ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XV (1971)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XV (1971)

S. H. Cox, Jr. and Sam B. Nadler, Jr. (Louisiana)

Supremum norm differentiability

1. Introduction. Restrepo [2], Šmulian [3]-[5], and others have related certain types of differentiability of a norm on a Banach space to some geometric properties (such as separability, uniform convexity, etc.) of the Banach space or its dual. In [1], p. 168-170, Banach proved that, for the supremum norm $\| \|_\infty$ on $C(X, R^1)$, where X is a compact metric space and R^1 denotes the reals,

$$\lim_{\lambda\to 0}\frac{\|f+\lambda g\|_\infty-\|f\|_\infty}{\lambda}\quad \text{exists for } f, g\in C(X, R^1)$$

iff f is a peaking function. In this paper we consider Banach spaces of the form $C(X, B) = \{f: X \to B | f \text{ is continuous}\}$, where X is in most cases a compact Hausdorff space and B is a Banach space, with the supremum norm $\| \|_\infty$. We give necessary and sufficient conditions that $\| \|_\infty$ be Fréchet differentiable at some $f\in C(X, B)$. When B itself is endowed with a Fréchet differentiable norm (such as when B is the reals) these conditions can be described completely in terms of topological properties of X. We now give some basic definitions and, in section 2, we state specifically the results described above and indicate some proofs.

Let X be a set and let $(B, \| \|_B)$ be a Banach space. A function $f: X \to B$ is said to be a *peaking function* iff there is a point $p\in X$ such that $\|f(p)\|_B > \|f(x)\|_B$ for all $x\in X$ such that $x\neq p$; the point p is called the *peak point for* f and f is said to *peak at* p.

Let E and F be Banach spaces, let U be an open subset of E, and let $f: U\to F$. If $x_0\in U$, then f is said to be *Fréchet differentiable*, or simply *differentiable*, at x_0 iff there is a continuous linear map $l: E\to F$ such that, for any $h\in E$ for which $(x_0+h)\in U$, we have:

$$f(x_0+h)-f(x_0) = l(h)+\omega(x_0, h),\qquad \text{where}\quad \lim_{\|h\|\to 0}\frac{\|\omega(x_0, h)\|}{\|h\|} = 0.$$

The mapping l is called the *differential of* f *at* x_0.

2. Throughout the remainder of this paper, unless otherwise stated, X will denote a compact Hausdorff space, $(B, \| \|_B)$ a Banach space, and A the Banach space of all continuous functions $f: X \to B$ with the supremum norm $\| \|_\infty$.

A proof of Lemma 1 can be obtained by modifying part of the proof of a lemma in [1], p. 168-170.

LEMMA 1. *If $\| \|_\infty$ is differentiable at $f \in A$, then f is a peaking function.*

The next lemma characterizes peaking functions in terms of a stability condition in A.

LEMMA 2. *A function $f \in A$ peaks at $p \in X$ iff, for each neighborhood V of p, there exists $\varepsilon > 0$ such that g attains its maximum (in norm) in V whenever $g \in A$ and $\|f-g\|_\infty < \varepsilon$.*

LEMMA 3. *Let $f \in A$ such that $\| \|_\infty$ is differentiable at f, let l denote the differential of $\| \|_\infty$ at f, and let $p \in X$ be the peak point for f (the point p exists by Lemma 1). If $h \in A$ and h vanishes on a neighborhood of p, then $l(h) = 0$.*

Proof. Let $h \in A$ such that h is the zero element of B on a neighborhood V of p. By Lemma 2 there exists $\varepsilon > 0$ such that if $0 < t < \varepsilon$, then $f + t \cdot h$ attains its maximum in norm in V. Since h vanishes on V, this implies that $\|f + t \cdot h\|_\infty \leqslant \|f\|_\infty$. However, since

$$\|f + t \cdot h\|_\infty \geqslant \|f(p) + t \cdot h(p)\|_B = \|f(p)\|_B = \|f\|_\infty,$$

it follows that $\|f + t \cdot h\|_\infty = \|f\|_\infty$. Therefore,

$$0 = \lim_{t \to 0+} \frac{\|f + t \cdot h\|_\infty - \|f\|_\infty - l(t \cdot h)}{\|t \cdot h\|_\infty} = \lim_{t \to 0+} \frac{-l(t \cdot h)}{\|t \cdot h\|_\infty} = -l(h),$$

i.e., $l(h) = 0$.

THEOREM. *If $f \in A$, then $\| \|_\infty$ is differentiable at f iff*

1. *f peaks at a point $p \in X$;*
2. *$\{p\}$ is an open subset of X;*
3. *$\| \|_B$ is differentiable at $f(p)$.*

Moreover, if $\| \|_\infty$ is differntiable at f, then $\|f\|'_\infty = \|f(p)\|'_B \cdot e_p$, where p is the peak point for f, $\|f\|'_\infty$ and $\|f(p)\|'_B$ are the differentials of $\| \|_\infty$ and $\| \|_B$ at f and $f(p)$ respectively, and $e_p: A \to B$ is given by $e_p(g) = g(p)$ for each $g \in A$.

Proof. Let $f \in A$ such that $\| \|_\infty$ is differentiable at f. By Lemma 1, f peaks at a point $p \in X$. To see that $\{p\}$ is an open subset of X suppose that $\{p\}$ is not open and let $\{p_a\}_{a \in D}$ be a net in $X - \{p\}$ such that $\{p_a\}_{a \in D}$ converges to p and $f(p_a) \neq 0$ for each $a \in D$. For each $a \in D$ let $h_a: X \to [0, 1]$ be continuous such that $h_a(p_a) = 1$ and h_a vanishes on a neighborhood W_a of p and define $g_a: X \to B$ by

$$\frac{2h_a(x)\|f(p) - f(p_a)\|_B}{\|f(p_a)\|_B} f_a(p)$$

for all $x \in X$. Using Lemma 3 it can be shown that

$$\lim_{a \in D} \frac{\|f+g_a\|_\infty - \|f\|_\infty}{\|g_a\|} = 0.$$

However, a simple computation shows that

$$\frac{\|f+g_a\|_\infty - \|f\|_\infty}{\|g_a\|_\infty} \geqslant 1/2$$

for all $a \in D$. This contradiction proves that $\{p\}$ is an open subset of X. Next we show that $\| \ \|_B$ is differentiable at $f(p)$. Let $c: B \to A$ be the linear isometry (into) given by

$$\big(c(b)\big)(x) = \begin{cases} b, & x = p, \\ 0, & x \neq p \end{cases}$$

for each $b \in B$. Let $\|f\|'_\infty$ and $\|c(f(p))\|'_\infty$ denote the differentials of $\| \ \|_\infty$ at f and $c(f(p))$ respectively. Using parts 1 and 2 of this theorem it is easy to see that there exists $\delta > 0$ such that if $\|h\|_\infty < \delta$, then

$$\|c(f(p))+h\|_\infty = \|c(f(p))(p)+h(p)\|_B = \|f(p)+h(p)\|_B$$

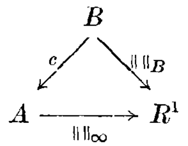
and

$$\|f+h\|_\infty = \|f(p)+h(p)\|_B.$$

It follows that

$$\lim_{\|h\|_\infty \to 0} \frac{\|c(f(p))+h\|_\infty - \|c(f(p))\|_\infty - \|f\|'_\infty(h)}{\|h\|_\infty} = 0$$

which proves that $\| \ \|_\infty$ is differentiable at $c(f(p))$ and, in fact, $\|c(f(p))\|'_\infty = \|f\|'_\infty$. Since c is a continuous linear mapping of B into A, c is differentiable at $f(p)$. Hence, since the diagram



commutes, the chain rule gives that $\| \ \|_B$ is differentiable at $f(p)$. In fact the differential $\|f(p)\|'_B$ of $\| \ \|_B$ at $f(p)$ is given by the formula $\|f(p)\|'_B = \|f\|'_\infty \circ c$. This completes the proof of the first half of the theorem.

To prove the second half of the theorem assume 1, 2, and 3 hold for f and let $\varepsilon > 0$. There exists $\delta > 0$ such that if $h \in A$ and $\|h\|_\infty < \delta$, then $\|f+h\|_\infty = \|f(p)+h(p)\|_B$ and

$$\frac{\|f(p)+h(p)\|_B - \|f(p)\|_B - \|f(p)\|'_B\big(h(p)\big)}{\|h(p)\|_B} < \varepsilon$$

where $\|f(p)\|_B'$ is the differential of $\| \ \|_B$ at $f(p)$. It follows that if $\|h\|_\infty < \delta$, then

$$\frac{\|f+h\|_\infty - \|f\|_\infty - \|f(p)\|_B'(h(p))}{\|h\|_\infty} < \varepsilon.$$

This proves that $\| \ \|_\infty$ is differentiable at f and, moreover, verifies the formula $\|f\|_\infty' = \|f(p)\|_B' \circ e_p$. This completes the proof of the theorem.

In the following three corollaries the Banach space $(B, \| \ \|_B)$ is specifically taken to be the real numbers R^1. However, the corollaries remain valid if the real numbers is replaced by a Banach space with a differentiable norm (i.e., with a norm differentiable on the space without zero).

COROLLARY 1. *The compact Hausdorff space X is perfect iff $\| \ \|_\infty$ is nowhere differentiable on $C(X, R^1)$.*

COROLLARY 2. *If X is a compact Hausdorff space such that each point is a G_δ, then $\| \ \|_\infty$ is differentiable at every peaking function in $C(X, R^1)$ iff X has only a finite number of points.*

COROLLARY 3. *If X is a compact Hausdorff space, then $\| \ \|_\infty$ is differentiable on a dense open subspace of $C(X, R^1)$ iff $\{x \in X | x$ is a limit point of $X\}$ is nowhere dense in X.*

Proof. Let $D = \{f \in C(X, R^1) | f$ peaks at an isolated point of $X\}$. The set D is open in X and, by the Theorem, D is precisely the points of differentiability of $\| \ \|_\infty$. It is not difficult to see that D is dense in $C(X, R^1)$ iff $\{x \in X | x$ is a limit point of $X\}$ is nowhere dense in X.

It is easy to give an example of a compact Hausdorff space X with an infinite number of points such that $\| \ \|_\infty$ is differentiable at every peaking function in $C(X, R^1)$. In fact, this differentiability condition is vacuously satisfied when X is a space in which no point is a G_δ (for example, the uncountable product of unit intervals). However, there are examples of compact Hausdorff spaces X with an infinite number of points such that $\| \ \|_\infty$ differentiable at every peaking function in $C(X, R^1)$ and the peaking functions form a dense open subspace of $C(X, R^1)$. Using the Theorem and Corollary 3 it is easy to see that this is the case if X is the one point compactification of an uncountable discrete set.

We now give a result concerning the differentiability of the supremum norm on spaces of bounded real-valued functions defined on a locally compact Hausdorff space.

COROLLARY 4. *Let X be a locally compact Hausdorff space and let $B(X, R^1) = \{f: X \to R^1 | f$ is continuous and bounded$\}$ with the supremum norm $\| \ \|_B$. If $f \in B(X, R^1)$, then $\| \ \|_B$ is differentiable at f iff*

1. *f peaks ar a point $p \in X$;*
2. *$\{p\}$ is an open subset of X;*
3. *$|f(p)|$ is bounded away from $\{|f(x)| \,|\, x \in (X - \{p\})\}$.*

Proof. Let $\beta(X)$ be the Stone–Čech compactification of X and define $\varphi\colon B(X, R^1) \to C(\beta(X), R^1)$, by extension. Since φ is an isometric isomorphism of $B(X, R^1)$ onto $C(\beta(X), R^1)$, it follows that $\| \ \|_B$ is differentiable at f iff the supremum norm on $C(\beta(X), R^1)$ is differentiable at $\varphi(f)$. The proof may now be completed by applying the Theorem several times together with the fact that a point of X is open in X iff it is open as a point of $\beta(X)$.

Bibliography

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] G. Restrepo, *Differentiable norms in Banach spaces*, Bull. Amer. Math. Soc. 70 (1964), p. 413-414.

[3] V. Šmulian, *On some geometrical properties of the sphere in a space of the type (B)*, Comptes Rendus (Doklady) de l'Academie des Sciences de l'URSS 24 (1939), p. 648-652.

[4] — *Sur la dérivalilite de la norme dans l'espace de Banach*, ibidem 27 (1940), p. 643-648.

[5] — *Sur la structure de la sphère unitaire dans l'espace de Banach*, Mat. Sbornik 9 (1941), p. 545-560.

LOUISIANA STATE UNIVERSITY
BATON ROUGE, LOUISIANA