) - h(x_2)| \leqslant M|h(x_1) - h(x_2)|.$$

In a similar manner we prove

Lemma 4. *Let the functions $g(x)$ and $h(x)$ be continuous on the interval $D$, finite or infinite, differentiable on $D^0$, and the function $g'(x)/h'(x)$ be bounded away from zero on $D^0$. If $\gamma = \operatorname{Inf}\{|g'(x)/h'(x)| : x \in D^0\}$, then $|g(x_1) - g(x_2)| \geqslant \gamma|h(x_1) - h(x_2)|$ for every $x_1, x_2 \in D$.*

Lemma 5. *Let function $g(x)$ be strictly increasing and continuous on the open interval $D$, finite or infinite. If $E$ is a Borel subset of $D$, then $\mu(g(E)) = \mu_g(E)$.*

Proof. Since $E$ is a Borel subset of $D$ it is $\mu_g$-measurable and it follows from Lemma 2 that $g(E)$ is $\mu$-measurable.

Given $\varepsilon > 0$, let $\{(a_n, b_n) : n = 1, 2, 3, \ldots\}$ be a sequence of open intervals contained in $D$ and such that $\bigcup\{(a_n, b_n) : n = 1, 2, 3, \ldots\} \supseteq E$ and $\sum_{n=1}^{\infty}[g(b_n) - g(a_n)] \leqslant \mu_g(E) + \varepsilon$; and let $c_n = g(a_n)$, $d_n = g(b_n)$ for $n = 1, 2, 3, \ldots$ Then $g$ being strictly increasing it follows that $\bigcup\{(c_n, d_n) : n = 1, 2, 3, \ldots\} \supseteq g(E)$.

Thus

$$\sum_{n=1}^{\infty} (d_n - c_n) = \sum_{n=1}^{\infty} \left|\big(g(b_n) - g(a_n)\big)\right| < \mu_g(E) + \varepsilon,$$

and since $\varepsilon$ is arbitrary, then $\mu\big(g(E)\big) \leqslant \mu_g(E)$.

Since $g$ is strictly increasing and continuous on $D$ it is an open mapping function there, hence $g(D)$ is an open set. Given $\varepsilon > 0$, let $\{(c_n, d_n): n = 1, 2, 3, \ldots\}$ be a sequence of open intervals such that $\bigcup \{(c_n, d_n): n = 1, 2, 3, \ldots\} \supseteq g(E)$, $[c_n, d_n] \subseteq g(D)$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty}(d_n - c_n) < \mu\big(g(E)\big) + \varepsilon$, and let $a_n = g^{-1}(c_n)$, $b_n = g^{-1}(d_n)$ for $n = 1, 2, 3, \ldots$ Then $g^{-1}$ being strictly increasing it follows that $\bigcup \{(a_n, b_n): n = 1, 2, 3, \ldots\} \supseteq E$, and $g$ being strictly increasing and continuous on $D$ it follows from $[c_n, d_n] \subseteq g(D)$ that $[a_n, b_n] = g^{-1}([c_n, d_n]) \subseteq D$ for $n = 1, 2, 3, \ldots$ Thus $\sum_{n=1}^{\infty} [g(b_n) - g(a_n)] = \sum_{n=1}^{\infty}(d_n - c_n) < \mu\big(g(E)\big) + \varepsilon$, and since $\varepsilon$ is arbitrary then $\mu_g(E) \leqslant \mu\big(g(E)\big)$.

Note. If $g(x)$ is merely strictly increasing and right continuous on $D$, then we have $\mu_g(E) \leqslant \mu\big(g(E)\big)$.

LEMMA 6. *Let the functions $g(x)$ and $h(x)$ be defined, strictly increasing, and right continuous on the open interval $D$, finite of infinite, and $|g(x_1) - g(x_2)| \geqslant \gamma |h(x_1) - h(x_2)|$ for some constant $\gamma > 0$ and every $x_1, x_2 \in D$. If $E \subseteq D$ is a Borel set, then $\mu_g(E) \geqslant \gamma \cdot \mu_h(E)$. Futhermore if $[a, b] \subseteq D$ is a finite interval,*

$$\eta = \frac{g(b) - g(a)}{\gamma[h(g) - h(a)]}, \qquad r \in (0, 1), \ \eta(1 - r) < 1,$$

*and $E \subseteq (a, b]$ is a Borel set such that $\mu_h(E) > \eta(1 - r)[h(b) - h(a)]$, then $\mu_g(E) > (1 - r)[g(b) - g(a)]$.*

*Finally, if $E \subseteq D$ is a Borel set of positive $\mu_h$-measure, then $\mu_g(E) > 0$.*

Proof. Since $E$ is a Borel subset of $D$ it is both $\mu_h$ and $\mu_g$-measurable. Let $\{(a_n, b_n): n = 1, 2, 3, \ldots\}$ be any sequence of open intervals, contained in $D$, and such that $\bigcup \{(a_n, b_n): n = 1, 2, 3, \ldots\} \supseteq E$. Then $\sum_{n=1}^{\infty}[g(b_n) - g(a_n)] \geqslant \gamma \sum_{n=1}^{\infty}[h(b_n) - h(a_n)] \geqslant \gamma \mu_h(E)$. Therefore $\mu_g(E) \geqslant \gamma \cdot \mu_h(E)$.

Let $E \subseteq (a, b] = I$ be a Borel set such that $\mu_h(E) > \eta(1 - r)[h(b) - h(a)]$. Thus

$$\begin{aligned}
\mu_g(I - E) &= \mu_g(I) - \mu_g(E) \leqslant \mu_g(I) - \gamma \cdot \mu_h(E) \\
&< \mu_g(I) - \gamma\eta(1 - r)\mu_h(I) \\
&= [g(b) - g(a)]\left[1 - \gamma\eta(1 - r)\left(\frac{h(b) - h(a)}{g(b) - g(a)}\right)\right] = r[g(b) - g(a)],
\end{aligned}$$

and consequently

$$\mu_g(E) = \mu_g(I) - \mu_g(I-E) > (1-r)[g(b)-g(a)].$$

Obviously if $E \subseteq D$ is a Borel set of positive $\mu_h$-measure, then $\mu_g(E) \geqslant \gamma \cdot \mu_h(E) > 0$.

Note 1. If $g(x)$ is strictly decreasing the lemma and proof remain valid with $\mu_g(S)$ replaced by $\mu(g(S))$ for $S = E$, $I$, and $I$-$E$ and $g(b) - g(a)$ replaced by $|g(b)-g(a)|$.

Note 2. Obviously the result is valid for all sets $E \subseteq D$ which are both $\mu_g$ and $\mu_h$-measurable, but for most of our applications $E$ will be a Borel set.

Note 3. If $g(x)$ and $h(x)$ are continuous at $x = a$, then we may replace $E \subseteq (a, b]$ by $E \subseteq [a, b]$ above, for then $\mu_g([a, b]) = \mu_g((a, b]) = g(b)-g(a)$ and $\mu_h([a, b]) = \mu_h((a, b]) = h(b)-h(a)$. Indeed it will be this form of the lemma which we make use of later.

Note 4. It also follows from the proof above that if $\mu_h(E) \geqslant \eta(1-r)[h(b)-h(a)]$, then $\mu_g(E) \geqslant (1-r)[g(b)-g(a)]$.

Note 5. Since the function $g(x)$ is strictly increasing, it is a simple matter to alter the proofs given above in such a manner that we can delete the requirement that $g(x)$ be right continuous and replace $\mu_g(E)$ by $\mu(g(E))$ throughout. Lemma 2 will assure that $g(E)$ is $\mu$-measurable

LEMMA 7. *Let the functions $g(x)$ and $h(x)$ be defined, strictly increasing, and right continuous on the open interval $D$, finite or infinite, and $|g(x_1) - g(x_2)| \leqslant M|h(x_1)-h(x_2)|$ for some constant $M > 0$ and every $x_1, x_2 \in D$. If $E \subseteq D$ is a Borel set, then $\mu_g(E) \leqslant M \cdot \mu_h(E)$. Furthermore if $[a, b] \subseteq D$ is a finite interval,*

$$\eta = \frac{[g(b)-g(a)]}{M[h(b)-h(a)]}, \qquad r \in (0, 1),$$

*and $E \subseteq (a, b]$ is a Borel set such that $\mu_h(E) > (1-\eta r)[h(b)-h(a)]$, then $\mu_g(E) > (1-r)[g(b)-g(a)]$. Finally, if $h(x)$ is continuous on $D$, and $E$ is a Borel subset of $D$ having positive $\mu_h$-measure, then $\mu_g(E) > 0$.*

Proof. Since $E$ is a Borel subset of $D$ it is both $\mu_h$ and $\mu_g$-measurable. Given $\varepsilon > 0$ let $\{(a_n, b_n) : n = 1, 2, 3, \ldots\}$ be a sequence of open intervals, contained in $D$, and such that

$$\bigcup \{(a_n, b_n): n = 1, 2, 3, \ldots\} \supseteq E \quad \text{and} \quad \sum_{n=1}^{\infty} [h(b_n) - h(a_n)] < \mu_h(E) + \frac{\varepsilon}{M}.$$

Thus

$$\sum_{n=1}^{\infty}[g(b_n)-g(a_n)] \leqslant M\sum_{n=1}^{\infty}[h(b_n)-h(a_n)] < M\cdot\mu_h(E)+\varepsilon,$$

and $\varepsilon$ being arbitrary, then $\mu_g(E) \leqslant M\cdot\mu_h(E)$.

Let $E \subseteq (a, b]$ be a Borel set such that $\mu_h(E) > (1-\eta r)[h(b)-h(a)]$, and $S = (a, b]-E$; then $S$ is also a Borel set and $\mu_h(S) = \mu_h\big((a, b]\big)-\mu_h(E) < \eta r[h(b)-h(a)]$. Therefore

$$\mu_g(E) = \mu_g\big((a, b]\big)-\mu_g(S) \geqslant [g(b)-g(a)]-M\cdot\mu_h(S) > [g(b)-g(a)]- -M\eta r[h(b)-h(a)] = (1-r)[g(b)-g(a)].$$

Suppose that $h(x)$ is continuous on $D$, and let $E \subseteq D$ be a Borel set having positive $\mu_h$-measure; then it follows from Lemma 5 that $\mu\big(h(E)\big) = \mu_h(E) > 0$. Since $h(E)$ is a set of positive $\mu$-measure it has a point of density $x_d$. Let $I_k = (x_d-k^{-1}, x_d]$ for $k = 1, 2, 3, \ldots$; then

$$\lim_k \frac{\mu\big(h(E) \cap I_k\big)}{\mu(I_k)} = 1.$$

Let $J_k = \big(h^{-1}(x_d-k^{-1}), h^{-1}(x_d)\big]$ for $k = 1, 2, 3, \ldots$, then $h(J_k) = I_k$ for $k = 1, 2, 3, \ldots$ Thus it now follows from Lemma 5 that

$$\lim_k \frac{\mu_h(E \cap J_k)}{\mu_h(J_k)} = \lim_k \frac{\mu\big(h(E \cap J_k)\big)}{\mu\big(h(J_k)\big)} = \lim_k \frac{\mu\big(h(E) \cap I_k\big)}{\mu(I_k)} = 1.$$

Let $r \in (0, 1)$ and choose $K$ so that $\mu_h(E \cap J_K) > (1-\eta r)\mu_h(J_K)$ and $J_K \subseteq D$. Then with $J_K = (a, b]$ above we obtain $\mu_g(E) \geqslant \mu_g(E \cap J_K) > (1-r)\mu_g(J_K) > 0$.

Note. The notes of Lemma 6 apply here as well (with the obvious alteration to Note 4).

DEFINITION 1. Let the functions $g(x)$ and $h(x)$ be defined on an interval $D$, finite or infinite. If corresponding to every finite interval $I \subseteq D$, there exists a finite positive constant $\gamma_I$ or $M_I$ such that either $|g(x_1)-g(x_2)| \geqslant \gamma_I |h(x_1)-h(x_2)|$ or $|g(x_1)-g(x_2)| \leqslant M_I |h(x_1)-h(x_2)|$ whenever $x_1, x_2 \in I$, then $g(x)$ is said to be *of type* $T_1$ with respect to $h(x)$ on $D$.

Note 1. In this paper, the application of this definition will be to functions $g(x)$ and $h(x)$ which are strictly monotonic and continuous on $D$ (with $h(x)$ actually strictly increasing).

Note 2. The property is obviously transitive.

Note 3. If $g(x)$ and $h(x)$ are defined on an interval $D$, finite or infinite, and $g(x) = c_1\cdot h(x)+c_2$ on $D$ for real constants $c_1, c_2$ with $c_1 \neq 0$, then $g(x)$ is of type $T_1$ with respect to $h(x)$ on $D$.

LEMMA 8. *Let the function $g(x)$ and $h(x)$ be defined, and $g(x)$ be one-to-one, on an interval $D$, finite or infinite. If $g(D)$ is an interval, then $g(x)$ is of type $T_1$ with respect to $h(x)$ on $D$ iff $h(g^{-1}(x))$ is of type $T_1$ with respect to $x$ on $g(D)$.*

Proof. For

$$|g(x_1)-g(x_2)| \geqslant \gamma_I |h(x_1)-h(x_2)| \quad \text{on } I$$

implies that

$$|x_1-x_2| = |g(g^{-1}(x_1))-g(g^{-1}(x_2))| \geqslant \gamma_I \cdot |h(g^{-1}(x_1))-h(g^{-1}(x_2))| \quad \text{on } g(I),$$

and

$$|h(g^{-1}(x_1))-h(g^{-1}(x_2))| \geqslant \gamma_I \cdot |x_1-x_2| \quad \text{on } g(I)$$

implies that

$$|h(x_1)-h(x_2)| = \left|h\big(g^{-1}(g(x_1))\big)-h\big(g^{-1}(g(x_2))\big)\right| \geqslant \gamma_I |g(x_1)-g(x_2)| \quad \text{on } I,$$

etc.

Note. In particular, if the function $h(g^{-1}(x))$ satisfies the Lipschitz condition on every finite interval $I \subseteq g(D)$, then $g(x)$ is of type $T_1$ with respect to $h(x)$ on $D$.

LEMMA 9. *Let the functions $g(x)$ and $h(x)$ be defined, strictly increasing, with $g(x)$ right continuous, and $h(x)$ continuous on an open interval $D$, finite or infinite. If there exists positive constants $\gamma$ or $M$ such that*

$$|g(x_1)-g(x_2)| \geqslant \gamma |h(x_1)-h(x_2)| \quad \text{or} \quad |g(x_1)-g(x_2)| \leqslant M |h(x_1)-h(x_2)|$$

*whenever $x_1, x_2 \in D$, and $E \subseteq D$ is both $\mu_g$ and $\mu_h$-measurable, then $\mu_g(E) > 0$ iff $\mu_h(E) > 0$.*

Proof. From Lemmas 6 and 7 it follows that if $\mu_h(E) > 0$, then $\mu_g(E) > 0$ (see Note 2 of Lemma 6). Furthermore since

$$|h(x_1)-h(x_2)| \leqslant \gamma^{-1} |g(x_1)-g(x_2)| \quad \text{or} \quad |h(x_1)-h(x_2)| \geqslant M^{-1} |g(x_1)-g(x_2)|$$

whenever $x_1, x_2 \in D$, it follows from Lemmas 6 and 7 that if $\mu_g(E) > 0$, then $\mu_h(E) > 0$.

Note 1. Lemma 5 is pertinent at this point.

Note 2. From Note 1 of Lemmas 6 and 7 it follows that the result holds with $g(x)$ strictly decreasing instead of strictly increasing.

Note 3. If $h(x)$ is only right continuous on $D$ and we have $|g(x_1)-g(x_2)| \geqslant \gamma |h(x_1)-h(x_2)|$ whenever $x_1, x_2 \in D$, then $\mu_g(E) > 0$. For in Lemma 6 we require only the right continuity of $h(x)$ in $D$.

Note 4. From 5 of Lemmas 6 and 7 it follows that if we delete the requirement that $g(x)$ be right continuous, then we may conclude that if $E$ is a $\mu_h$-measurable subset of $D$ such that $g(E)$ is $\mu$-measurable, then $\mu(g(E)) > 0$ whenever $\mu\ (E) > 0$.

LEMMA 10. *Let the functions $g(x)$ and $h(x)$ be defined, strictly increasing with $g(x)$ right continuous, and $h(x)$ continuous on an open interval $D$, finite or infinite, and $g(x)$ be of type $T_1$ with respect to $h(x)$ on $D$. If $E \subseteq D$ is both $\mu_g$ and $\mu_h$-measurable, then $\mu_g(E) > 0$ iff $\mu_h(E) > 0$.*

Proof. This follows directly from Lemma 9 and the fact that every set of positive $\mu_h$-measure contains a bounded Borel subset of positive $\mu_h$-measure, which is consequently contained in a finite interval.

Note 1. In view of Note 2 of Lemma 9 the result holds with $g(x)$ strictly decreasing instead of strictly increasing.

Note 2. In view of Note 4 of Lemma 9 above, if we delete the requirement that $g(x)$ be right continuous, then we may conclude that if $E$ is a $\mu_h$-measurable subset of $D$ such that $g(E)$ is $\mu$ -measurable, then $\mu(g(E)) > 0$ whenever $\mu_h(E) > 0$.

DEFINITION 2. Let the function $g(x)$ be defined except possibly at a finite number of points of an interval $D$, finite or infinite, and $h(x)$ be strictly increasing and continuous on $D$. Then $g(x)$ is said to be *of type $T_2$* with respect to $h(x)$ on $D$, if both $g'(x)$ and $g'(x)/h'(x)$ exist and are non-zero except on a finite set $S \subseteq D$, and both $g'(x)$ and $g'(x)/h'(x)$ are continuous on $D$-$S$.

Note. Polynomial and rational functions are of type $T_2$ with respect to (a continuous strictly increasing) $h(x)$ on $D$ whenever $h'(x)$ is continuous and non-zero except at finitely many points in $D$.

DEFINITION 3. Let the functions $g(x)$ and $h(x)$ be defined on an interval $D$, finite or infinite, and $h(x)$ be strictly increasing and continuous on $D$. Then $g(x)$ is said to be *of type $T$* witht respect to $h(x)$ on $D$, if it is of type $T_2$ with respect to $h(x)$ on $D$, or it is strictly monotonic and right continuous on $D$ and of type $T_1$ with respect to $h(x)$ on $D$.

Note 1. Since $g(x)$ is either strictly monotonic on $D$ or else strictly monotonic on finitely many subintervals of $D$, then $g(x)$ is $\mu_h$-measurable.

Note 2. If $h(x)$ is strictly increasing and continuous on $D$ and $g(x) = c_1 h(x) + c_2$ on $D$ for real constants $c_1, c_2$ with $c_1 > 0$, then $g(x)$ is of type $T$ with respect to $h(x)$ on $D$ (see Note 3 of Definition 1).

LEMMA 11. *Let the function $g(x)$ be of type $T_2$ with respect to $h(x)$ on the open interval $D$, finite or infinite. If $E \subseteq D$ is a Borel set, then $g(E)$ is a $\mu$-measurable set, and if $\mu_h(E) > 0$, then $\mu(g(E)) > 0$.*

Proof. Since $g(x)$ is of type $T_2$ with respect to $h(x)$ on $D$; then $D = \bigcup \{I_k : k = 1, \ldots, m\}$ is a union of intervals in whose interior $g'(x)$ and $g'(x)/h'(x)$ are continuous and non-zero. Thus in each interval $I_k$ the function $g(x)$ is strictly monotonic and continuous. Since $E \cap I_k^0$

is a Borel set; then it follows from Lemma 2 that $g(E \cap I_k^0)$ is $\mu$-measurable for $k = 1, \ldots, m$. Thus $g(E)$ as a union of finitely many $\mu$-measurable sets is $\mu$-measurable.

Since $\sum_{k=1}^{m} \mu_h(E \cap I_k^0) = \mu_h(E) > 0$; then $\mu_h(E \cap I_k^0) > 0$ for some $k$. Thus there exists a closed interval $I \subseteq I_k^0$ such that $\mu_h(E \cap I) > 0$. Since $g'(x)/h'(x)$ is continuous and non-zero on $I$; then there exists some $\gamma > 0$ such that $|g'(x)/h'(x)| \geqslant \gamma$ on $I$. Therefore it now follows from Lemmas 4 and 6 that $\mu\big(g(E)\big) \geqslant \mu\big(g(E \cap I)\big) \geqslant \gamma \mu_h(E \cap I) > 0$ (see Notes 1 and 5 of Lemma 6).

LEMMA 12. *Let the function $g(x)$ be of type $T$ with respect to $h(x)$ on the open interval $D$, finite or infinite. If $E \subseteq D$ is a Borel set and $\mu_h(E) > 0$, then $\mu\big(g(E)\big) > 0$.*

Proof. Follows directly from Lemmas 2, 10 and 11 (see also Note 1 of Lemma 10).

THEOREM 13. *Let $f(x)$ be continuous, $p$ be a fixed real or complex number, and $\{t_n\}$ be a convergent sequence of real numbers with $t_0 = \lim_n t_n$. Let the function $g(x)$ be strictly monotonic and continuous on the open interval $D$, finite or infinite, with $g$ of type $T_1$ with respect to $h(x) = x$ on the interval $D$. Then a necessary condition that the sequence $\{f(\lambda_n \cdot g(x) + t_n)\}$ converges to $p$ on a subset of $D$ having positive $\mu$-measure for no bounded real sequence $\{\lambda_n\}$ is:*

$$\text{(ii)} \qquad \mu\big[\big(\{x - t_0 : f(x) = p\}\big) \cap \big(g(D)\big)\big] = 0.$$

Proof. First of all we note from Lemma 8 that $g^{-1}(x)$ is of type $T_1$ with respect to $h(x) = x$ on $g(D)$.

Let $E = \big[\big(\{x - t_0 : f(x) = p\}\big) \cap \big(g(D)\big)\big]$ and suppose that $\mu_h(E) = \mu(E) > 0$. Since sets of positive $\mu_h$-measure have Borel subsets of positive $\mu_h$-measure, there exists a Borel set $E_0 \subseteq E$ such that $\mu_h(E_0) > 0$. Let $\lambda_n = 1$ for $n = 1, 2, 3, \ldots$; then $\lim_n f(\lambda_n \cdot y + t_n) = f(y + t_0) = p$ on $E_0$. Since $E_0 \subseteq g(D)$ and $g^{-1}$ is of type $T_1$ with respect to $h(x)$ on $g(D)$; then it follows from Lemma 12 that $\mu\big(g^{-1}(E_0)\big) > 0$. Thus we now have $\lim_n f\big(\lambda_n g(x) + t_n\big) = p$ on $g^{-1}(E_0) \subseteq D$ with $\mu\big(g^{-1}(E_0)\big) > 0$. Therefore (ii) is necessary.

Note. If $t_n = t_0$ for $n = 1, 2, 3, \ldots$ the hypothesis $f(x)$ is continuous may be replaced by $f(x)$ is a Baire function. For continuity is no longer required, and this is enough to ensure that $E$ is a $\mu_h$-measurable set and $f\big(\lambda_n g(x) + t_n\big)$ is a $\mu_h$-measurable function.

THEOREM 14. *Let $f(x)$ be continuous, $p$ be a fixed real or complex number, $\{t_n\}$ be any bounded sequence of real numbers, and $S$ be the set of all limit* *doints of* $\{t_n\}$. *Let* $D$ *be an open interval, finite or infinite, and* $g(x)$ *be of type T with respect to* $h(x)$ *on* $D$. *Then sufficient conditions that the sequnce* $\{f(\lambda_n \cdot g(x)+t_n)\}$ *converges to* $p$ *on a subset of* $D$ *having positive* $\mu_h$*-measure for no bounded real seuqence* $\{\lambda_n\}$ *are*:

$$\text{(iii)} \qquad f(t) \neq p \quad \textit{whenever} \quad t \in S;$$

$$\text{(iv)} \qquad \mu(\{x : f(x) = p\}) = 0.$$

Proof. Suppose that there exists a subset $E$ of $D$ having positive $\mu_h$-measure and a bounded sequence $\{\lambda_n\}$ such that $\lim_n f(\lambda_n \cdot g(x)+t_n) = p$ on $E$, then since sets of positive $\mu_h$-measure have bounded Borel subsets of positive $\mu_h$-measure, we may assume that $E$ is a bounded Borel set. Thus it now follows from Lemma 12 that $\mu(g(E)) > 0$. For simplicity of notation we denote $g(E)$ by $E$ so that we now have $\lim_n f(\lambda_n x+t_n) = p$ on $E$, $E \subseteq g(D)$, and $\mu(E) > 0$. Simultaneously suppose that both (iii) and (iv) are true.

First let us assume that $\{\lambda_n\}$ has zero as a limit point; then there exists a subsequence $\{n_k\}$ such that $\lim_k \lambda_{n_k} = 0$ and $\lim_k t_{n_k} = t_0$ for some $t_0 \in S$. Let $x$ be any point in $E$; then $\lim_k f(\lambda_{n_k} x+t_{n_k}) = p$. But since $f$ is continuous $\lim_k f(\lambda_{n_k} x+t_{n_k}) = f(t_0)$. Thus $f(t_0) = p$, and from this contradiction of (iii) it follows that $\{\lambda_n\}$ does not have zero as a limit point.

Now assume that $\{\lambda_n\}$ has a finite non-zero limit $\lambda$, then there exists a subsequence $\{n_k\}$ such that $\lim_k \lambda_{n_k} = \lambda$ and $\lim_k t_{n_k} = t_0$ for some $t_0 \in S$. Let $x$ be any point in $E$; then $\lim_k f(\lambda_{n_k} \cdot x+t_{n_k}) = p$. But since $f$ is continuous $\lim_k f(\lambda_{n_k} \cdot x+t_{n_k}) = f(\lambda x+t_0)$. Thus $f(\lambda x+t_0) = p$ on $E$, so that $\lambda \cdot E+t_0 \subseteq \{x : f(x) = p\}$, and consequently $\mu(\{x : f(x) = p\}) > 0$. From this contradiction of (iv) the theorem now follows.

THEOREM 15. *Let* $f(x)$ *be continuous*, $p$ *be a fixed real or complex number*, $\{t_n\}$ *be a convergent sequence of real numbers, and* $t_0 = \lim_n t_n$. *Let the function* $g(x)$ *be strictly monotonic and continuous on the open interval* $D$, *finite or infinite, with* $g(D) = R_1$ *and* $g$ *be of type* $T_1$ *with respect to* $h(x) = x$ *on* $D$. *Then necessary and sufficient conditions that the sequence* $\{f(\lambda_n \cdot g(x)+ + t_n)\}$ *converges to* $p$ *on a subset of* $D$ *having positive* $\mu$*-measure for no bounded real sequence* $\{\lambda_n\}$ *are*: (i) *and* (iv).

Proof. Since $g(D) = R_1$ and $h(x) = x$, this follows from Theorems, 1, 13, and 14.

Note. Theorem 4 of [1] follows from Theorem 15 on letting $g(x) = x$ and $D = (-\infty, \infty)$.

DEFINITION 4. The $\mu_h$-measurable function $f(x)$ is said to have the *property* $P_h(\theta, \delta, l, r)$ with respect to the real or complex number $p$, if corresponding to every $\theta > 0$ there exist positive numbers $\delta, l, r$ $(r < 1)$ such that for every interval $J = [a, b]$, $a \neq 0$, of length greater than $l$ and such that $(b-a)/|a| \geqslant \theta$ we have $\mu_h(J_{\delta,p}) \geqslant r\mu_h(J)$, where $J_{\delta,p} = \{x : x \in J, |f(x)-p| \geqslant \delta\}$. If $h(x) = x$ merely denote the property by $P(\theta, \delta, l, r)$.

DEFINITION 5. The $\mu_h$-measurable function $f(x)$ is said to have the *property* $Q_h(r, \theta, \delta, l)$ with respect to the real or complex number $p$, if there exists $r$ in $(0, 1)$ such that corresponding to every $\theta > 0$ there exist positive numbers $\delta, l$ such that for every interval $J = [a, b]$, $a \neq 0$, of length greater than $l$ such that $(b-a)/|a| \geqslant \theta$ we have $\mu_h(J_{\delta,p}) \geqslant r \cdot \mu_h(J)$. If $h(x) = x$ merely denote the property by $Q(r, \theta, \delta, l)$.

Note. Comparing Definitions 4 and 5, since $r$ is independent of $\theta$ in Definition 5 we see that if $f(x)$ has the property $Q_h(r, \theta, \delta, l)$ with respect to $p$, then $f(x)$ has the property $P_h(\theta, \delta, l, r)$ with respect to $p$.

DEFINITION 6. The $\mu_h$-measurable function $f(x)$ is said to have the *property* $P_h(\delta, l, r)$ with respect to the real or complex number $p$, if there exist positive numbers $\delta, l, r$ $(r < 1)$ such that for every interval $J$ of length greater than $l$ we have $\mu_h(J_{\delta,p}) \geqslant r \cdot \mu_h(J)$.

Note 1. Comparing Definitions 4, 5 and 6 if $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$, then it also has the properties $P_h(\theta, \delta, l, r)$ and $Q_h(r, \theta, \delta, l)$ with respect to $p$.

Note 2. Lemma 3 of [3] and Lemma 26 below are of interest at this point.

LEMMA 16. *Let the functions $g(x)$ and $h(x)$ be defined strictly increasing and right continuous on $R_1$. Let $f(x)$ be defined and both $\mu_g$ and $\mu_h$-measurable on $R_1$. If there exist constants $0 < \gamma \leqslant M < \infty$ such that*

$$\gamma|h(x_1)-h(x_2)| \leqslant |g(x_1)-g(x_2)| \leqslant M|h(x_1)-h(x_2)| \quad \text{for every } x_1, x_2 \in R_1,$$

*then $f(x)$ has the property $P_g(\delta, l, r)$ with respect to $p$ iff $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$.*

Proof. Suppose that $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$ and let $\delta, l, r$ be the $\delta, l, r$ of $P_h(\delta, l, r)$ for $f(x)$ with respect to $p$. Let $J$ be any interval of length greater than $l$; then $\mu_h(J_{\delta,p}) \geqslant r \cdot \mu_h(J)$. Let

$$\eta_J = \frac{\mu_g(J)}{\gamma\mu_h(J)} \quad \text{and} \quad r_J^* = 1-\eta_J^{-1}r;$$

then

$$1 \leqslant \eta_J \leqslant \frac{M}{\gamma}, \quad 1-r \leqslant r_J^* \leqslant 1-\frac{\gamma}{M}r,$$

and $\eta_J(1-r_J^*) = r$. Thus it now follows from Lemma 6 that $\mu_g(J_{\delta,p}) \geqslant (1-$ $-r_J^*)\cdot\mu_g(J)$ whenever $\mu_h(J_{\delta,p}) \geqslant \eta_J(1-r_J^*)\mu_h(J) = r\mu_h(J)$. Letting $r_J^{**}$ $= 1-r_J^* = \eta_J^{-1}\cdot r$ and $r^{**} = \gamma\cdot M^{-1}r$; then $r^{**} \leqslant r_J^{**} \leqslant r$ so that $\mu_g(J_{\delta,p})$ $\geqslant r_J^{**}\cdot\mu_g(J) \geqslant r^{**}\cdot\mu_g(J)$ whenever $\mu_h(J_{\delta,p}) \geqslant r\cdot\mu_h(J)$, that is whenever $J$ is an interval length greater than $l$. Therefore there exist positive numbers $\delta$, $l$, $r^{**}$ ($r^{**} < 1$) such that for every interval $J$ of length greater than $l$ we have $\mu_g(J_{\delta,p}) \geqslant r^{**}\cdot\mu_g(J)$, hence $f(x)$ has the property $P_g(\delta, l, r)$ with respect to $p$.

Since $M^{-1}|g(x_1)-g(x_2)| \leqslant |h(x_1)-h(x_2)| \leqslant \gamma^{-1}|g(x_1)-g(x_2)|$ for every $x_1, x_2 \in R$, the converse also follows in the same manner.

Note. Obviously we may replace $P_h(\delta, l, r)$ by either $Q_h(\theta, \delta, l, r)$ or $P_h(r, \theta, \delta, l)$ in the above lemma and its proof.

LEMMA 17. *Let the $\mu$-measurable function $f(x)$ be defined in $R_1$ and $p$ be any real or complex number. Let $\bar{a} = \underline{\lim}_{x\to\infty} f(x)$, $\bar{b} = \overline{\lim}_{x\to\infty} f(x)$, $\underline{a} = \underline{\lim}_{x\to-\infty} f(x)$, and $\underline{b} = \overline{\lim}_{x\to-\infty} f(x)$. Then if $p \notin \{[\underline{a}, \underline{b}] \cup [\bar{a}, \bar{b}]\}$ the function $f(x)$ has the property $P(\delta, l, r)$ with respect to $p$.*

Proof. Suppose that $p \notin \{[\underline{a}, \underline{b}] \cup [\bar{a}, \bar{b}]\}$ and let $\delta$ be one half of the distance from the point $p$ to the set $\{[\underline{a}, \underline{b}] \cup [\bar{a}, \bar{b}]\}$. Now choose $K$ large enough so that $f(x) \in \{[\underline{a}-\delta, \underline{b}+\delta] \cup [\bar{a}-\delta, \bar{b}+\delta]\}$ whenever $|x|$ $\geqslant K$, for then we have $|f(x)-p| \geqslant \delta$ whenever $|x| \geqslant K$. Now let $l = 4K$ and $r = \frac{1}{2}$; then if $J$ is an interval of length greater than $l$ we have $|f(x)-$ $-p| \geqslant \delta$ over at least half of the interval $J$, that is $\mu(J_{\delta,p}) \geqslant \frac{1}{2}\mu(J)$. Thus $f(x)$ has the property $P(\delta, l, r)$ with respect to $p$.

Note. Thus every $\mu$-measurable real valued function $f(x)$ defined on $R_1$ has the property $P(\delta, l, r)$ with respect to every non-real complex number $p$.

THEOREM 18. *Let $f(x)$ be a $\mu$-measurable function, $p$ be a fixed real or complex number, and $\mathscr{C}$ be the class of all pairs $(\{\lambda_n\}, \{t_n\})$ of sequences of real numbers with $\{\lambda_n\}$ unbounded. Let $g(x)$ be strictly monotonic and continuous on the open interval $D$, finite or infinite, with $g(x)$ of type $T_1$ with respect to $h(x) = x$ on $D$, and $D \subseteq g(D)$.*

*Then a necessary condition that the sequence $\{f(\lambda_n\cdot g(x)+t_n)\}$ converges to $p$ on a subset of $D$ having positive $\mu$-measure for no $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ is:*

(v) *$f(x)$ has the property $P(\theta, \delta, l, r)$ with respect to $p$.*

Proof. First of all we note from Lemma 8 that $g^{-1}(x)$ is of type $T_1$ with respect to $h(x) = x$ on $g(D)$.

Suppose that (v) is a necessary condition that the sequence $\{f(\lambda_n\cdot y+$ $+t_n)\}$ converges to $p$ on a subset of $D$ having positive $\mu$-measure for no $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$. Then if (v) is false there exists a $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ and a subset $E$

of $D$ having positive $\mu$-measure such that $\lim_n f(\lambda_n \cdot y + t_n) = p$ on $E$. But as a set of positive $\mu$-measure, $E$ has a bounded Borel subset $E_0$ of positive $\mu$-measure. Since $E_0 \subseteq D \subseteq g(D)$ and $g^{-1}(x)$ is of type $T_1$ with respect to $h(x)$ on $g(D)$; then it follows from Lemma 12 that $\mu(g^{-1}(E_0)) > 0$. Thus we now have $\lim_n f(\lambda_n \cdot g(x) + t_n) = p$ on $g^{-1}(E_0) \subseteq D$ with $\mu(g^{-1}(E_0)) > 0$. Therefore we have only to prove the assertion made in the first sentence of this paragraph.

Suppose that (v) is false. Then there exists a $\theta > 0$ such that corresponding to every set of positive numbers $\delta$, $l$, $r$ $(r < 1)$ there is an interval $J = [a, b]$, $a \neq 0$, having length greater than $l$ and $(b-a)/|a| \geqslant \theta$, for which $\mu(J_{\delta,p}) < r \cdot \mu(J)$.

Let $r_n = 10^{-n/3}$, $\{\delta_n\}$ be a strictly decreasing sequence of real numbers such that $\lim_n \delta_n = 0$, and $\{l_n\}$ be a strictly increasing sequence of real numbers such that $\lim_n l_n = \infty$ and $l_n > \operatorname{Sup}(r_n^{-1}+1, 2^{n+1}|t_n|)$ for $n = 1, 2, 3, \ldots$ Then for every $n$ there exists an interval $J_n = [X_n, Y_n]$, $X_n \neq 0$, having length greater than $l_n$, with $(Y_n - X_n)/|X_n| \geqslant \theta$, for which $\mu_h((J_n)_{\delta_n,p}) < r_n \cdot \mu_h(J_n)$. Consider the sequence $\{X_n\}$. There is a monotonic subsequence $\{X_{n_k}\}$ which tends to a limit $X$ which is either finite or $\pm\infty$. If $X$ is finite we may choose $\{X_{n_k}\}$ so that for $k = 1, 2, 3, \ldots$ we have $|X_{n_k} - X| < 2^{-k}$ and $|X_{n_k} - X| < 2^{-k}|X|$ if $X \neq 0$. If $\lim_n |Y_n/X_n| = \infty$ and $\lim_n |Y_n/X| = \infty$, which certainly occurs if $X$ is finite and non-zero, then we may further require that $|Y_{n_k}/X_{n_k}| > 2^k$ and $|Y_{n_k}/X| > 2^k$ for $k = 1, 2, 3, \ldots$ For simplicity we denote the subsequence by $\{X_n\}$, then if $X$ is finite and non-zero we have $|X_n - X| < 2^{-n} \operatorname{Inf}\{|X|, 1\}$, $|Y_n/X_n| > 2^n$, and $|Y_n/X| > 2^n$; and if $X = 0$ we have $|X_n - X| < 2^{-n}$; all for $n = 1, 2, 3, \ldots$

Suppose that $X$ is finite. Let $J'_n = [X_n - \varepsilon_n, Y_n]$, then $\mu((J_n)_{\delta_n,p}) < r_n \mu(J_n) + |\varepsilon_n|$. Then letting $\varepsilon_n = X_n - X$ we have $|\varepsilon_n| < 2^{-n}$ and

$$r((J'_n)_{\delta n,p}) < r_n \cdot \mu(J_n) + |\varepsilon_n| \leqslant \left[r_n + \frac{|\varepsilon_n|(r_n+1)}{\mu(J'_n)}\right] \cdot \mu(J'_n)$$

$$< \left[r_n + \frac{2}{l_n - 1}\right] \mu(J'_n)_{\delta_n,p} < 3r_n \cdot \mu(J'_n) \quad \text{for } n = 1, 2, 3, \ldots$$

Note here that $J'_n = [X, Y_n]$.

If $X$ is finite, then since $\lim_n l_n = \infty$ there exists an integer $N_1$ such that $l_n - 2 > 2|X|$ whenever $n \geqslant N_1$. Thus

$$|Y_n| = |(Y_n - X_n) + X_n| > |Y_n - X_n| - |X_n| \geqslant l_n - |(X_n - X) + X|$$
$$\geqslant l_n - |X| - |X_n - X| \geqslant l_n - |X| - 1 \geqslant l_n/2$$

whenever $n \geqslant N_1$. But $l_n \geqslant 2^{n+1}|t_n|$ for $n = 1, 2, 3, \ldots$, hence $|Y_n| \geqslant 2^n|t_n|$ for $n \geqslant N_1$.

If $X = +\infty$, then there exists an integer $N_1$ such that $X_n > 0$ whenever $n \geqslant N_1$, so that $Y_n = (Y_n - X_n) + X_n > l_n \geqslant 2^{n+1}|t_n|$ for $n \geqslant N_1$.

Suppose first that $X = 0$. Let $N_2 = \mathrm{Sup}(N_1, 3)$; then $|t_n/Y_n| < 2^{-n}$, for $n \geqslant N_2$. Let

$$A_n = \left\{x : x \in \left[-\frac{t_n}{Y_n}, 1 - \frac{t_n}{Y_n}\right],\quad |f(Y_n \cdot X + t_n) - p| < \delta_n\right\}$$

$$\text{for } n = 1, 2, 3, \ldots$$

and $A = \bigcap_{n=N_2}^{\infty} A_n$; then $\mu(A_n) > 1 - 3r_n$ for $n = 1, 2, 3, \ldots$ Since

$$\mu([0,1]-A)$$

$$\leqslant \sum_{n=N_2}^{\infty} \mu([0,1]-A_n) \leqslant \sum_{n=N_2}^{\infty} \{1-\mu(A_n)\} + \sum_{n=N_2}^{\infty} \mu(A_n - A_n \cap [0,1])$$

$$\leqslant \sum_{n=N_2}^{\infty} 3r_n + \sum_{n=N_2}^{\infty} \left|\frac{t_n}{Y_n}\right| < \sum_{n=N_2}^{\infty} (10^{-n} + 2^{-n}) \leqslant \sum_{n=3}^{\infty} (10^{-n} + 2^{-n}) < \tfrac{1}{2};$$

then $\mu(A) > \frac{1}{2}$. Furthermore $|f(Y_n \cdot X + t_n) - p| < \delta_n$ on $A$ for $n \geqslant N_2$, hence $\lim_n f(Y_n X + t_n) = p$ on $A$.

Now suppose that $X > 0$ is finite, and let $\sigma_n = Y_n/X_n$; then

$$\left|\frac{t_n}{\sigma_n}\right| = \left|\frac{t_n}{Y_n}\right| |X_n| < 2^{-n}(X+1) \qquad \text{for } n \geqslant N_1.$$

Let

$$A_n = \left\{x : x \in \left[\frac{X - t_n}{\sigma_n},\ X_n - \frac{t_n}{\sigma_n}\right], |f(\sigma_n \cdot x + t_n) - p| < \delta_n\right\}$$

$$\text{for } n = 1, 2, 3, \ldots$$

Choose $N$ so that $(6X+2)2^{-N+1} < \frac{1}{2}X$, and let $N_2 = \mathrm{Sup}(N_1, N)$; then

$$\mu(A_n) > (1 - 3r_n)\left[\frac{\sigma_n X_n - X}{\sigma_n}\right] \qquad \text{for } n \geqslant N_2.$$

Let $A = \bigcap_{n=N_2}^{\infty} A_n$. Since

$$\mu([0,\ X]-A)\leqslant \sum_{n=N_2}^{\infty}\mu([0,X]-A_n) = \sum_{n=N_2}^{\infty}(X-\mu(A_n))+$$

$$+\sum_{n=N_2}^{\infty}\mu(A_n-A_n\cap[0,X])\leqslant \sum_{n=N_2}^{\infty}\left(|X-X_n|+\left|\left(\frac{1-3r_n}{\sigma_n}\right)X\right|+|3r_nX_n|\right)+$$

$$+\sum_{n=N_2}^{\infty}\left(\left|\frac{X}{\sigma_n}\right|+|X_n-X|+\left|\frac{t_n}{\sigma_n}\right|\right)$$

$$\leqslant \sum_{n=N_2}^{\infty}\left(2|X-X_n|+2\left|\frac{X}{\sigma_n}\right|+\left|\frac{t_n}{\sigma_n}\right|+\frac{|X|+1}{10^n}\right)$$

$$\leqslant \sum_{n=N_2}^{\infty}[5\cdot 2^{-n}X+10^{-n}(X+1)+2^{-n}] < (6X+2)2^{-N+1} < \tfrac{1}{2}X,$$

then $\mu(A) > X/2$. Futhermore $|f(\sigma_n X+t_n)-p| < \delta_n$ on $A$ for $n\geqslant N_2$, hence $\lim_n f(\sigma_n\cdot X+t_n) = p$ on $A$.

Next suppose that $X<0$ is finite, and let $\sigma_n = Y_n/X$; then $|t_n/\sigma_n| = |t_n/Y_n||X| < 2^{-n}|X|$ for $n\geqslant N_1$. Let $N_2 = \min(N_1, 5)$,

$$A_n = \left\{x\colon\ x\in\left[\frac{-X+t_n}{\sigma_n},\ -X+\frac{t_n}{\sigma_n}\right],\ |f(-\sigma_n x+t_n)-p|<\delta_n\right\}$$

for $n = 1,2,3,\ldots$, and $A = \bigcap_{n=N_2}^{\infty} A_n$; then $\mu(A_n)>(1-3r_n)(\sigma_n^{-1}/\sigma_n)(-X)$ for $n = 1,2,3,\ldots$ Since

$$\mu([0,\ -X]-A)\leqslant \sum_{n=N_2}^{\infty}\mu([0,\ -X]-A_n)\leqslant \sum_{n=N_2}^{\infty}[-X-\mu(A_n)]+$$

$$+\sum_{n=N_2}^{\infty}\mu(A_n-A_n\cap[0,\ -X])$$

$$\leqslant \sum_{n=N_2}^{\infty}\left(2\left|\frac{X}{\sigma_n}\right|+3r_n|X|\right)+\sum_{n=N_2}^{\infty}\left(\left|\frac{X}{\sigma_n}\right|+\left|\frac{t_n}{\sigma_n}\right|\right)$$

$$\leqslant \sum_{n=5}^{\infty}(4\cdot 2^{-n}+10^{-n})|X| < \tfrac{1}{2}|X|;$$

then $\mu(A) > \frac{1}{2}|X|$.

Furthermore $|f(-\sigma_n x+t_n)-p| < \delta_n$ on $A$ for $n\geqslant N_2$, hence $\lim_n f(-\sigma_n\cdot x+t_n) = p$ on $A$.

If $X = +\infty$ there is no loss of generality in also assuming that $X_n > 0$ $(n = 1, 2, 3, \ldots)$, for otherwise merely delete the finite number of non-positive terms. Letting $\sigma_n = Y_n/X_n$ we have $\{\sigma_n\}$ as a sequence of numbers greater than one, hence there exists a monotonic subsequence $\{\sigma_{n_k}\}$ which converges either to a finite limit $b \geqslant 1$ or to $b = +\infty$. Furthermore since $\lim\limits_k \left(\dfrac{1}{\sigma_{n_k}} - \dfrac{1}{b}\right) = 0$ we may choose $\{\sigma_{n_k}\}$ so that $|\sigma_{n_k}^{-1} - b^{-1}| < 2^{-n}$ for $k = 1, 2, 3, \ldots$ which may be given an obvious interpretation when $b = +\infty$. For simplicity denote this subsequence by $\{\sigma_n\}$, then we have $|\sigma_n^{-1} - b^{-1}| < 2^{-n}$ for $n = 1, 2, 3, \ldots$, with $b^{-1}$ replaced by zero in case $b = +\infty$. Finally notice that since our intervals $J_n = [X_n, Y_n]$ are such that $(Y_n - X_n)/X_n \geqslant 0$, then $\sigma_n \geqslant 1 + \theta$ for $n = 1, 2, 3, \ldots$ and consequently $b > 1$.

Note that the sequences chosen above are subsequences of those chosen earlier in this proof, so that the pertinent properties obtained there are still in force.

Suppose that $X = +\infty$ and $b > 1$ is finite. Let

$$A_n = \left\{x\colon\ x \in \left[\frac{1}{\sigma_n} - \frac{t_n}{Y_n}, 1 - \frac{t_n}{Y_n}\right], |f(Y_n x + t_n) - p| < \delta_n\right\}$$

$$\text{for } n = 1, 2, 3, \ldots,$$

then $\mu(A_n) > (1 - r_n)(1 - \sigma_n^{-1})$ for $n = 1, 2, 3, \ldots$ Choose $N$ such that $2^{-3N+3} < (1 - b^{-1})/2$, let $N_2 = \operatorname{Sup}(N, N_1, 1)$, and $A = \bigcap\limits_{n=N_2}^{\infty} A_n$. Since

$$\mu([b^{-1}, 1] - A) \leqslant \sum_{n=N_2}^{\infty} \mu([b^{-1}, 1] - A_n) \leqslant \sum_{n=N_2}^{\infty} [(1 - b^{-1}) - \mu(A_n)] +$$

$$+ \sum_{n=N_2}^{\infty} \mu(A_n - A_n \cap [b^{-1}, 1]) \leqslant \sum_{n=N_2}^{\infty} (|\sigma_n^{-1} - b^{-1}| + r_n |1 - \sigma_n^{-1}|) +$$

$$+ \sum_{n=N_2}^{\infty} (|\sigma_n^{-1} - b^{-1}| + |t_n Y_n^{-1}|) < 3 \sum_{n=N_2}^{\infty} 2^{-n} +$$

$$+ \sum_{n=N_2}^{\infty} r_n (|1 - b^{-1}| + |b^{-1} - \sigma_n^{-1}|) \leqslant 4 \sum_{n=N_2}^{\infty} 2^{-n} +$$

$$+ (1 - b^{-1}) \sum_{n=1}^{\infty} r_n \leqslant 2^{-n+3} + \tfrac{1}{4}(1 - b^{-1}) < \tfrac{3}{4}(1 - b^{-1}).$$

Then $\mu(A) > (1-b^{-1})/4$. Furthermore $|f(Y_n\cdot x+t_n)-p| < \delta_n$ on $A$ for $n \geqslant N_2$, hence $\lim\limits_n f(Y_n x+t_n) = p$ on $A$.

Now suppose that $X = +\infty$ and $b = +\infty$. Let

$$A_n = \left\{x\colon x\in\left[\frac{1}{\sigma_n}-\frac{t_n}{Y_n}, 1-\frac{t_n}{Y_n}\right] |f(Y_n\cdot x+t_n)-p| < \delta_n\right\}$$

for $n = 1, 2, 3, \ldots$

and $A = \bigcap\limits_{n=4}^{\infty} A_n$. Then $\mu(A_n) \geqslant (1-r_n)\cdot(1-\sigma_n^{-1})$ for $n = 1, 2, 3, \ldots$ Since

$$\mu([0,1]-A) \leqslant \sum_{n=4}^{\infty}\mu([0,1]-A_n) \leqslant \sum_{n=4}^{\infty}[1-\mu(A_n)]+ + \sum_{n=4}^{\infty}\mu(A_n-A_n\cap[0,1])$$

$$\leqslant \sum_{n=4}^{\infty}\left(\frac{1+r_n}{\sigma_n}+r_n\right)+\sum_{n=4}^{\infty}\left(\frac{1}{\sigma_n}+\left|\frac{t_n}{Y_n}\right|\right) < \sum_{n=4}^{\infty}(3\cdot 2^{-n}+10^{-n}) < \tfrac{1}{2};$$

then $\mu(A) > \frac{1}{2}$. Furthermore $|f(Y_n x+t_n)-p| < \sigma_n$ on $A$ for $n \geqslant 4$, hence $\lim\limits_n f(Y_n x+t_n) = p$ on $A$.

If $X = -\infty$ there is no loss of generality in assuming that $X_n < 0$ $(n = 1, 2, 3, \ldots)$, for otherwise merely delete the finite number of non-negative terms. Let $\sigma_n = Y_n/X_n$ $(n = 1, 2, 3, \ldots)$; then from $(Y_n - X_n)/|X_n| \geqslant \theta$ $(n = 1, 2, 3, \ldots)$ it follows that $|\sigma_n-1| \geqslant \theta$ for $n = 1, 2, 3, \ldots$ Now choose a monotonic subsequence $\{\sigma_{n_k}\}$ which converges either to a finite limit $b$ such that $|b-1| \geqslant \theta$ or to $b = -\infty$. Since $\lim\limits_n(\sigma_n-b) = 0$ if $b$ is finite, and $\lim\limits_n\big(1/(\sigma_n-1)\big) = 0$ if $b = -\infty$; then we may choose a subsequence $\{\sigma_{n_k}\}$ so that $|\sigma_{n_k}-b| < 2^{-k}(b-1)^2$ and $|\sigma_{n_k}-1| > |b-1|/2$ for $k = 1, 2, 3, \ldots$ if $b$ finite, and $|(\sigma_{n_k}-1)^{-1}| < 2^{-k}$ $(k = 1, 2, 3, \ldots)$ if $b = -\infty$. For simplicity denote these subsequences by $\{\sigma_n\}$, then we have $|\sigma_n-b| < 2^{-n}(b-1)^2$ and $|\sigma_n-1| > |b-1|/2$ for $n = 1, 2, 3, \ldots$ if $b$ finite and $|(\sigma_n-1)^{-1}| < 2^{-n}$ $(n = 1, 2, 3, \ldots)$ if $b = -\infty$.

Note that the sequence chosen above are subsequences of those chosen earlier in the proof, so that the pertinent properties obtained there are still in force.

Now suppose that $X = -\infty$ and $b$ is finite. Let

$$A_n = \left\{x\colon\ x\in\left[\frac{X_n-t_n}{Y_n-X_n}, \frac{Y_n-t_n}{Y_n-X_n}\right] \big|f\big((Y_n-X_n)x+t_n\big)-p\big| < \delta_n\right\}$$

for $n = 1, 2, 3, \ldots$

and $A = \bigcap_{n=4}^{\infty} A_n$; then $\mu(A_n) > 1 - r_n$ for $n \geqslant 4$.

Since

$$\mu\big([(b-1)^{-1}, (1-b^{-1})^{-1}] - A\big) \leqslant \sum_{n=4}^{\infty} \mu\big([(b-1)^{-1}, (1-b^{-1})^{-1}] - A_n\big)$$

$$\leqslant \sum_{n=4}^{\infty} \Big\{[(1-b^{-1})^{-1} - (b-1)^{-1}] - \mu(A_n)\Big\} + \sum_{n=4}^{\infty} \mu\big(A_n - A_n \cap [(1-b^{-1})^{-1},$$

$$(b-1)^{-1}]\big) \leqslant \sum_{n=4}^{\infty} r_n + \sum_{n=4}^{\infty} \left(\frac{4}{(b-1)^2}\,|b-\sigma_n| + \frac{|t_n|}{Y_n - X_n}\right)$$

$$\leqslant \sum_{n=4}^{\infty} \left(\tfrac{1}{3}\cdot 10^{-n} + 4\cdot 2^{-n} + |t_n/l_n|\right) < \sum_{n=4}^{\infty} (5\cdot 2^{-n} + 2^{-(n+1)})$$

$$< \sum_{n=4}^{\infty} 6\cdot 2^{-n} = 3/4;$$

then $\mu(A) > \frac{1}{4}$. Furthermore $\big|f\big((Y_n - X_n)x + t_n\big) - p\big| < \delta_n$ on $A$ for $n \geqslant 4$, hence $\lim f[(Y_n - X_n)x + t_n] = p$ on $A$.

Finally suppose that $X = -\infty$ and $b = -\infty$. Let

$$A_n = \left\{x\colon x \in \left[\frac{X_n - t_n}{Y_n - X_n}, \frac{Y_n - t_n}{Y_n - X_n}\right], \big|f\big((Y_n - X_n)x + t_n\big) - p\big| < \delta_n\right\}$$

$$\text{for } n = 1, 2, 3, \ldots$$

and $A = \bigcap_{n=4}^{\infty} A_n$; then $\mu(A_n) > 1 - r_n$ for $n \geqslant 4$. Since

$$\mu([0,1] - A_n) \leqslant \sum_{n=4}^{\infty} \mu([0,1] - A_n) \leqslant \sum_{n=4}^{\infty} [1 - \mu(A_n)] + \sum_{n=4}^{\infty} \mu(A_n - A_n \cap [0,1])$$

$$\leqslant \sum_{n=4}^{\infty} r_n + \sum_{n=4}^{\infty} \left[2(\sigma_n - 1)^{-1} + \frac{|t_n|}{Y_n - X_n}\right] \leqslant \sum_{n=4}^{\infty} \left(\frac{10^{-n}}{3} + 2\cdot 2^{-n} + \frac{|t_n|}{l_n}\right)$$

$$< \sum_{n=4}^{\infty} 4\cdot 2^{-n} = \tfrac{1}{2},$$

then $\mu(A) > \frac{1}{2}$. Furthermore $\big|f\big((Y_n - X_n)x + t_n\big) - p\big| < \delta_n$ on $A$ for $n \geqslant 4$, hence $\lim_n f\big((Y_n - X_n)x + t_n\big) = p$ on $A$.

Thus if (iii) is false there exists, in each case, an unbounded real sequence $\{\lambda_n\}$ and a set $A$ of positive $\mu$-measure on which $\lim_n f(\lambda_n x + t_n) = p$. Therefore (iii) is necessary.

Note 1. Theorem 1 of [3] follows directly from Theorem 18, Theorem 1, and Theorem 13 on letting $g(x) = h(x) = x$ and $D = (-\infty, \infty)$.

LEMMA 19. *Let the function $h(x)$ be strictly increasing and continuous on $R_1$ and there exist a constant $\gamma > 0$ such that $|h(x_1) - h(x_2)| \geqslant \gamma |x_1 - x_2|$ for every $x_1, x_2 \in R_1$. If $f(x)$ is a Baire function and $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$, then $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$.*

Proof. If $I$ is an interval of length $\lambda$, then from the inequality $|h(x_1) - h(x_2)| \geqslant \gamma |x_1 - x_2|$ for every $x_1, x_2 \in R_1$ it follows that $h(I)$ is an interval of length $\geqslant \gamma\lambda$.

Suppose that $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$ and let $\delta$, $l$, $r$ be of the $\delta$, $l$, $r$ of $P_{h^{-1}}(\delta, l, r)$ for $f(x)$ with respect to $p$. Let $J$ be of any interval of length $\geqslant \gamma^{-1} l$ and $J^* = h(J)$; then $J^*$ is an interval of length $\geqslant l$. Since $f(x)$ is a Baire function, $J^*_{\delta,p}$ is a Borel set, and it follows from Lemma 5 that $\mu(h^{-1}(J^*_{\delta,p})) = \mu_{h^{-1}}(J^*_{\delta,p}) \geqslant r \cdot \mu_{h^{-1}}(J^*) = r \cdot \mu(h^{-1}(J^*)) = r \cdot \mu(J)$. But $\{x : x \in J, |f(h(x)) - p| \geqslant \delta\} = h^{-1}(J^*_{\delta,p})$, hence $\mu(\{x : x \in J, |f(h(x)) - p| \geqslant \delta\}) \geqslant r \cdot \mu(J)$. Let $l^* = \lambda^{-1} l$, then for every interval $J$ of length $\geqslant l^*$ we have $\mu(\{x : x \in J, |f(h(x)) - p| \geqslant \delta\}) \geqslant r \cdot \mu(J)$, hence the function $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$.

LEMMA 20. *Let the function $h(x)$ be strictly increasing and continuous on $R_1$ and there exists a constant $M > 0$ such that $|h(x_1) - h(x_2)| \leqslant M |x_1 - x_2|$ for every $x_1, x_2 \in R_1$. If $f(x)$ is a Baire function and $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$, then $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$.*

Proof. If $I$ is an interval of length $\lambda$, then from the inequality, $|h^{-1}(x_1) - h^{-1}(x_2)| \geqslant M^{-1} |x_1 - x_2|$ for every $x_1, x_2 \in R_1$ it follows that $h^{-1}(I)$ is an interval of length $\geqslant M^{-1}\lambda$.

Suppose that $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$ and let $\delta$, $l$, $r$ be the $\delta$, $l$, $r$ of $P(\delta, l, r)$ for $f(h(x))$ with respect to $p$. Let $J$ be any interval of length $\geqslant Ml$ and $J^* = h^{-1}(J)$, then $J^*$ is an interval of length $\geqslant l$. Since $f(x)$ is a Baire function, $J^*_{\delta,p}$ is a Borel set, and it follows from Lemma 5 that $\mu_{h^{-1}}(J_{\delta,p}) = \mu(h^{-1}(J_{\delta,p}))$. But $\{x : x \in J^*, |f(h(x)) - p| \geqslant \delta\} = h^{-1}(J_{\delta,p})$, hence $\mu_{h^{-1}}(J_{\delta,p}) = \mu(\{x : x \in J^*, |f(h(x)) - p| \geqslant \delta\}) \geqslant r \cdot \mu(J^*) = r \cdot \mu(h^{-1}(J)) = r \cdot \mu_{h^{-1}}(J)$. Let $l^* = Ml$; then for every interval $J$ of length $\geqslant l^*$ we have $\mu_{h^{-1}}(J_{\delta,p}) \geqslant r \cdot \mu_{h^{-1}}(J)$, hence the function $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$.

LEMMA 21. *Let the function $h(x)$ be strictly increasing and continuous on $R_1$ and there exist constants $0 < \gamma \leqslant M < \infty$ such that $\gamma |x_1 - x_2| \leqslant |h(x_1) - h(x_2)| \leqslant M |x_1 - x_2|$ for every $x_1, x_2 \in R_1$. If $f(x)$ is a Baire function, then $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$, iff $f(x)$ has the* *property* $P_h(\delta, l, r)$ *with respect to* $p$ *(and also iff* $f(x)$ *has the property* $P_{h^{-1}}(\delta, l, r)$ *with respect to* $p$*).*

Proof. From Lemmas 19 and 20 it follows that $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$ iff $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$. But since $\gamma^2|h^{-1}(x_1)-h^{-1}(x_2)| \leqslant |h(x_1)-h(x_2)| \leqslant M^2|h^{-1}(x_1)-h^{-1}(x_2)|$ for every $x_1, x_2 \epsilon R_1$; then it follows from Lemma 16 that $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$ iff $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$.

LEMMA 22. *Let the function* $h(x)$ *be strictly increasing and continuous on* $R_1$, *and there exists a constant* $\gamma > 0$ *such that* $|h(x_1)-h(x_2)| \geqslant \gamma|x_1-x_2|$ *for every* $x_1, x_2 \epsilon R_1$. *If* $f(x)$ *is a Baire function and* $f(h^{-1}(x))$ *has the property* $P(\delta, l, r)$ *with respect to* $p$, *then* $f(x)$ *has the property* $P_h(\delta, l, r)$ *with respect to* $p$.

Proof. Let $g(x) = h^{-1}(x)$; then $g(x)$ is strictly increasing and continuous on $R_1$, and $|g(x_1)-g(x_2)| \leqslant \gamma^{-1}|x_1-x_2|$ for every $x_1, x_2 \epsilon R_1$, thus the result now follows from Lemma 20.

LEMMA 23. *Let the function* $h(x)$ *be strictly increasing and continuous on* $R_1$ *and there exists a constant* $M > 0$ *such that* $|h(x_1)-h(x_2)| \leqslant M|x_1-x_2|$ *for every* $x_1, x_2 \epsilon R_1$. *If* $f(x)$ *is a Baire function and* $f(x)$ *has the property* $P_h(\delta, l, r)$ *with respect to* $p$, *then* $f(h^{-1}(x))$ *has the property* $P(\delta, l, r)$ *with respect to* $p$.

Proof. Let $g(x) = h^{-1}(x)$; then $g(x)$ is strictly increasing and continuous on $R_1$ and $|g(x_1)-g(x_2)| \geqslant M^{-1}|x_1-x_2|$ for every $x_1, x_2 \epsilon R_1$. The result now follows from Lemma 19.

LEMMA 24. *Let the function* $h(x)$ *be strictly increasing and continuous on* $R_1$ *and there exist constants* $0 < \gamma \leqslant M < \infty$ *such that* $\gamma|x_1-x_2| \leqslant |h(x_1)-h(x_2)| \leqslant M|x_1-x_2|$ *for every* $x_1, x_2 \epsilon R_1$. *If* $f(x)$ *is a Baire function, then* $f(h^{-1}(x))$ *has the property* $P(\delta, l, r)$ *with respect to* $p$, *iff* $f(x)$ *has the property* $P_{h^{-1}}(\delta, l, r)$ *with respect to* $p$ *(and also iff* $f(x)$ *has the property* $P_h(\delta, l, r)$ *with respect to* $p$*).*

Proof. From Lemmas 22 and 23 it follows that $f(h^{-1}(x))$ has the property $P(\delta, l, r)$ with respect to $p$ iff $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$. But since $\gamma^2|h^{-1}(x_1)-h^{-1}(x_2)| \leqslant |h(x_1)-h(x_2)| \leqslant M^2|h^{-1}(x_1)-h^{-1}(x_2)|$ for every $x_1, x_2 \epsilon R_1$; then it follows from Lemma 16 that $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$, iff $f(x)$ has the property $P_{h^{-1}}(\delta, l, r)$ with respect to $p$.

LEMMA 25. *Let the function* $h(x)$ *be defined, strictly increasing, and continuous on* $R_1$. *Let* $f(x)$ *be a Baire function on* $R_1$. *If there exist constants* $0 < \gamma \leqslant M < \infty$ *such that* $\gamma|x_1-x_2| \leqslant |h(x_1)-h(x_2)| \leqslant M|x_1-x_2|$ *for every* $x_1, x_2 \epsilon R_1$, *then* $f(x)$ *has the property* $P(\delta, l, r)$ *with respect to* $p$, *iff* $f(h(x))$

*has the property $P(\delta, l, r)$ with respect to $p$ (and iff $f(h^{-1}(x))$ has the property $P(\delta, l, r)$ with respect to $p$).*

Proof. From Lemma 16 it follows that $f(x)$ has the property $P(\delta, l, r)$ with respect to $p$ iff $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$. But from Lemma 21 it follows that $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$ iff $f(h(x))$ has the property $P(\delta, l, r)$ with respect to $p$, and from Lemma 24 it follows that $f(x)$ has the property $P_h(\delta, l, r)$ with respect to $p$ iff $f(h^{-1}(x))$ has the property $P(\delta, l, r)$ with respect to $p$.

LEMMA 26. *Let the function $h(x)$ be defined, strictly increasing, and continuous on $R_1$, and there exist constants $0 < \gamma \leqslant M < \infty$ such that $\gamma |x_1 - x_2| \leqslant |h(x_1) - h(x_2)| \leqslant M |x_1 - x_2|$ for every $x_1, x_2 \in R_1$. Let $f(x)$ be a non-constant, continuous, almost periodic function. Then $f(x)$, $f(h(x))$, and $f(h^{-1}(x))$ all have the property $P(\delta, l, r)$ with respect to every real or complex $p$.*

Proof. This follows directly from Lemma 3 of [3] and Lemma 25 above.

THEOREM 27. *Let $f(x)$ be a Baire function, $p$ be any real or complex number, and $c$ be the class of all pairs $(\{\lambda_n\}, \{t_n\})$ of sequences of real numbers with $\{\lambda_n\}$ unbounded. Let $D$ be an open interval, finite or infinite, and $g(x)$ be of type $T$ with respect to (a continuous strictly increasing) $h(x)$ on $D$. Then a sufficient condition that the sequence $\{f(\lambda_n \cdot g(x) + t_n)\}$ converge to $p$ on a subset of $D$ having positive $\mu_h$-measure for no $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ is:*

(vi) *$f(x)$ has the property $P(\delta, l, r)$ with respect to $p$.*

Proof. Suppose that there exists a subset $E$ of $D$ having positive $\mu_h$-measure and $a(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ such that $\lim\limits_n f(\lambda_n \cdot g(x) + t_n) = p$ on $E$; then since sets of positive $\mu_h$-measure have bounded subsets of positive $\mu_h$-measure we may assume that $E$ is a bounded set. In fact we can arrange to have $E$ as a subset of a finite closed subinterval of $D$. Thus it now follows from Egoroff's Theorem that $\lim\limits_n f(\lambda_n \cdot g(x) + t_n) = p$ uniformly on a subset $E_0$ of $E$ having positive $\mu_h$-measure. Since every set of positive $\mu_h$-measure has a bounded Borel subset of positive $\mu_h$-measure; then we may assume that $E_0$ is a bounded Borel set. From Lemma 12 it now follows that $\mu(g(E_0)) > 0$. For simplicity of notation we denote $g(E_0)$ by $E$ so that we finally have $\lim\limits_n f(\lambda_n \cdot x + t_n) = p$ uniformly on $E \subseteq g(D)$ with $\mu(E) > 0$. Since $E$ is a set of positive $\mu$-measure it has a point of density $x_d$, and since almost every point of $E$ is a point of density; then we may arrange to have $x_d \neq 0$.

Let $r$ be the $r$ of $P(\delta, l, r)$ and $I_k = [x_d, x_d + k^{-1}]$ for $k = 1, 2, 3, \ldots$; then

$$\lim_k \frac{\mu(E \cap I_k)}{\mu(I_k)} = 1.$$

Now choose $K$ so that $\mu(E \cap I_K) > (1-r)\mu(I_K)$. Let $\delta, l$ be the $\delta, l$ of $P(\delta, l, r)$ and choose $N$ so that $|\lambda_N|\cdot K^{-1} > l$ and $|f(\lambda_N \cdot x + t_N) - p| < \delta$ on $E$. Let $J = \lambda_N \cdot I_K + t_N$; then $|f(y)-p| < \delta$ on $\lambda_N \cdot E + t_N$ and $\mu[(\lambda_N \cdot E + t_N) \cap J] = \lambda_N \cdot \mu(E \cap I_K) > (1-r)\mu(J)$ so that $\mu(J_{\delta,p}) < r\mu(J)$. But $J = [\lambda_N \cdot x_d + t_N, \lambda_N \cdot x_d + \lambda_N K^{-1} + t_N]$ is an interval of length $|\lambda_N|\cdot K^{-1} > l$, hence we have the desired contradiction of (vi).

Note 1. If the class $\mathscr{C}$ is further restricted by requiring that the sequence $\{\lambda_n^{-1}\cdot t_n\}$ converge to zero, then condition (vi) may be replaced by condition

(vii) $f(x)$ has the property $Q(r, \theta, \delta, l)$ with respect to $p$.

For then we have

$$\lim_n \left| \frac{\lambda_n K^{-1}}{\lambda_n x_d + t_n} \right| = \lim_n \frac{K^{-1}}{|x_d + \lambda_n^{-1}\cdot t_n|} = \frac{K^{-1}}{|x_d|} > 0.$$

Thus if we let $2\theta = K^{-1}|x_d|$ we may choose $N$ so that

$$\left| \frac{\lambda_N K^{-1}}{\lambda_N \cdot x_d + t_N} \right| > \theta.$$

See Theorem 2 of [3].

Note 2. If $g(x) = \gamma_1 x + \gamma_2$ for real constants $\gamma_1, \gamma_2$ with $\gamma_1 \neq 0$, and $h(x) = x$, then the result holds for any $\mu$-measurable function $f(x)$. For then $f(\lambda_n \cdot g(x) + t_n)$ is a $\mu$-measurable function without requiring that $f(x)$ be a Baire function.

Note 3. Theorem 2 of [3] follows directly from Note 1 of Theorems 27 and 14 on letting $h(x) = g(x) = x$ and $D = (-\infty, \infty)$.

Note 4. Note 2 of Definition 3 is pertinent here.

Note 5. If $g(x)$ is strictly monotonic and of type $T_1$ with respect to $h(x)$ on $D$, then the theorem holds without any continuity restriction on $g(x)$. This follows from Note 5 of Lemmas 6 and 7. Futhermore, if this $T_1$ condition is the type used in Lemma 6 we can weaken this continuity condition on $h(x)$ to right continuity since Lemma 6 requires no more.

THEOREM 28. *Let $f(x)$ be a Baire function, $p$ be any real or complex number, and $\mathscr{C}^*$ be the class of all pairs $(\{\lambda_n\}, \{t_n\})$ of sequences of real numbers with $\{\lambda_n\}$ having no finite points of accumulation. Let $D$ be an open interval, finite or infinite, and $g(x)$ be of the type $T$ with respect to (a continuous strictly increasing) $h(x)$ on $D$. If $f(x)$ has the property $P(\delta, l, r)$ with respect to $p = 0$, and $\lim_n a_n \cdot f(\lambda_n \cdot g(x) + t_n) = 0$ on a subset $E$ of $D$ having positive $\mu_h$-measure for some $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}^*$, then $\lim_n a_n = 0$.*

Proof. Suppose that $\lim_n a_n \cdot f(\lambda_n \cdot g(x) + t_n) = 0$ on a subset $E$ of $D$ having positive $\mu_h$-measure for some $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}^*$, and suppose that $\lim_n a_n \neq 0$. Then there exists a subsequence $\{n_j\}$ such that $a_{n_j} \neq 0$ for $j = 1, 2, 3, \ldots$ and either $\lim_j a_{n_j} = a \neq 0$ for some finite $a$ or $\lim_j |a_{n_j}| = \infty$. Thus we have $\lim_j a_{n_j}^{-1} = a^{-1}$ or zero respectively, and in either case $\lim_j f(\lambda_{n_j} \cdot g(x) + t_{n_j}) = \lim_j (a_{n_j}^{-1})(a_{n_j} f(\lambda_{n_j} \cdot g(x) + t_{n_j})) = 0$ on $E$. But since $(\{\lambda_{n_j}\}, \{t_{n_j}\})$ is in the class $\mathscr{C}$ of Theorem 27 it then follows from Theorem 27 that we have a contradiction, hence $\lim_n a_n = 0$.

Note 1. Obviously a similar theorem may be had in conjunction with Theorems 14, 31, 32 and 35.

Note 2. Note 5 of Theorem 27 applies here.

Definition 7. The $\mu_h$-measurable function $f(x)$ is said to have the *property* $P_h(\theta, \delta, l, r)$ uniformly with respect to the subset $S$ of the complex plane, if corresponding to every $\theta > 0$ there exist positive numbers, $\delta, l, r$ $(r < 1)$, independent of $p$, such that for every interval $J$ of length greater than $l$ and such that $(b-a)/|a| \geqslant \theta$ we have $\mu_h(J_{\delta,p}) \geqslant r\mu_h(J)$ for all $p$ in $S$.

If $h(x) = x$ merely denote the property by $P(\theta, \delta, l, r)$.

Theorem 29. *Let $f(x)$ be a $\mu$-measurable function, and $\mathscr{C}$ be the class of all pairs $(\{\lambda_n\}, \{t_n\})$ of sequences of real numbers with $\{\lambda_n\}$ unbounded. Let $g(x)$ be strictly monotonic and continuous on the open interval $D$, finite or infinite, with $g(x)$ of type $T_1$ with respect to $h(x) = x$ on the interval $D$, and $D \subseteq g(D)$. Then a necessary condition that the sequence $f(\lambda_n \cdot g(x) + t_n)$ converges to a finite limit function on a subset of $D$ having positive $\mu$-measure for no $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ is*

(viii) *$f(x)$ has the property $P(\theta, \delta, l, r)$ uniformly with respect to every bounded subset $S$ of the complex plane.*

Proof. First of all we note from Lemma 8 that $g^{-1}(x)$ is of type $T_1$ with respect to $h(x) = x$ on $g(D)$.

Suppose that (viii) is a necessary condition that the sequence $\{f(\lambda_n \cdot y + t_n)\}$ converges to a finite limit function on a subset of $D$ having positive $\mu_h$-measure for no $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$. Then if (viii) is false their exists a $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ and a subset $E$ of $D$ having positive $\mu_h$-measure, and a finite (real or complex) valued function $\Phi(y)$ such that $\lim_n f(\lambda_n \cdot y + t_n) = \Phi(y)$ on $E$. But as a set of positive $\mu_h$-measure $E$ has a bounded Borel subset $E_0$ of positive $\mu_h$-measure. Since $E_0 \subseteq D \subseteq g(D)$ and $g^{-1}$ is of type $T_1$ with respect to $h(x)$ on $g(D)$; then it follows from Lemma 12 that $\mu(g^{-1}(E_0))$

$> 0$. Thus we now have $\lim_n f(\lambda_n \cdot g(x) + t_n) = \Phi(g(x))$ on $g^{-1}(E_0) \subseteq D$ with $\mu(g^{-1}(E_0)) > 0$. Therefore we have only to prove the assertion made in the first sentence of this paragraph, recalling that $\mu_h = \mu$. Let $\{t_n\}$ be any fixed sequence of real numbers and proceed to alter the proof of Theorem 18 as outlined in Theorem 5 of [3].

Note 1. Lemma 5 of [3] is of interest here.

Note 2. Theorem 5 of [3] follows from Theorem 29 on letting $g(x) = x$ and $D = (-\infty, \infty)$.

THEOREM 30. *Let $f(x)$ be a Baire function and $\mathscr{C}$ be the class of all pairs $(\{\lambda_n\}, \{t_n\})$ of sequences of real numbers with $\{\lambda_n\}$ unbounded. Let $D$ be an open interval, finite or infinite, and $g(x)$ be of type $T$ with respect to (a continuous strictly increasing) $h(x)$ on $D$. Then a sufficient condition that sequence $\{f(\lambda_n \cdot g(x) + t_n)\}$ converges to a finite limit function on a subset of $D$ having positive $\mu_h$-measure for no $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ is*

(ix) *$f(x)$ has the property $P(\delta, l, r)$ with respect to every real or complex $p$.*

Proof. Suppose that there exists a subset $E$ of $D$ having positive $\mu_h$-measure, $a(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$, and a function $\chi(x)$, defined and finite valued on $E$, such that $\lim_n f(\lambda_n g(x) + t_n) = \chi(x)$ on $E$; then since sets of positive $\mu_h$-measure have bounded subsets of positive $\mu_h$-measure, we may assume that $E$ is a bounded set. In fact we can arrange to have $E$ as a subset of a finite closed subinterval of $D$. Thus it follows from Egoroff's Theorem that $\lim_n f(\lambda_n \cdot g(x) + t_n) = \chi(x)$ uniformly on a subset $E_0$ of $E$ having positive $\mu_h$-measure. Since every set of positive $\mu_h$-measure has a bounded Borel subset of positive $\mu_h$-measure; then we may assume that $E_0$ is a bounded Borel set. From Lemma 12 it now follows that, $\mu(g(E_0)) > 0$. For simplicity of notation we denote $g(E_0)$ by $E$ so that we finally have that the sequence $\{f(\lambda_n \cdot y + t_n)\}$ converges uniformly on $E$ to a finite valued limit function which we denote by $\Phi(y)$. Thus it now follows from Lusin's Theorem that there exists a closed subset $F$ of $E$ having positive $\mu$-measure such that $\Phi(y)$ is continuous on $F$. For simplicity of notation denote $F$ by $E$. Since $E$ is a set of positive $\mu$-measure it contains a point of density $y_d$, and since almost every point of $E$ is a point of density of $E$; then we may arrange to have $y_d \neq 0$.

Let $p = \Phi(y_d)$ and $\delta, l, r$ $(r < 1)$ be the $\delta, l, r$, of $P(\delta, l, r)$ for $f(x)$ with respect to $p$. Since $\Phi(y)$ is continuous on $E$, there exists $\eta(\delta) > 0$ such that $|\Phi(y) - p| < \delta/2$ whenever $y \in S = E \cap (y_d - \eta, y_d + \eta)$. Let $I_k = [y_d, y_d + k^{-1}]$ for $k = 1, 2, 3, \ldots$; then $\lim_k \dfrac{\mu(E \cap I_k)}{\mu(I_k)} = 1$. Now choose $K$ so that $K^{-1} < \eta$ and $\mu(E \cap I_K) > (1 - r)\mu(I_K)$. Choose $N$ so that $|\lambda_N| \cdot K^{-1} > l$ and $|f(\lambda_N \cdot y + t_N) - \Phi(y)| < \delta/2$ on $E$. Then $|f(\lambda_N \cdot y +$

$+t_N)-p| \leqslant |f(\lambda_N y+t_N)-\Phi(y)|+|\Phi(y)-p| < \delta$ on $S$. Let $J = \lambda_N\cdot I_K+$ $+t_N$; then $|f(Z)-p| < \delta$ on $\lambda_N\cdot S+t_N$ and $\mu[(\lambda_N\cdot S+t_N) \cap J] = \lambda_N\cdot\mu(S \cap$ $\cap I_K) = \lambda_N\cdot\mu(E \cap I_K) > (1-r)\mu(J)$, so that $\mu(J_{\delta,p}) < r\cdot\mu(J)$. But $J = [\lambda_N\cdot y_d+t_N, \lambda_N\cdot y_d+\lambda_N\cdot K^{-1}+t_N]$ is an interval of length $|\lambda_N|\cdot K^{-1}$ $> l$, hence we have the desired contradiction of (ix).

Note 1. If the class $\mathscr{C}$ is further restricted by requiring that the sequence $\{\lambda_n^{-1}\cdot t_n\}$ converge to zero, then condition (ix) may be replaced by condition:

(x) *$f(x)$ has the property $Q(r, \theta, \delta, l)$ with respect to every real or complex $p$.*

This follows exactly as in Note 1 of Theorem 27.

Note 2. Notes 2, 4 and 5 of Theorem 27 apply here also.

Note 3. Theorem 6 of [3] follows directly from Notes 1 and 2 of Theorem 30 on letting $g(x) = h(x) = x$ and $D = (-\infty, \infty)$.

THEOREM 31. *Let $f$ be a Baire function, $p$ be any real or complex number, $m$ be a positive integer, and $\mathscr{C}$ be the class of all $(m+1)$-tuples $(\{\lambda_{m,n}\}, \ldots$ $\ldots, \{\lambda_{0,n}\})$ of sequences of positive numbers, at least one of which is unbounded, and such that for some $\theta_0$ the sequences $\{\lambda_{m,n}/\lambda_{\theta_0,n}\}, \ldots, \{\lambda_{0,n}/\lambda_{\theta_0,n}\}$, are all bounded. Let $h(x)$ be strictly increasing, continuous, and of type $T_1$ with respect to $x$ on an open interval $D_1$ finite or infinite, differentiable with $h'(x) > 0$ except possibly on a set of $\mu$-measure zero in $D$; and $h'(x)$ either right continuous*[1] *or left continuous except possibly on a set of $\mu$-measure zero in $R_1$. Let $g_m(x), \ldots, g_0(x)$ be functions defined on $D$, differentiable in $D$, and satisfy the following conditions:*

(a) *$g_\theta'(x) > 0$ in $D$ for $\theta = 0, 1, \ldots, m$.*

(b) *$g_\theta'(x)$ is right (left) continuous whenever $h'(x)$ is right (left) continuous in $D$ for $\theta = 0, 1, \ldots, m$ (excepting, of course, sets of $\mu$-measure zero).*

*Then a sufficient condition that the sequence $\{f(\lambda_{m,n}\cdot g_m(x)+\ldots+$ $+\lambda_{0,n}\cdot g_0(x))\}$ does not converge to $p$ on a subset of $D$ having positive $\mu_h$-measure for any $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\})$ in $\mathscr{C}$ is:*

(vi) *$f(x)$ has the property $P(\delta, l, r)$ with respect to $p$.*

Proof. Suppose that there exists a subset $E$ of $D$ having positive $\mu_h$-measure and a $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\})$ in $\mathscr{C}$ such that

$$\lim_n f(\lambda_{m,n}g_m(x)+\ldots+\lambda_{0,n}g_0(x)) = p \quad \text{on } E;$$

then since sets of positive $\mu_h$-measure have bounded subsets of positive $\mu_h$-measure, we may assume that $E$ is a bounded set. In fact we can

[1] When we refer to the continuity of a derivative from the right (left) we tacitly assume that the derivative exists in an interval to the right (left) of the point in question.

arrange to have $E$ as a subset of a finite closed subinterval of $D$. Thus it now follows from Egoroff's Theorem that $\lim_n f\big(\lambda_{m,n}g_m(x)+\ldots+$ $+\lambda_{0,n}g_0(x)\big) = p$, uniformly on a subset $E_0$ of $E$ having positive $\mu_h$-measure. For simplicity denote $E_0$ by $E$. Since sets of positive $\mu_h$-measure have bounded Borel subsets of positive $\mu_h$-measure, we may assume that $E$ is a bounded Borel set. From Lemma 5 it follows that $\mu\big(h(E)\big)$ $= \mu_h(E) > 0$, and consequently the set $h(E)$ has a point of density $y_d$, let $x_d = h^{-1}(y_d)$. Since almost every point of $h(E)$ is a point of density of $h(E)$ it now follows from Lemmas 8 and 12 that we may arrange to have $x_d$ a point such that $h'(x_d) > 0$, $h'(x)$ is either right continuous or left continuous at $x_d$, $g_\theta(x_d) \neq 0$ for $\theta = 0, 1, \ldots, m$ and $g_0'(x)$ is either right continuous or left continuous at $x_d$ $(\theta = 0, 1, \ldots, m)$ with this of the same type as $h'(x)$ has there. Furthermore, by taking subsequences we may arrange to have the sequences $\{\lambda_{\theta,n}\}$ $(\theta = 0, 1, \ldots, m)$ tending to limits, at least one of which is infinite, and by a further choice of subsequences we may for some $\theta_0$ arrange to have $\lim_n |\lambda_{\theta,n}/\lambda_{\theta_0,n}| = \lambda_\theta$ finite for $\theta = 0, 1, \ldots, n$.

Let $\lambda = \operatorname{Max}\{\lambda_\theta : \theta = 0, 1, \ldots, m\}$ and note that $\lambda_{\theta_0} = 1$.

If $h'(x)$ is right continuous at $x_d$ let $J_k = [y_d, y_d+k^{-1}]$, but if $h'(x)$ is left continuous at $x_d$ let $J_k = [y_d-k^{-1}, y_d]$. For the sake of the argument let us assume the former. Then $h^{-1}$ being strictly increasing and continuous at $x_d$, we have $I_k = h^{-1}(J_k) = [x_d, x_k]$ with $x_d < x_{k+1} < x_k$ for

$$k = 1, 2, 3, \ldots,$$

and

$$\lim_k x_k = x_d.$$

Furthermore since $x_d \in D$, there exists an integer $K_0$ such that $I_k \subseteq D$ whenever $k \geqslant K_0$. Now choose an integer $N_0$ such that $\lambda_\theta/2 \leqslant |\lambda_{\theta,n}/\lambda_{\theta_0,n}| < 2\lambda$ for $\theta = 0, 1, \ldots, m$ and $n \geqslant N_0$.

Let $\varphi_n(x) = \lambda_{\theta_0 n}^{-1}\big(\lambda_{m,n}\cdot g_m(x)+\ldots+\lambda_{0,n}\cdot g_0(x)\big)$ for $n = 1, 2, 3, \ldots$; then $h'(x)$ and $\varphi_n'(x)$ being right continuous at $x_d$; we may choose $K_1 \geqslant K_0$, $\varepsilon_k$, and $\delta_k$ so that $h(x)$ and $\varphi_n(x)$ for $n = 1, 2, 3, \ldots$ are differentiable in $I_k$ with

$$|\varphi_n'(x)/h'(x)| \leqslant |\varepsilon_k\cdot\varphi_n'(x_d)/\delta_k h'(x_d)|$$

uniformly for $n = 1, 2, 3, \ldots$ and every $x \in I_k$ $(k \geqslant K_1)$, and where $\lim_k \varepsilon_k$ $= \lim_k \delta_k = 1$. Thus letting $M_{n,k} = \varepsilon_k\cdot\varphi_n'(x_d)/\delta_k\cdot h'(x_d)$, it follows from Lemma 3 that

$$|\varphi_n(x')-\varphi_n(x'')| \leqslant M_{n,k}|h(x')-h(x'')| \qquad \text{for every } x', x'' \in I_{K_1}.$$

Now let

$$\eta_{n,k} = \frac{\varphi_n(x_k)-\varphi_n(x_d)}{M_{n,k}[h(x_k)-h(x_d)]} = \frac{\delta_k\cdot h'(x_d)[\varphi_n(x_k)-\varphi_n(x_d)]}{\varepsilon_k\cdot\varphi_n'(x_d)[h(x_k)-h(x_d)]},$$

then $\lim_k \eta_{n,k} = 1$ (uniformly with respect to $n \geqslant N_0$). Furthermore, using the Cauchy Mean Value Theorem we obtain formula

$$(*)\qquad |\eta_{n,k}-1|$$

$$= \left|\frac{\delta_k\cdot h'(x_d)\varphi_n'(\xi_{n,k})-\varepsilon_k\cdot\varphi_n'(x_d)h'(\xi_{n,k})}{\varepsilon_k\cdot\varphi_n'(x_d)\cdot h'(\xi_{n,k})}\right|$$

$$\leqslant \frac{|\lambda_{m,n}|\cdot|\lambda_{\theta_0,n}^{-1}|\cdot|\delta_k\cdot h'(x_d)\cdot g_m'(\xi_{n,k})-\varepsilon_k\cdot h'(\xi_{n,k})\cdot g_m'(x_d)|+\ldots+|\lambda_{0,n}|\cdot|\lambda_{\theta_0,n}^{-1}|\cdot|\delta_k\cdot h'(x_d)\cdot g_0'(\xi_{n,k})-\varepsilon_k\cdot h'(\xi_{n,k})g_0'(x_d)|}{|\varepsilon_k\cdot h'(\xi_{n,k})|\cdot[|\lambda_{m,n}|\cdot|\lambda_{\theta_0,n}^{-1}|\,|g_m'(x_d)|+\ldots+|\lambda_{0,n}|\cdot|\lambda_{\theta_0,n}^{-1}|\cdot|g_0'(x_d)|]}$$

$$\leqslant \frac{4\lambda}{|\varepsilon_k|\,|h'(\xi_{n,k})|}\left[\frac{|\delta_k\cdot h'(x_d)\cdot g_m'(\xi_{n,k})-\varepsilon_k\cdot h'(\xi_{n,k})\cdot g_m'(x_d)|+\ldots+|\delta_k\cdot h'(x_d)\cdot g_0'(\xi_{n,k})-\varepsilon_k\cdot h'(\xi_{n,k})g_0'(x_d)|}{\lambda_m\cdot|g_m'(x_d)|+\ldots+\lambda_0\cdot|g_0'(x_d)|}\right]$$

$$\leqslant \frac{8\lambda}{|h'(\xi_{n,k})|\,|g_{\theta_0}'(x_d)|}[|\delta_k\cdot h'(x_d)\cdot g_m'(\xi_{n,k})-\varepsilon_k\cdot h'(\xi_{n,k})\cdot g_m'(x_d)|+\ldots+|\delta_k\cdot h'(x_d)\cdot g_0'(\xi_{n,k})-\varepsilon_k\cdot h'(\xi_{n,k})\cdot g_0'(x_d)|]$$

whenever $n \geqslant N_0$ and $k \geqslant K_2 \geqslant K_1$, where $K_2$ is choosen large enough so that $|\varepsilon_k| > \frac{1}{2}$ whenever $k \geqslant K_2$. Thus we may choose $K_3 \geqslant K_2$ such that $|\eta_{n,k}-1| < \frac{1}{2}$ whenever $n \geqslant N_0$ and $k \geqslant K_3$.

Since $\varphi_n(x)$ is continuous on $D$ and $I_k$ is a compact connected subset of $D$ for $k \geqslant K_0$; then $\varphi_n(I_k)$ is a compact connected set for $k \geqslant K_0$, that is a closed interval. Let $r$ be the $r$ of $P(\delta, l, r)$ for $f(x)$ with respect to $p$; then if $B$ is a Borel subset of $I_k$ ($k \geqslant K_0$) such that $\mu_h(B) > (1-\eta_{n,k}r)\mu_h(I_k)$ it follows from Lemmas 5 and 7 that

$$\mu\big(\varphi_n(B)\big) = \mu_{\varphi_n}(B) > (1-r)\mu_{\varphi_n}(I_k) = (1-r)\mu\big(\varphi_n(I_k)\big).$$

Since $\eta_{n,k} > \frac{1}{2}$ whenever $n \geqslant N_0$ and $k \geqslant K_3$; then $\mu_h(B) > (1-\frac{1}{2}r)\mu\big(h(I_k)\big)$ implies that $\mu_h(B) > (1-\eta_{n,k}r)\mu_h(I_k)$ whenever $n \geqslant N_0$ and $k \geqslant K_3$.

Since $y_d$ is a point of density of $h(E)$; then

$$\lim_k \frac{\mu\big(h(E)\cap J_k\big)}{\mu(J_k)} = 1.$$

But $h(I_k) = J_k$ for $k = 1, 2, 3, \ldots$, hence it follows from Lemma 5 that

$$\lim_k \frac{\mu_h(E\cap I_k)}{\mu_h(I_k)} = \lim_k \frac{\mu\big(h(E\cap I_k)\big)}{\mu\big(h(I_k)\big)}$$

$$= \lim_k \frac{\mu\big(h(E\cap J_k)\big)}{\lambda(J_k)} = 1.$$

Choose $K \geqslant K_3$ such that $\mu_h(E \cap I_K) > (1-\frac{1}{2}r)\mu_h(I_K)$; then $E \cap I_K$ being a Borel subset of $I_K$ such that $\mu_h(E \cap I_K) > (1-\eta_{n,K}r)\mu_h(I_K)$ whenever $n \geqslant N_0$, it follows that $\mu(\varphi_n(E \cap I_K)) > (1-r)\mu(\Phi_n(I_K))$ whenever $n \geqslant N_0$. Letting $\delta$, $l$ be the $\delta$, $l$ of $P(\delta, l, r)$ choose $N \geqslant N_0$ such that

$$\lambda_{m,N}[g_m(x_K)-g_m(x_d)]+\ldots+\lambda_{0,N}[g_0(x_K)-g_0(x_d)] > l$$

and on $E$.

$$|f(\lambda_{m,N}\cdot g_m(x)+\ldots+\lambda_{0,N}\cdot g_0(x))-p| < \delta$$

Let $\Phi_n(x) = \lambda_{\theta_0,n}\cdot\varphi_n(x)$ for every $n$, $J = \Phi_N(I_K)$, and $S = \Phi_N(E)$; then $S \cap J = \Phi_N(E \cap I_K)$.

Since $|f(y)-p| < \delta$ on $S$; then $J_{\delta,p} \subseteq J-(S \cap J)$, hence from

$$\mu(S \cap J) = \mu(\Phi_N(E \cap I_K)) > (1-r)\cdot\mu(\Phi_N(I_K)) = (1-r)\mu(J)$$

it follows that $\mu(J_{\delta,p}) < r\cdot\mu(J)$. But $J$ is an interval of length

$$\mu(J) = \mu(\Phi_N(I_K)) = \mu(\Phi_N[x_d, x_K]) = \lambda_{m,N}[g_m(x_K)-g_m(x_d)]+ \\ +\ldots+\lambda_{0,N}[g_0(x_K)-g_0(x_d)] > l,$$

hence we have the desired contradiction of (vi).

Note 1. The theorem also remains valid if some or all of the sequences $\{\lambda_{\theta,n}\}$ are sequences of negative numbers, provided that the corresponding functions $(-g_\theta(x))$ have properties (a) and (b). In view of Note 1 of Lemma 7 of the sequences $\{\gamma_{\theta,n}\}$ could all be sequences of negative numbers, and the theorem also remains valid if some of the sequences $\{\lambda_{\theta,n}\}$ are sequences of positive numbers, provided the corresponding functions $(-g_\theta(x))$ have properties (a) and (b). It is for this reason that certain seemingly unnecessary absolute value signs occur in the proof above.

Note 2. If the class $\mathscr{C}$ is further restricted so that for some $\theta_0$ the sequences $\{\lambda_{\theta,n}/\lambda_{\theta_0,n}\}$ converge to zero for every $\theta \neq \theta_0$, then conditions (vi) may be replaced by condition.

(vii) $f(x)$ has the property $Q(r, \theta, \delta, l)$ with respect to $p$.

For then we have:

$$\frac{\Phi_n(x_k)-\Phi_n(x_d)}{\Phi_n(x_d)} = \frac{\lambda_{m,n}[g_{m,n}(x_K)-g_m(x_d)]+\ldots+\lambda_{0,n}[g_0(x_K)-g_0(x_d)]}{\lambda_{m,n}\cdot g_m(x_d)+\ldots+\lambda_{0,n}\cdot g_0(x_d)}$$

$$= \frac{\lambda_{m,n}\cdot\lambda_{\theta_0,n}^{-1}[g_m(x_K)-g_m(x_d)]+\ldots+\lambda_{0,n}\cdot\lambda_{\theta_0,n}^{-1}[g_0(x_K)-g_0(x_d)]}{\lambda_{m,n}\cdot\lambda_{\theta_0,n}^{-1}\cdot g_m(x_d)+\ldots+\lambda_{0,n}\cdot\lambda_{\theta_0,n}^{-1}\cdot g_0(x_d)}$$

tending to

$$\frac{g_{\theta_0}(x_K)-g_{\theta_0}(x_d)}{g_{\theta_0}(x_d)} \quad \text{as } n\to\infty.$$

Thus as in the proof of Theorem 2 of [1], if we let

$$2\bar{\theta} = \frac{g_{\theta_0}(x_K)-g_{\theta_0}(x_d)}{g_{\theta_0}(x_d)}$$

we may choose $N$ so that

$$\frac{\Phi_n(x_K)-\Phi_n(x_d)}{\Phi_n(x_d)} \geqslant \bar{\theta}$$

whenever $n\geqslant N$.

Note 3. Condition (a) may be replaced by conditions (c) $g'_\theta(x)\geqslant 0$ for $x\in D$ and $\theta=0,1,\ldots,m$; and (d) $g'_j(x)>0$ for $x\in D$ and at least one value of $j$, if we further restrict the class $\mathscr{C}$ so that for one of these $j$, the sequence $\{\lambda_{j,n}\}$ is unbounded and the sequences $\{\lambda_{\theta,n}/\lambda_{j,n}\}$, $\theta=0,1,\ldots$ $\ldots,m$, are bounded. For then we may let $\theta_0$ be one of the $j$'s such that $g'_j(x_k)>0$. Furthermore $\Phi_n(x)$ is strictly increasing on $D$ and we may choose $N\geqslant N_0$ so that

$$\mu(J) = \mu\big(\Phi_N(I_K)\big) = \lambda_{m,N}[g_m(x_K)-g_m(x_d)]+\ldots+\lambda_{0,N}[g_0(x_K)-g_0(x_d)]$$
$$\geqslant \lambda_{\theta,N}[g_\theta(x_K)-g_\theta(s_d)]>l$$

just as before.

Note 4. In view of Note 1 of Lemma 7 it follows that the theorem remains valid if condition (a) is replaced by condition (a′) $g'_\theta(x)<0$ in $D$ for $\theta=0,1,\ldots,m$.

THEOREM 32. *Let $f$ be a Baire function, $p$ be any real or complex number, and $\mathscr{C}$ be the class of all pairs $(\{\lambda_n\},\{t_n\})$ of sequences of real numbers with $\{\lambda_n\}$ positive and unbounded. Let $h(x)$ be strictly increasing on $R_1$, continuous on $R_1$, differentiable except on a set of $\mu$-measurez eros in $R_1$, $h'(x)$ be either right continuous or left continuous*(2) *except possibly on a set of $\mu$-measure zero in $R_1$, and $h'(x)>0$ except possibly on a set of $\mu$-measure zeros in $R_1$. Let $D$ be an open interval finite or infinite; $g(x)$ be a function defined on $D$, differentiable in $D$, and satisfying the following conditions:*

(a) $g'(x)>0$ *for* $x\in D$.

(b) $g'(x)$ *is (right) left continuous wherever $h'(x)$ is right (left) continuous in $D$ (excepting, of course, sets of $\mu$-measure zero).*

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(2) When we refer to the continuity of a derivative from the right (left) we tacitly assume that the derivative exists at the point and in an interval to the right (left) of the point in question.

*Then a sufficient condition that the sequence* $\{f(\lambda_n \cdot g(x)+t_n)\}$ *does not converge to p on a subset of D having positive* $\mu_h$*-measure for any* $(\{\lambda_n\}, \{t_n\})$ *in* $\mathscr{C}$ *is:*

(vi) $f(x)$ *has the property* $P(\delta, l, r)$ *with respect to* $p$.

Proof. This follows from Theorem 31 with $m=1$, $g_1(x)=g(x)$, $g_0(x)=1$, $\lambda_{1,n}=\lambda_n$, and $\lambda_{0,n}=t_n$ for $n=1,2,3,\ldots$ despite the fact that the function $g_0(x)$ does not satisfy (a) and the sequential restrictions of Note 3, Theorem 31 are not satisfied either. For $\Phi_n(x)=\lambda_n \cdot g(x)+t_n$ for $n=1,2,3,\ldots$, and consequently

$$|\eta_{n,k}-1| = \frac{\delta_k \cdot h'(x_d)\cdot g'(\xi_{n,k}) - \varepsilon_k \cdot h'(\xi_{n,k})\cdot g'(x_d)}{\varepsilon_k h'(\xi_{n,k})\cdot g'(x_d)}$$

whenever $k \geqslant K_1$ and $n \geqslant N_0$. Thus we may choose $K_2 \geqslant K_1$ such that $|\eta_{n,k}-1| < \frac{1}{2}$ whenever $n \geqslant N_0$ and $k \geqslant K_2$. Furthermore $\Phi_n(x)$ is strictly increasing in $D$ and we may choose $N \geqslant N_0$ so that $\mu(J) = \mu(\Phi_N(I_K)) = \lambda_N[g(x_K)-g(x_d)] > l$.

Note 1. We may alter the class $\mathscr{C}$ by having $\{\lambda_n\}$ negative and unbounded if we require that the function $-g(x)$ have properties (a) and (b).

Note 2. If the class $\mathscr{C}$ is further restricted so that the sequence $\{\lambda_n/t_n\}$ converges to zero condition (vi) may be replaced by condition:

(vii) $f(x)$ has the property $Q(r, \theta, \delta, l)$ with respect to $p$.

For then we have

$$\frac{\Phi_n(x_K)-\Phi_n(x_d)}{\Phi_n(x_d)} = \frac{g(x_K)-g(x_d)}{g(x_d)+\lambda_n^{-1}\cdot t_n} \quad \text{tends to} \quad \frac{g(x_K)-g(x_d)}{g(x_d)}$$

as $n \to \infty$. Thus as in the proof of Theorem 2 of [3], if we let

$$2\theta = \frac{g(x_K)-g(x_d)}{g(x_d)}$$

we may choose $N$ so that

$$\frac{\Phi_n(x_K)-\Phi_n(x_d)}{\Phi_n(x_d)} \geqslant \theta \quad \text{whenever } n \geqslant N.$$

Note 3. In view of Note 1 of Lemma 7 it follows that the theorem remain valid if condition:

(a) $g'(x) > 0$ for $x \in D$ is replaced by condition

(a′) $g'(x) < 0$ for $x \in D$.

THEOREM 33. *Under the condition given in Theorem 31 a sufficient condition that the sequence* $\{f(\lambda_{m,n}\cdot g_m(x)+\ldots+\lambda_{0,n}\cdot g_0(x))\}$ *does not converge to a finite limit function on a subset of D having positive* $\mu_h$*-measure for any* $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\}) \in \mathscr{C}$ *is:*

(ix) $f(x)$ *has the property* $P(\delta, l, r)$ *with respect to every real or complex* $p$.

Proof. Suppose that there exists a subset $E$ of $D$ having positive $\mu_h$-measure, a $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\}) \in \mathscr{C}$, and a function $\chi(x)$ defined and finite valued on $E$ such that

$$\lim_n f\big(\lambda_{m,n}\cdot g_m(x) + \ldots + \lambda_{0,n}\cdot g_0(x)\big) = \chi(x) \quad \text{on } E.$$

In fact we can arrange to have $E$ as a subset of a finite closed subinterval of $D$. By using Egoroff's Theorem, recalling that sets of positive $\mu_h$-measure have bounded Borel subsets of positive $\mu_h$-measure, we may obtain the above limit uniformly on $E$ and $E$ as a bounded Borel set of positive $\mu_h$-measure. Furthermore by using Lusin's Theorem we may arrange to have $E$ a closed set and $\chi(x)$ continuous on $E$. From Lemma 5 it follows that $\mu\big(h(E)\big) = \mu_h(E) > 0$, and consequently the set $h(E)$ has a point of density $y_d$, let $x_d = h^{-1}(y_d)$. Since almost every point of $h(E)$ is a point of density of $h(E)$; then it follows from Lemmas 8 and 12 that we may arrange to have $x_d$ a point such that $h'(x_d) > 0$, $h'(x)$ either right or left continuous at $x_d$, $g_\theta(x_d) \neq 0$ for $\theta = 0, 1, \ldots, m$ and $g'_\theta(x)$ is either right continuous or left continuous at $x_d$ $(\theta = 0, 1, \ldots, m)$ with this of the same type as $h'(x)$ has there. Now define $i_0$, $\lambda_\theta$, $\lambda$, $J_k$, $I_k$, $x_k$, $K_0$, $N_0$, $\varphi_n(x)$, $K_1$, $\Phi_n(x)$, $\delta_k$, $M_{n,k}$, $\eta_{n,k}$, $K_2$ and proceed with them exactly as in Theorem 31. Then we will have from Lemma 3 that $|\varphi_n(x') - \varphi_n(x'')| \leqslant M_{n,k}|h(x') - h(x'')|$ fore very $x', x'' \in I_k$ $(k \geqslant K_1)$, $|\eta_{n,k} - 1| < \frac{1}{2}$ whenever $n \geqslant N_0$ and $k \geqslant K_2$, and if $B$ is a Borel subset of $I_k$ $(k \geqslant K_1)$ such that $\mu_h(B) > (1 - \eta_{n,k} r)\mu_h(I_k)$ it follows from Lemmas 5 and 7 that

$$\mu\big(\varphi_n(B)\big) = \mu_{\varphi_n}(B) > (1-r)\mu_{\varphi_n}(I_k) = (1-r)\mu\big(\varphi_n(I_k)\big).$$

Let $p = \chi(x_d)$ and $\delta, l, r$ $(r < 1)$ be the $\delta$, $l$, $r$ of $P(\delta, l, r)$ for $f(x)$ with respect to $p$. Since $\chi(x)$ is continuous on $E$ there exists $\eta(\delta) > 0$ such that $|\chi(x) - p| < \delta/2$ whenever $x \in A = E \cap (x_d - \eta, x_d + \eta)$. Since $y_d$ is a point of density of $h(E)$ and $h(I_k) = J_k$ for $k = 1, 2, 3, \ldots$; then it follows from Lemma 5 that

$$\lim_k \frac{\mu_h(E \cap I_k)}{\mu_h(I_k)} = \lim_k \frac{\mu\big(h(E \cap I_k)\big)}{\mu\big(h(I_k)\big)} = \lim_k \frac{\mu\big(h(E) \cap J_k\big)}{\mu(J_k)} = 1.$$

Choose $K \geqslant K_3$ such that $\mu_h(E \cap I_k) > (1 - \frac{1}{2}r)\mu_h(I_K)$ and $K^{-1} < \eta$, then $E \cap I_K$ is a Borel subset of $I_K$ such that $\mu_h(E \cap I_K) > (1 - \eta_{n,k}\cdot r)\mu_h(I_K)$ whenever $n \geqslant N_0$, hence it follows from Lemmas 5 and 7 that $\mu\big(\varphi_n(E \cap I_K)\big) > (1-r)\mu\big(\varphi_n(I_K)\big)$ whenever $n \geqslant N_0$. Choose $n \geqslant N_0$ such that $\lambda_{m,N}[g_m(x_K) - g_m(x_d)] + \ldots + \lambda_{0,N}[g_0(x_K) - g(x_d)] > l$ and $\big|f\big(\Phi_N(x)\big) - \chi(x)\big| < \delta/2$ on $E$. Then $\big|f\big(\Phi_N(x)\big) - p\big| \leqslant \big|f\big(\Phi_N(x)\big) - \chi(x)\big| + |\chi(x) - p| < \delta$ on $E \cap I_K \subseteq A$. Let $J = \Phi_N(I_K)$ and $S = \Phi_N(E)$; then $S \cap J = \Phi_N(E \cap I_K)$. Since $|f(y) - p| < \delta$ on $S \cap J$; then $J_{\delta,p} \subseteq J - (S \cap J)$, hence from $\mu(S \cap J) = \mu\big(\Phi_N(E \cap I_K)\big) > (1-r)\mu\big(\Phi_N(I_K)\big)$

$= (1-r)\mu(J)$ it follows that $\mu(J_{\delta,p}) < r\mu(J)$. But $J = \Phi_N(I_K)$ is an interval of length $\lambda_{m,N}[g_m(x_K)-g_m(x_d)]+\ldots+\lambda_{0,N}[g_0(x_K)-g_0(x_d)] > l$, hence we have the desired contradiction of (ix).

Note 1. If the class $\mathscr{C}$ is further restricted by requiring that for some $\theta_0$ the sequences $\{\lambda_{\theta,n}/\lambda_{\theta_0,n}\}$ converge to zero for every $\theta \neq \theta_0$, then condition (ix) may be replaced by the condition

(x) $f(x)$ has the property $Q(r, \theta, \delta, l)$ with respect to every real or complex $p$.

This follows exactly as in Note 2 of Theorem 31.

Note 2. Notes 1, 3, and 4 of Theorem 31 also apply here.

THEOREM 34. *Under the conditions given in Theorem 32 a sufficient condition that the sequence $\{f(\lambda_n \cdot g(x)+t_n)\}$ does not converge to a finite limit function on a subset of $D$ having positive $\mu_h$-measure for any $(\{\lambda_n\}, \{t_n\}) \in \mathscr{C}$ is:*

(ix) *$f(x)$ has the property $P(\delta, l, r)$ with respect to every real or complex $p$.*

Proof. This follows from Theorem 33 with $m = 1$, $g_1(x) = g(x)$, $g_0(x) = 1$, $\lambda_{1,n} = \lambda_n$, and $\lambda_{0,n} = t_n$ for $n = 1, 2, 3, \ldots$, despite the fact that the function $g_0(x)$ does not satisfy (a) and the sequential restrictions of Note 3, Theorem 31 are not satisfied either. The proof follows from the proof of Theorem 33 exactly as the proof of Theorem 32 followed from the proof of Theorem 31.

Note 1. If the class $\mathscr{C}$ is further restricted by requiring that the sequence $\{\lambda_n/t_n\}$ converges to zero, then condition (ix) may be replaced by condition

(x) $f(x)$ has the property $Q(r, \theta, \delta, l)$ with respect to every real or complex $p$.

This follows exactly as in Note 2 of Theorem 32.

Note 2. Note 1 and 3 of Theorem 32 apply here.

THEOREM 35. *Let $f$ be a Baire function, $p$ be any real or complex, number, $m$ be a positive integer, and $\mathscr{C}$ be the class of all $(m+1)$-tuples $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\})$ of sequences of positive numbers, at least one of which is unbounded, and such that for some $\theta_0$ the sequences $\{\lambda_{m,n}/\lambda_{\theta_0,n}\}, \ldots, \{\lambda_{0,n}/\lambda_{\theta_0,n}\}$ are all bounded. Let $h(x)$ be strictly increasing, continuous, and of type $T_1$ with respect to $x$ on an open interval $D$, finite or infinite, differentiable with $h'(x) > 0$ except possibly on a set of $\mu$-measure zero in $D$; and $h'(x)$ either right continuous*(3) *or left continuous except possibly on a set of $\mu$-measure zero in $D$. Let $\{g_{\theta,n}(x)\}$ $(\theta = 0, 1, \ldots, m)$ be sequences of functions*

(3) When we refer to the continuity of a derivative from the right (left) we tacitly assume that the derivative exists at the point and in an interval to the right (left) of the point in question.

*defined and differentiable, on $D$, and satisfying the following conditions:*

(a) *Corresponding to every $x$ in $D$ there exists a $\varrho_x > 0$ and an integer $N_x$ such that $g'_{\theta_0,n}(x) \geqslant \varrho_x$ for every $n \geqslant N_x$, and $g'_{\theta,n}(x) \geqslant 0$ in $D$ for every $(\theta, n)$.*

(b) *The set of functions $\{g'_{\theta,n}(x) : n = 1, 2, 3, \ldots\}$ is equicontinuous on $D$ for $\theta = 0, 1, \ldots, m$.*

*Then a sufficient condition that the sequence $\{f(\lambda_{m,n}\cdot g_{m,n}(x) + \ldots + \lambda_{0,n}\cdot g_{0n}(x))\}$ does not converge to $p$ on a subset of $D$ having positive $\mu_h$-measure for any $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\})$ in $\mathscr{C}$ is:*

(vi) *$f(x)$ has the property $P(\delta, l, r)$ with respect to $p$.*

Proof. Suppose that there exists a subset $E$ of $D$ having positive $\mu_h$-measure and a $(\{\lambda_{m,n}\}, \ldots, \{\lambda_{0,n}\})$ in $\mathscr{C}$ such that

$$\lim_n f(\lambda_{m,n}\cdot g_{m,n}(x) + \ldots + \lambda_{0,n}\cdot g_{0,n}(x)) = p \quad \text{on } E;$$

then since sets of positive $\mu_h$-measure have compact subsets of positive $\mu_h$-measure, we may assume that $E$ is a compact subset of $D$. Thus it now follows from Egoroff's Theorem that

$$\lim_n f(\lambda_{m,n}\cdot g_{m,n}(x) + \ldots + \lambda_{0,n}\cdot g_{0,n}(x)) = p$$

uniformly on a subset $E_0$ of $E$ having positive $\mu_h$-measure. For simplicity of notation denote $E_0$ by $E$. Since sets of positive $\mu_h$-measure have compact subsets of positive $\mu_h$-measure we may assume that $E$ is a compact set. From Lemma 5 it follows that $\mu(h(E)) = \mu_h(E) > 0$, and consequently the set $h(E)$ has a point of density $y_d$, let $x_d = h^{-1}(y_d)$. Since almost every point of $h(E)$ is a point of density of $h(E)$ it now follows from Lemmas 8 and 12 that we may arrange to have $x_d$ such that $h'(x_d) > 0$ and $h'(x)$ is either right continuous or left continuous at $x_d$. Furthermore, by taking subsequences we may arrange to have the sequences $\{\lambda_{\theta,n}\}$ $(\theta = 0, 1, \ldots, m)$ tending to limits, at least one of which is infinite, and by a further choice of subsequences we may for some $\theta_0$ arrange to have $\lim_n |\lambda_{\theta,n}/\lambda_{\theta_0,n}| = \lambda_\theta$ finite for $\theta = 0, 1, \ldots, m$. Let $\lambda$-Sup$\{\lambda_\theta : \theta = 0, 1, \ldots, m\}$ and note that $\lambda_{\theta_0} = 1$.

If $h'(x)$ is right continuous at $x_d$ let $J_k = [y_d, y_d + k^{-1}]$, but if $h'(x)$ is left continuous at $x_d$ let $J_k = [y_d - k^{-1}, y_d]$. For the sake of the argument let us assume the former. Then $h^{-1}$ being strictly increasing and continuous at at $x_d$ we have $I_k = h^{-1}(J_k) = [x_d, x_k]$ with $x_d < x_{k+1} < x_k$ for $k = 1, 2, 3, \ldots$, and $\lim_k x_k = x_d$. Furthermore since $x_d \in D$ there exists an integer $K_0$ such that $I_k \subseteq D$ whenever $k \geqslant K_0$. Now choose an integer $N_0$ such that

$$\frac{\lambda_\theta}{2} \leqslant \left|\frac{\lambda_{\theta,n}}{\lambda_{\theta_0,n}}\right| < 2\lambda \quad \text{for } \theta = 0, 1, \ldots, m \text{ and } n \geqslant N_0.$$

Let $\varphi_n(x) = \lambda_{\theta_0,n}^{-1}(\lambda_{m,n}\cdot g_{m,n}(x)+\ldots+\lambda_{0,n}\cdot g_{0,n}(x))$ and $\Phi_n(x) = \lambda_{\theta_0,n}\cdot\varphi_n(x)$ for $n = 1, 2, 3, \ldots$ Since the set of functions $\{g'_{\theta,n}(x): n = 1, 2, 3, \ldots\}$ is equicontinuous on $D$ for $\theta = 0, 1, \ldots, m$, then given $\varepsilon > 0$ there exists a $\delta > 0$ such that $[x_d, x_d+\delta] \subseteq D$ and $|g'_{\theta,n}(x) - g'_{\theta,n}(x_d)| < \varepsilon$ for every $(\theta, n)$ whenever $x \in [x_d, x_d+\delta]$. Thus

$$|\varphi'_n(x)-\varphi'_n(x_d)| \leqslant \left|\frac{\lambda_{m,n}}{\lambda_{\theta_0,n}}\right|\cdot|g'_{m,n}(x)-g'_{m,n}(x_d)|+\ldots+\left|\frac{\lambda_{0,n}}{\lambda_{\theta_0 n}}\right|\cdot|g'_{0,n}(x)- g'_{0,n}(x_d)| \leqslant 2\lambda(m+1)\varepsilon$$

whenever $x\in[x_d, x_d+\delta]$ and $n \geqslant N_0$. Furthermore letting $\varrho = \varrho_{x_d}$ we have

$$\varphi'_n(x_d) \geqslant \tfrac{1}{2}[\lambda_m g'_{m,n}(x_d)+\ldots+\lambda_0 g'_{0,n}(x_d)] \geqslant \tfrac{1}{2}\lambda_{\theta_0}\cdot g'_{\theta_0,n}(x_d) > \tfrac{1}{2}\varrho$$

whenever $n \geqslant N_1 = \operatorname{Sup}\{N_0, N_{x_d}\}$. Therefore

$$-2\lambda(m+1)\varepsilon+\varphi'_n(x_d) \leqslant \varphi'_n(x) \leqslant \varphi'_n(x_d)+2\lambda(m+1)\varepsilon,$$

and consequently

$$1-\frac{4\lambda(m+1)\varepsilon}{\varrho} \leqslant 1-\frac{2\lambda(m+1)\varepsilon}{\varphi_n(x_d)} \leqslant \frac{\varphi'_n(x)}{\varphi_n(x_d)} \leqslant 1+\frac{2\lambda(m+1)\varepsilon}{\varphi'_n(x_d)} \leqslant 1+\frac{4\lambda(m+1)\varepsilon}{\varrho}$$

whenever $x\in[x_d, x_d+\delta]$ and $n \geqslant N_1$. From this and the fact that $h'(x)$ is right continuous at $x_d$ it follows that we may choose $K_1 \geqslant K_0$, $\varepsilon_k$, and $\delta_k$ so that $h(x)$ and $\varphi_n(x)$ are differentiable in $I_k$ and

$$|\varphi'_n(x)/h'(x)| \leqslant \frac{\varepsilon_k\cdot\varphi'_n(x_d)}{\delta_k\cdot h'(x_d)}$$

whenever $x\in I_k$ for $k \geqslant K_1$ and $n \geqslant N_1$, where $\lim\limits_k \varepsilon_k = \lim\limits_k \delta_k = 1$. Thus letting

$$M_{n,k} = \frac{\varepsilon_k\cdot\varphi'_n(x_d)}{\delta_k\cdot h'(x_d)}$$

it follows from Lemma 3 that $|\varphi_n(x')-\varphi_n(x'')| \leqslant M_{n,k}|h(x')-h(x'')|$ for every $x'$, $x''\in I_k$ $(k \geqslant K_1)$. Now let

$$\eta_{n,k} = \frac{\varphi_n(x_k)-\varphi_n(x_d)}{M_{n,k}\cdot[h(x_k)-h(x_d)]};$$

then $\lim_k \eta_{n,k} = 1$ and it follows as in the proof of Theorem 31 that we may choose $K_2 \geqslant K_1$ so that $|\eta_{n,k}-1| < \frac{1}{2}$ whenever $n \geqslant N_1$ and $k \geqslant K_2$. The remainder of the proof now follows exactly as in Theorem 31.

Note 1. Notes 1 and 2 of Theorem 31 apply here as well.

Note 2. Analogues of Theorem 32, 33, and 34 may be proved.

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