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On an inequality in the general additive theory of numbers

Let A and B be two increasing sequences of integers

$$a_0 = 0,\ a_1, a_2, \ldots; \qquad b_0 = 0,\ b_1, b_2, \ldots$$

The increasing sequence formed by the sums a_i+b_j $(i \geqslant 0, j \geqslant 0)$ is denoted by C. Moreover, let us denote the number of terms of A, B and C which do not exceed n (exclusive of 0) by $A(n)$, $B(n)$ and $C(n)$, respectively.

Finally,

$$\inf_k \frac{A(k)}{k} \quad \text{and} \quad \inf_k \frac{B(k)}{k}$$

are denoted by a and β. If $C(n) < n$, then according to the theorem of Mann, $C(n) \geqslant (a+\beta)n$. The inequality $C(n) \geqslant A(n)+\beta n$ for $n \notin C$ — from which this theorem would easily follow — is not generally true as was shown by Mann [1] by the following example:

$$\begin{array}{llr} A & 0,\ 1,\ 2,\ 6,\ 7,\ 8,\ 12,\ \ldots & A(11) = 5 \\ B & 0,\ 1,\ 2,\ 6,\ 7,\ 8,\ 12,\ \ldots & 11\cdot\beta = 11\cdot\frac{2}{5} = 4.4 \\ C & 0,\ 1,\ 2,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10,\ 12,\ \ldots & C(11) = 9 < 5+4.4 \end{array}$$

There arises a problem of finding an inequality for $n \notin C$ of the type $C(n) \geqslant A(n)+f(\beta)n$. We prove below this inequality for $f(\beta) = \beta/(1+\beta)$. The proof has some common elements with that of the theorem of Besicovitch [2]: $C(n) \geqslant A(n)+\beta' n$ for $n \notin C$, where $\beta' = \inf_k \frac{B(k)}{k+1}$.

THEOREM. *If $n \notin C$, then*

$$C(n) \geqslant A(n) + \frac{\beta}{1+\beta} n.$$

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(1) Henry B. Mann, *Addition theorems*, New York, London, Sydney 1965, p. 22.
(2) Ibidem, p. 20.

Proof. Since the theorem is obvious for $\beta = 0$ we can confine the proof to the case $1 \in B$.

Let us consider the segments of the sequence C in the interval $(0, n)$ which contain at least one number $a \in A$. Let us number the segments. The first segment 1 will start from $a_0 = 0$. The number of segments will be denoted by m. Let us now denote the least a of the segment s by a_s', the greatest a of this segment by a_s'' and the greatest c of the segment by c_s'.

From $1 \in B$ it follows directly $a_s'' + 1 \in C$ and thus

$$C(n) \geqslant A(n) + m. \tag{1}$$

Next $c_s' + 1 \notin C$ and consequently $c_s' + 1 - b \notin A$. It follows that the numbers $c = c_s' + 1 - b$ satisfying the condition $a_s' < c_s' + 1 - b \leqslant c_s'$ do not belong to A. The number of such numbers is $B(c_s' - a_s')$. We have

$$\begin{aligned} (2)\quad & C(a_{s+1}' - 1) - C(a_s' - 1) \\ &= \big(A(a_{s+1}' - 1) - A(a_s' - 1)\big) + B(c_s' - a_s') + \big(B(a_{s+1}' - a_s' - 1) - B(c_s' - a_s')\big) \\ &= A(a_{s+1}' - 1) - A(a_s' - 1) + B(a_{s+1}' - a_s' - 1) \\ &\geqslant A(a_{s+1}' - 1) - A(a_s' - 1) + \beta(a_{s+1}' - a_s') - \beta. \end{aligned}$$

For the least segment we obtain analogously, taking into consideration $n \notin C$,

$$C(n) - C(a_m' - 1) \geqslant A(n) - A(a_m' - 1) + \beta(n - a_m' + 1) - \beta. \tag{3}$$

It follows now from (2) and (3)

$$C(n) \geqslant A(n) + \beta n - \beta m \tag{4}$$

and multiplying (1) by β and adding it to (4),

$$C(n) \geqslant A(n) + \frac{\beta}{1+\beta} n. \tag{5}$$

For the sequence quoted at the beginning of the paper we obtain

$$9 > 5 + \frac{\frac{2}{5}}{1+\frac{2}{5}} \cdot 11 = 8.14 \ldots$$