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Some properties of M-variations

1. Introduction. We shall denote by $M(u), N(u)$ and $M_1(u), N_1(u)$ the pairs of non-negative continuous convex functions complementary in the sense of Young ([1], p. 16-20 or [6], p. 16). For the inverse functions the symbols $M^{-1}(v), N^{-1}(v)$ etc. will be used.

Let $f(t)$ be an arbitrary real function defined in a finite interval $\langle a, b\rangle$. Consider partitions

$$P = \{a = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_n = b\}$$

together with the sequences $U = (u_0, u_1, \ldots, u_{n-1})$ of non-negative numbers u_i such that

$$\sum_{i=0}^{n-1} N(u_i) \leqslant 1.$$

Putting $\Delta f(t_i) = f(t_{i+1}) - f(t_i)$, we shall write $V_M(f; a, b)$ and $V_M^*(f; a, b)$ for the upper bounds of the sums

$$\sum_{i=0}^{n-1} M(|\Delta f(t_i)|) \quad \text{and} \quad \sum_{i=0}^{n-1} |\Delta f(t_i)| u_i,$$

respectively. These quantities are called the *first* and the *second M-variation* of f in $\langle a, b\rangle$. Clearly, the ordinary variation

$$V(f; a, b) = \sup_P \sum_{i=0}^{n-1} |\Delta f(t_i)|$$

with the factor $N^{-1}(1)$ majorises $V_M^*(f; a, b)$. Moreover, by inequality of W. H. Young ([1], p. 24),

$$V_M^*(f; a, b) \leqslant V_M(f; a, b) + 1.$$

In this note we shall give some properties of M-variations. Most of them concern the second M-variation. Our investigations completes the results announced in [4].

2. Basic properties. An argument similar to that of [6], p. 171, leads to

2.1. *Let* $\gamma = V_M^*(f; a, b)$ *be a positive number. Then,*

$$V_M(f/\gamma; a, b) \leqslant 1.$$

The assumption $V_M^*(f; a, b) < \infty$ implies

$$V_M^*(kf; a, b) = |k|\, V_M^*(f; a, b)$$

for any real number k. Further, as in [5], p. 191 we obtain

2.2. *Suppose that the functions* $f(t), g(t)$ *are bounded in* $\langle a, b\rangle$, *and set*

$$\varphi(t) = f(t)+g(t), \qquad \psi(t) = f(t)g(t),$$
$$A = \sup_{a\leqslant t\leqslant b}|f(t)|, \qquad B = \sup_{a\leqslant t\leqslant b}|g(t)|.$$

Then

$$V_M^*(\varphi; a, b) \leqslant V_M^*(f; a, b)+V_M^*(g; a, b),$$
$$V_M^*(\psi; a, b) \leqslant BV_M^*(f; a, b)+AV_M^*(g; a, b).$$

Also, it is easily seen,

2.3. *If* $a < c < b$, *then*

$$V_M^*(f; a, b) \leqslant V_M^*(f; a, c)+V_M^*(f; c, b).$$

2.4. *If* $a_1 < a < b < b_1$, *then*

$$V_M^*(f; a, b) \leqslant V_M^*(f; a_1, b_1).$$

2.5. *If* $N(u) \leqslant N_1(u)$ *for* $u \in \langle 0, N_1^{-1}(1)\rangle$, *then*

$$V_M^*(f; a, b) \geqslant V_{M_1}^*(f; a, b).$$

Finally, we extend the corresponding propositions of [2], p. 36 and 38, to the first M-variation of 2π-periodic measurable $f(t)$ defined for all t (cf. [3], p. 12).

2.6. (i) *The assumption*

$$\sup_{0\leqslant t\leqslant 2\pi} M\big(|f(t+h)-f(t)|\big) = O(h) \quad \text{as } h\to 0$$

implies

$$V_M(f; 0, 2\pi) < \infty. \tag{1}$$

(ii) *If condition* (1) *holds, then*

$$\int_0^{2\pi} M\big(|f(t+h)-f(t)|\big)\,dt = O(h) \quad \text{as } h\to 0.$$

Proof. Case (i) is trivial. To prove (ii), we choose a positive integer k such that $2\pi/(k+1) < |h| \leqslant 2\pi/k$. Considering $h>0$, we have

$$\int_0^{2\pi} M\big(|f(t+h)-f(t)|\big)\,dt \leqslant \sum_{\nu=0}^{k} \int_{\nu h}^{(\nu+1)h} M\big(|f(t+h)-f(t)|\big)\,dt$$

$$\leqslant \int_{-h}^{h} \sum_{\nu=0}^{k} M\left(\left|f\left(z+\frac{2\pi}{k}\nu+h\right)-f\left(z+\frac{2\pi}{k}\nu\right)\right|\right)dz$$

$$\leqslant 2h\cdot V_M(f;\,-h,\,2\pi+2h)$$

$$\leqslant 2h\{4V_M(f;\,0,\,2\pi)+3M(w)\},$$

where

$$w = 2\sup_{0\leqslant t\leqslant 2\pi}|f(t)| < \infty.$$

Further it is enough to observe that

$$\int_0^{2\pi} M\big(|f(t-h)-f(t)|\big)\,dt = \int_0^{2\pi} M\big(|f(t+h)-f(t)|\big)\,dt.$$

3. Special theorems. A function $N_1(u)$ is said to be the *strong majorant* for $N(u)$ in $\langle 0, l\rangle$ if, for every positive integer k and all integers r large enough $(r \geqslant k)$,

$$rN(u) \leqslant N_1(ru/k) \qquad \text{whenever } 0 < ru/k \leqslant l.$$

For example,

$$N(u) = \frac{u^a}{\left(\log\dfrac{c}{u}\right)^{\beta}} \qquad (a>1,\ \beta\geqslant 0,\ c\geqslant 1)$$

possesses the strong majorant $N_1(u) \equiv N(u)$ in $\langle 0, c/2\rangle$. If

$$N(u) = \frac{u}{\left(\log\dfrac{c}{u}\right)^{\beta}} \qquad (\beta>0,\ c\geqslant 1),$$

the strong majorant, in the last interval, is of the form

$$N_1(u) = \frac{u}{\left(\log\dfrac{c}{u}\right)^{\beta_1}} \qquad (0<\beta_1<\beta).$$

Now, some generalizations of Marcinkiewicz's results ([2], p. 38-40) will be given.

3.1. *Suppose that* $V_M^*(f; a, b) < \infty$ *and that* $N(u)$ *has the strong majorant* $N_1(u)$ *in the interval* $\langle 0, l\rangle$. *Then, for any* $\varepsilon > 0$ *and any* $x \in \langle a, b)$, *there is a* $\sigma > 0$ *such that for every positive* $\delta < \sigma$,

$$V_{M_1}^*(f; x+\delta, x+\sigma) < \varepsilon.$$

Proof. Given a number $c \geqslant 1$ satisfying the inequality $N_1^{-1}(1/c) \leqslant l$, let us consider the modified pairs of functions

$$\bar{M}(u) = cM(u), \quad \bar{N}(u) = cN\left(\frac{u}{c}\right)$$

and

$$\bar{M}_1(u) = cM_1(u), \quad \bar{N}_1(u) = cN_1\left(\frac{u}{c}\right)$$

coplementary in the sense of Young, too. The function $\bar{N}_1(u)$ is the strong majorant for $\bar{N}(u)$ in $\langle 0, \bar{N}_1^{-1}(1)\rangle$; the inequality

$$V_{\bar{M}}^*(f; a, b) \leqslant cV_M^*(f; a, b)$$

implies

$$V_{\bar{M}}^*(f; a, b) < \infty.$$

First we shall prove that our thesis holds for the variation $V_{\bar{M}_1}^*$ instead of $V_{M_1}^*$. Supposing the contrary we could find, for an $\varepsilon > 0$ and an $x \in \langle a, b\rangle$, a sequence of non-overlapping intervals $\langle x+\delta_n, x+\sigma_n\rangle$ tending right-sidely to x such that

$$V_{\bar{M}_1}^*(f; x+\delta_n, x+\sigma_n) \geqslant \varepsilon \quad (n = 1, 2, \ldots).$$

Let k, r be some positive integers ($r \geqslant k$) for which

$$\frac{k\varepsilon}{2} > V_{\bar{M}}^*(f; a, b) \tag{2}$$

and $r\bar{N}(u) \leqslant \bar{N}_1(\lambda u)$ if $0 < \lambda u \leqslant \bar{N}_1^{-1}(1)$, where $\lambda = r/k$. Consider a partition $\{x = x_0 < x_1 < x_2 < \ldots < x_s = b\}$ such that the pairs $x_{j_\nu}, x_{j_{\nu+1}}$ ($\nu = 2r-1, 2r-3, \ldots, 5, 3, 1$) coincide with the end-points of r successive intervals $\langle x+\delta_n, x+\sigma_n\rangle$ ($n = 1, 2, \ldots, r$). Choose some non-negative u_i satisfying the condition

$$\sum_{i=j_\nu}^{j_{\nu+1}} \bar{N}_1(\lambda u_i) \leqslant 1 \quad (\nu = 1, 3, 5, \ldots, 2r-3, 2r-1).$$

Evidently, we may suppose that

$$\sum_{i=j_\nu}^{j_{\nu+1}} |\Delta f(x_i)|\lambda u_i \geqslant V_{\bar{M}_1}^*(f; x_{j_\nu}, x_{j_{\nu+1}}) - \frac{\varepsilon}{2} \geqslant \frac{\varepsilon}{2}.$$

Taking $u_i = 0$ for other i, we have

$$\sum_{i=0}^{s-1} \overline{N}(u_i) \leqslant \frac{1}{r}\Big\{\sum_{i=j_1}^{j_2} \overline{N}_1(\lambda u_i) + \ldots + \sum_{i=j_{2r-1}}^{j_{2r}} \overline{N}_1(\lambda u_i)\Big\} \leqslant 1.$$

Therefore,

$$V_{\overline{M}}^*(f; a, b) \geqslant \sum_{i=0}^{s-1} |\Delta f(x_i)| u_i$$

$$= \frac{1}{\lambda}\Big(\sum_{i=j_1}^{j_2} + \sum_{i=j_3}^{j_4} + \ldots + \sum_{i=j_{2r-1}}^{j_{2r}}\Big) |\Delta f(x_i)| \lambda u_i \geqslant \frac{k\varepsilon}{2},$$

which contradicts (2). Hence

$$V_{\overline{M}_1}^*(f; x+\delta, x+\sigma) < \varepsilon \qquad (a \leqslant x < b)$$

if σ is small enough and $0 < \delta < \sigma$.

Futher, in view of 2.1,

$$V_{\overline{M}_1}(\theta f; x+\delta, x+\sigma) \leqslant 1 \qquad \text{when } 1/\theta = V_{\overline{M}_1}^*(f; x+\delta, x+\sigma).$$

Consequently,

$$V_{M_1}(\theta f; x+\delta, x+\sigma) \leqslant \frac{1}{c}, \qquad V_{M_1}^*(\theta f; x+\delta, x+\sigma) \leqslant \frac{1}{c} + 1.$$

Thus

$$V_{M_1}^*(f; x+\delta, x+\sigma) \leqslant 2 V_{\overline{M}_1}^*(f; x+\delta, x+\sigma) < 2\varepsilon,$$

and ε being arbitrary, we get the desired assertion.

Clearly, the related inequality

$$V_{M_1}^*(f; x-\sigma, x-\delta) < \varepsilon \qquad (a < x \leqslant b)$$

is valid, too.

3.2. *Let $f(t)$ be continuous at every point of the interval $\langle a, b\rangle$, and let $V_M^*(f; a-\eta, b+\eta) < \infty$ for some $\eta > 0$. Then, under the assumption on $N(u)$ and $N_1(u)$ as in 3.1,*

$$\lim_{h\to 0+} V_{M_1}^*(f; x-h, x+h) = 0$$

uniformly in $x \in \langle a, b\rangle$.

Proof. Otherwise, there are an $\varepsilon > 0$ and a certain $x \in \langle a, b\rangle$ such that

(3) $$V_{M_1}^*(f; x-\sigma, x+\sigma) \geqslant \varepsilon$$

for arbitrary positive σ. On the other hand, in view of 3.1,

$$V_{M_1}^*(f; x-\sigma, x-\delta) + V_{M_1}^*(f; x+\delta, x+\sigma) < \varepsilon/4$$

for sufficiently small $\sigma > 0$ and $\delta \epsilon (0, \sigma)$. Obviously, we may also suppose

$$|f(x \pm \varrho) - f(x)| < \frac{\varepsilon}{8N_1^{-1}(1)} \quad \text{when } 0 \leqslant \varrho \leqslant \sigma.$$

Let $\{x - \sigma = x_0 < x_1 < \ldots < x_k < x < x_{k+1} < \ldots < x_m = x + \sigma\}$ be a partition of $\langle x - \sigma, x + \sigma \rangle$, and let u_i be some positive numbers such that

$$\sum_{i=0}^{m-1} N_1(u_i) \leqslant 1, \quad \sum_{i=0}^{m-1} |\Delta f(x_i)| u_i \geqslant V_{M_1}^*(f; x - \sigma, x + \sigma) - \frac{\varepsilon}{2}.$$

Since

$$\begin{aligned} \sum_{i=0}^{m-1} |\Delta f(x_i)| u_i &= \Big(\sum_{i=0}^{k-1} + \sum_{i=k+1}^{m-1} \Big) |\Delta f(x_i)| u_i + |f(x_{k+1}) - f(x_k)| u_k \\ &\leqslant V_{M_1}^*(f; x - \sigma, x_k) + V_{M_1}^*(f; x_{k+1}, x + \sigma) + \{|f(x_{k+1}) - \\ &\quad - f(x)| + |f(x) - f(x_k)|\} N_1^{-1}(1), \end{aligned}$$

we have

$$V_{M_1}^*(f; x - \sigma, x + \sigma) - \frac{\varepsilon}{2} < \frac{\varepsilon}{2}.$$

This contradicts (3) (see also [2], p. 40).

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