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Exponential analogues of a generalized Lambert series

1. Introduction. The F-series

$$\sum_{n=1}^{\infty} a_n \frac{z^{pn}}{1-z^{qn}},$$

where p and q are positive integers and the a_n are complex valued constants, was introduced by Garvin [1] as a generalization of previous work in the theory of Lambert series. If the transformation $t = e^{-z}$ is applied to Garvin's F-series as a series in t, there results the series

$$\sum_{n=1}^{\infty} a^n \frac{e^{-pnz}}{1-e^{-qnz}}.$$

If, in addition, the sequences $\{pn\}$ and $\{qn\}$ are generalized to $\{\lambda_n\}$ and $\{\mu_n\}$ respectively, where $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences of real numbers which are strictly monotone increasing and unbounded, this last series becomes

$$\sum_{n=1}^{\infty} a_n \frac{e^{-\lambda_n z}}{1-e^{-\mu_n z}}, \tag{1.1}$$

hereinafter called simply the *S-series*. Kennedy [2] considered the special case of the S-series when $\lambda_n = \mu_n = \ln n$.

The purpose of this paper is to determine the regions of convergence of the S-series, and the expansion and inversion relationships between the S-series and the associated Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}. \tag{1.2}$$

A special S-series, the P-Q series,

$$\sum_{n=2}^{\infty} a_n \frac{n^{-pz}}{1-n^{-qz}},$$

resulting when $\lambda_n = p(\ln n)$ and $\mu_n = q(\ln n)$, where p and q are fixed positive integers, will also be considered and explicit formulas for relating ordinary Dirichlet series and P-Q series will be obtained. Finally, conditions under which the axis of imaginaries is a natural boundary of the function represented by a P-Q series will be determined.

Hereafter, unless otherwise indicated, all summations will be understood to range from $n = 1$ to ∞. Since all but a finite number of the elements of the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are positive, we may assume that λ_n and μ_n are positive for all n.

2. Convergence of the S-series. The following are given without proof:

THEOREM 2.1. *If z is such that $R(z) > 0$, then the S-series* (1.1)

(i) *converges for all such z if $\sum a_n$ converges;*

(ii) *converges and diverges at the same z points as the associated Dirichlet series* (1.2) *if $\sum a_n$ diverges.*

If z is such that $R(z) < 0$, then the S-series

(iii) *converges and diverges at the same z points as the series $\sum a_n e^{-(\lambda_n-\mu_n)z}$.*

In order to simplify notation in the statements of the following theorems, set $\mu_n - \lambda_n = \nu_n$ and $\lambda_n - \mu_n = \varrho_n$.

THEOREM 2.2. (i) *If the S-series converges at $z' = x'+iy'$, $x' > 0$, the S-series converges uniformly in each bounded region in the half plane $R(z) > x'$.* (ii) *If the S-series converges at $z' = x'+iy'$, $x' < 0$, and*

(a) *if $\nu_n < \nu_{n+1} \to \infty$, then the S-series converges uniformly in each bounded region in the half plane $R(z) < x'$. If $\{\nu_n\}$ is a positive, monotonely increasing or decreasing, bounded sequence, then the S-series converges uniformly in each bounded region in $-L < R(z) < x'$, where L is a positive constant;*

(b) *if $\{\varrho_n\}$ is a positive, monotonely increasing or decreasing sequence, then the S-series converges uniformly in each bounded region within $\{x' < R(z) < -H\} \cup \{R(z) > H\}$, where H is a positive constant.*

THEOREM 2.3. (i) *If $\sum |a_n|$ converges, then the S-series converges absolutely for all z for which $R(z) > 0$.*

(ii) *If $\sum |a_n|$ diverges, then the S-series converges absolutely for those values of z for which the associated Dirichlet series* (1.2) *converges absolutely and for which $R(z) > 0$.*

(iii) *Whether $\sum |a_n|$ diverges or converges, the S-series converges absolutely for those values of z for which the series $\sum a_n e^{-\varrho_n z}$ converges absolutely and for which $R(z) < 0$.*

THEOREM 2.4. (i) *If the S-series converges absolutely at $z' = x'+iy'$, $x' > 0$, then it converges absolutely and uniformly for all z with $R(z) \geqslant x'$.*

(ii) *If the S-series converges absolutely at $z' = x'+iy'$, $x' < 0$, and*

(a) *if $\{\nu_n\}$ is a positive sequence, then the S-series converges absolutely and uniformly for all z with $R(z) \leqslant x'$;*

(b) *if $\{\varrho_n\}$ is a positive sequence, then the S-series converges absolutely and uniformly for all z with $x' \leqslant R(z) \leqslant -H$, where H is an arbitrary but fixed positive constant.*

3. Relationship between the S-series and general Dirichlet series. Each term of the S-series can be written

$$a_n \frac{e^{-\lambda_n z}}{1-e^{-\mu_n z}} = \sum_{j=0}^{\infty} a_n e^{-\lambda_n z} e^{-j\mu_n z}$$

and therefore

$$\sum_{n=1}^{\infty} a_n \frac{e^{-\lambda_n z}}{1-e^{-\mu_n z}} = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} a_n e^{-(\lambda_n+j\mu_n)z}. \tag{3.1}$$

From this double sum form a single series by summing the terms in ascending order of the numbers $\lambda_n+j\mu_n$; $n = 1, 2, 3, \ldots$; $j = 0, 1, 2, \ldots$ Should any two or more terms of the set $\{\lambda_n+j\mu_n\colon n = 1, 2, 3, \ldots;$ $j = 0, 1, 2, \ldots\}$ be equal, factor out the common $e^{-\tau_k z}$, where

$$\tau_k = \lambda_{n_1}+j_1\mu_{n_1} = \lambda_{n_2}+j_2\mu_{n_2} = \ldots = \lambda_{n_s}+j_s\mu_{n_s},$$

and write its coefficient as the sum

$$h_k = \sum_{m=1}^{s} a_{n_m}.$$

This process utilizes all of the terms of the double series and gives a Dirichlet series

$$\sum_{k=1}^{\infty} h_k e^{-\tau_k z}.$$

Assume the S-series converges absolutely at z, $R(z) > 0$, and consider the array of absolute values of the double series (3.1). The r-th row of this array converges uniformly for all $x \geqslant x' > 0$ to

$$|a_r| \frac{e^{-\lambda_r x}}{1-e^{-\mu_r x}}.$$

The sum of these "row-sums" is

$$\sum |a_n| \frac{e^{-\lambda_n x}}{1-e^{-\mu_n x}},$$

which is the S-series in absolute values at x; from Theorem 2.4 (i) this series converges absolutely and uniformly. Thus the double series (3.1) is absolutely convergent and can be deranged without affecting its convergence. We have therefore

THEOREM 3.1. *Given an S-series absolutely convergent in a region of the half plane $R(z) > 0$, there is a Dirichlet series which converges absolutely and represents the same analytic function in that region.*

Again by use of the double series (3.1) it can be shown that it is possible to express a given Dirichlet series $\sum_{k=1}^{\infty} h_k e^{-\tau_k z}$ as an S-series in a formal manner. The resulting S-series will not be unique. If the S-series so obtained is absolutely convergent in the half plane $R(z) > 0$, it represents again the same analytic function as the given Dirichlet series. Hence we have

THEOREM 3.2. *A given Dirichlet series can be formally expressed as an S-series. In its region of absolute convergence in the half plane $R(z) > 0$, the resulting S-series represents the same analytic function as the given Dirichlet series.*

4. A special S-series; its relation to ordinary Dirichlet series. The P-Q series

$$\sum_{n=2}^{\infty} a_n \frac{n^{-pz}}{1-n^{-qz}} \tag{4.1}$$

is the special case of the S-series when λ_n is taken to be $p(\ln n)$ and μ_n to be $q(\ln n)$, where p and q are positive integers. By Theorem 3.1 the P-Q series (4.1) which can be written

$$\sum_{n=2}^{\infty} \sum_{j=0}^{\infty} a_n (n^{jq+p})^{-z} \tag{4.2}$$

may, in any region of its absolute convergence in the half plane $R(z) > 0$, be expressed as the series $\sum_{k=2}^{\infty} h_k k^{-z}$ by summing the terms of the double series (4.2) according to increasing values of n^{jq+p}, $n = 2, 3, 4, \ldots$; $j = 0, 1, 2, \ldots$ Then h_k will be the sum of those a_n for which $n^{jq+p} = k$; that is, $n^d = k$ for some positive integer $d = jq+p$. Using the notation $d \equiv r(\mathrm{mod}\, q)+$ to indicate that $d = jq+r$, where j is a non-negative integer and q and r are positive integers, we can write

$$h_k = \sum_{\substack{r^d = k \\ d \equiv p(\mathrm{mod}\, q)+}} a_r.$$

Conversely, a given Dirichlet series

$$\sum_{k=2}^{\infty} h_k k^{-pz} \tag{4.3}$$

can be expressed as a P-Q series. Choose q such that $q = cp$, c a positive integer. Again we use the double series (4.2) as an intermediate step where now the sequence $\{a_n\}$ is to be determined. Equating coefficients in (4.2) and (4.3) gives

$$a_n = \sum_{\substack{s^d = n \\ d \equiv 1 (\mathrm{mod}\, c)+}} R\left(\frac{\ln n}{\ln s}\right) h_s,$$

where $R(n) = 0$ for $n \not\equiv 1 \pmod{c}$ and for $n \equiv 1 \pmod{c}$, $R(n)$ is defined recursively by

$$\sum_{k|n} R(k) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

$$\frac{n}{k} \equiv 1 (\mathrm{mod}\, c)$$

the symbol $\frac{n}{k} \equiv 1 \pmod{c}$ indicating that k is a c-divisor of n; that is, k is a positive integer which divides n and $k \equiv 1 \pmod{c}$. The function $R(n)$ is an extension of Garvin's inversion function [1]; if $c = 1$, $R(n)$ reduces to the Möbius function.

It can be shown that if q does not satisfy the condition $q = cp$, where c is a positive integer, the representation of (4.3) is impossible by this method. The region in which the above expansion is valid is given in the theorem which follows.

The above relationships are expressed in

THEOREM 4.1. *Any P-Q series*

$$\sum_{n=2}^{\infty} a_n \frac{n^{-pz}}{1-n^{-qz}}$$

can be expressed as an ordinary Dirichlet series $\sum_{k=2}^{\infty} h_k k^{-z}$, *where*

$$h_k = \sum_{\substack{n^d = k \\ d \equiv p (\mathrm{mod}\, q)+}} a_n .$$

Conversely a Dirichlet series $\sum_{k=2}^{\infty} h_k k^{-pz}$ *can be expressed as a P-Q series, where* p *is determined from the given Dirichlet series, and* $q = cp$ *for a* *positive integer c. The coefficients a_n of the P-Q series are then given by*

$$a_n = \sum_{\substack{s^d = n \\ d \equiv 1 \pmod{c}+}} R\left(\frac{\ln n}{\ln s}\right) h_s .$$

Both representations are valid in the regions of absolute convergence of the respective P-Q series in the half plane $R(z) > 0$.

5. An Abelian theorem and natural boundary theorems. We shall determine the behavior of a function represented by a *P-Q* series as the variable approaches a boundary point of the region of convergence, and shall establish conditions under which the axis of imaginaries is a natural boundary for such a function using both a "real" approach and a Stolz path approach; that is, along any analytic curve which terminates at a point z on the boundary of the region of convergence and lies entirely within the intersection of the region of convergence and the region bounded by two half-lines which originate at z such that they make non-zero angles with the tangent to the boundary at z and lie on the same side of the tangent as does the region of convergence.

Throughout the remainder of this paper square brackets shall designate the Greatest Integer function and

$$[r]' = \begin{cases} 0, & \text{if } r < 1, \\ [r], & \text{if } r \geqslant 1. \end{cases}$$

In the proof of Theorem 5.1 we shall require the following lemmas:

LEMMA 5.1. *If k and n are natural numbers, $k, n \geqslant 2$, and $k^{c+1} \leqslant n$ for a fixed positive c, then*

$$\left| \frac{1}{[\ln n]} \left[\frac{\left[\frac{\ln n}{\ln k} - 1 \right]}{c} \right]' - \frac{1}{c \ln k} \right| \leqslant \frac{3+c}{c\{(\ln n) - 1\}} .$$

LEMMA 5.2. *If $\sum (a_n/\alpha_n)$ converges absolutely, where $0 < \alpha_n < \alpha_{n+1} \to \infty$, and $\lim_{n\to\infty} \beta_n/\alpha_n = 1$, then $s_n = \frac{1}{\beta_n} \sum_{k=1}^{n} |a_k|$ is a null sequence* ([2], p. 453).

THEOREM 5.1. *If the P-Q series (4.1), where $q = cp$, c a positive integer, converges absolutely for $R(z) > 0$, and if the series of constants $\sum_{n=2}^{\infty} (a_n/\ln n)$ converges absolutely, then for any approach to the origin in the half plane $R(z) > 0$ along any Stolz path*

$$\lim_{z \to 0} \left\{ z \sum_{n=2}^{\infty} a_n \frac{n^{-pz}}{1 - n^{-qz}} \right\} = \frac{1}{cp} \sum_{n=2}^{\infty} \left(\frac{a_n}{\ln n} \right). \tag{5.1}$$

Proof. If in the left-hand side of (5.1) the P-Q series is replaced by its equivalent Dirichlet series and pz is set equal to t, the limit to be evaluated is

$$\frac{1}{p}\lim_{t\to 0}\left\{\frac{1}{t^{-1}}\sum_{k=2}^{\infty} h_k k^{-t}\right\} = \frac{1}{p}\lim_{t\to 0}\left\{\sum_{k=2}^{\infty} h_k k^{-t}\Big/\sum_{k=1}^{\infty}(t+1)^{-k}\right\}.$$

By a theorem of Knopp ([3], p. 122), this last limit has the same value as

$$\frac{1}{p}\lim_{n\to\infty}\left\{\sum_{k=2}^{n} h_k/[\ln n]\right\}$$

which can be shown to be equal to

$$\lim_{n\to\infty}\left\{\left(\sum_{k=2}^{n} a_k\left[\frac{\left[\dfrac{\ln n}{\ln k}-1\right]}{c}\right]'\right)\Big/p[\ln n]\right\}.$$

Let $\varepsilon > 0$ be given. For each n, let $L(n)$ be the largest integer such that $\{L(n)\}^{c+1} \leqslant n$ for a fixed c. Let H be a fixed integer such that

$$\sum_{k=H+1}^{\infty} |a_k/\ln k| < \varepsilon pc/2$$

and let $M' = H^{c+1}$. Then for all $n > M'$

$$(5.2)\qquad \left|\frac{\sum_{k=2}^{n} a_k\left[\dfrac{\left[\dfrac{\ln n}{\ln k}-1\right]}{c}\right]'}{p[\ln n]} - \frac{1}{pc}\sum_{k=2}^{\infty} a_k\frac{1}{\ln k}\right|$$

$$\leqslant \frac{1}{p}\left|\sum_{k=2}^{L(n)} a_k\left\{\frac{\left[\dfrac{\left[\dfrac{\ln n}{\ln k}-1\right]}{c}\right]'}{[\ln n]} - \frac{1}{c(\ln k)}\right\}\right| + \frac{1}{pc}\sum_{k=L(n)+1}^{\infty} |a_k/\ln k|$$

$$\leqslant \frac{1}{p}\,\frac{3+c}{c\{(\ln n)-1\}}\sum_{k=2}^{L(n)} |a_k| + \varepsilon/2$$

by Lemma 5.1. Since $\lim_{n\to\infty}\{(\ln n)-1\}/\ln n = 1$ and

$$\sum_{k=2}^{L(n)} |a_k| < \sum_{k=2}^{n} |a_k|,$$

(5.2) becomes arbitrarily small with increasing n by Lemma 5.2. The theorem then follows.

In order to establish the axis of imaginaries as a natural boundary by "real" approach along lines parallel to the x-axis, which is analogous to radial approach for the Lambert series, we shall require the following adaption of the Abelian theorem of Garvin ([1], p. 511) and the following lemma.

THEOREM 5.2. *If the coefficients of the series*

$$\sum a_n \frac{x^{pn}}{1-x^{qn}}$$

are so chosen that the series $\sum a_n/n$ *converges, then for real* x

$$\lim_{x\to 1-}\left\{(1-x)\sum a_n \frac{x^{pn}}{1-x^{qn}}\right\} = \frac{1}{q}\sum (a_n/n).$$

LEMMA 5.3. *If* q, n, *and* k *are natural numbers,* $n, k \geqslant 2$ *and* n *is not an integral power of* k, $y' = (2\pi)/(\ln k)$, *then*

$$2\,|e^{iy'|k'|q(\ln n)} - n^{-qx}| \geqslant y'\{\ln(1+n^{-q|k'|})\}$$

for all x, $0 \leqslant x \leqslant L$, *where* $k' = \pm 1, \pm 2, \pm 3, \ldots$ *and* L *is an arbitrary but fixed constant. If* $k = d^f$ *for some integers* d *and* f *we further require that* $(k'q, f) = 1$.

The following theorem represents the main development for the natural boundary theorem of the P-Q series.

THEOREM 5.3. *Let* $z = x + iy''$, *where* $x > 0$ *and* y'' *is a fixed element of the set* $\{2\pi k'/(\ln k) : k' = \pm 1, \pm 2, \pm 3, \ldots;\ k = 2, 3, 4, \ldots,$ *where if* k *is of the form* $k = d^f$ *for some natural numbers* d *and* f, $k'q$ *and* f *are relatively prime for fixed* $q\}$. *If the coefficients* a_n *are so chosen that*

$$\sum_{n=2}^{\infty} a_n/\ln(1+n^{-q|k'|})$$

converges absolutely, then

$$\lim_{x\to 0+}\left\{x\sum_{n=2}^{\infty} a_n \frac{n^{-pz}}{1-n^{-qz}}\right\} = \frac{1}{q(\ln k)}\sum_{r=1}^{\infty} a_{k^r}/r.$$

Proof. Under the hypothesis $\sum |a_n|$ also converges and thus by Theorem 2.3, the series

$$\sum a_n \frac{n^{-pz}}{1-n^{-qz}} \tag{5.3}$$

converges absolutely for $R(z) > 0$.

Consider now the subset of terms of the series (5.3) for which $n = k^r$, where n, k and r are natural numbers with k fixed. Designate the sum of these terms by $\sum_1$; the sum of all other terms denote by $\sum_2$. $\sum_1$

includes all those terms and only those terms for which $1-n^{-qz}$ vanishes at $z = iy''$.

If we set $e^{-(\ln k)x} = k^{-x} = \xi$, then

$$\sum_1 = \sum_{r=1}^{\infty} a_{k^r} \frac{\xi^{pr}}{1-\xi^{qr}}$$

which is Garvin's series in real variable. The convergence of $\sum |a_n|$ implies that of $\sum_{r=1}^{\infty} |(a_{k^r}/r)|$ and hence by Theorem 5.2

$$\lim_{\xi\to 1^-} \left\{(1-\xi) \sum_{r=1}^{\infty} a_{k^r} \frac{\xi^{pr}}{1-\xi^{qr}}\right\} = \frac{1}{q} \sum_{r=1}^{\infty} a_{k^r}/r$$

or

$$\lim_{x\to 0^+} \left\{(1-k^{-x}) \sum_{r=1}^{\infty} a_{k^r} \frac{k^{-rpx}}{1-k^{-rqx}}\right\} = \frac{1}{q} \sum_{r=1}^{\infty} a_{k^r}/r.$$

Since $x(\ln k)$ is asymptotically equal to $(1-k^{-x})$ as $x \to 0$, this last limit can be rewritten to give

$$\lim_{x\to 0^+} \left\{x \sum_{n=2}^{\infty}{}_1\, a_n \frac{n^{-pz}}{1-n^{-qz}}\right\} = \frac{1}{q(\ln k)} \sum_{r=1}^{\infty} a_{k^r}/r.$$

Now $\sum = \sum_1 + \sum_2$; hence if

$$\lim_{x\to 0^+} (x\sum_2) = 0, \tag{5.4}$$

the theorem is complete. In order to verify (5.4) it suffices to prove the uniform convergence of $\sum_2$ in some closed interval $0 \leqslant x \leqslant M$, where M is a finite constant.

Since $\sum |a_n|$ converges, by Theorem 2.3 $\sum_2$ converges for all $x > 0$. To show uniform convergence in $0 \leqslant x \leqslant M$ consider

$$\sum_2 \left| a_n \frac{n^{-pz}}{1-n^{-qz}} \right| \leqslant \sum_2 \frac{|a_n|}{|1-n^{-qz}|}$$
$$= \sum_2 \frac{|a_n|}{|e^{\pm iy'|k'|q(\ln n)} - n^{-qx}|} \leqslant \frac{2}{y'} \sum_2 \frac{|a_n|}{\ln(1+n^{-q|k'|})}$$

by Lemma 5.3. By hypothesis this latter series of constants converges; hence by the Weierstrass M-test the series $\sum_2$ converges absolutely and uniformly in $0 \leqslant x \leqslant M$, and the theorem follows.

A set of integers will be called *dense* if it is unbounded above and below and there is a positive number l such that every interval of length l contains an integer of the set.

THEOREM 5.4. *If to each positive integer k of an infinite set there corresponds a dense set of k' for which the hypotheses of Theorem 5.3 are fulfilled, and if for each such k*

$$\frac{1}{\ln k}\sum_{r=1}^{\infty} a_{k^r}/r \neq 0,$$

then $x = 0$ is a natural boundary of the function represented by the P-Q series.

The proof consists in showing that the set of singularities $\{i2\pi k'/\ln k\}$ is everywhere dense on the axis of imaginaries.

A result similar to Theorem 5.3 can be obtained for approach to the singular points along any Stolz path by the variable z of the P-Q series with the restriction that $q = cp$, where c is a natural number.

THEOREM 5.5. *If the coefficients of the P-Q series, where $q = cp$ for some natural number c, are so chosen that*

$$\sum_{n=2}^{\infty} \frac{a_n}{\ln(1+n^{-cp|k'|})}$$

converges, then for $R(z) > 0$ and for an approach along any Stolz path to $z' = iy''$, where y'' is a fixed element of the set $\{2\pi k'/\ln k:\ k' = \pm1, \pm2, \pm3, \ldots;\ k = 2, 3, 4, \ldots;$ and when $k = d^f$, where d and f are natural numbers, $(cpk', f) = 1\}$, then

$$\lim_{z\to z'}\left\{(z-z')\sum_{n=2}^{\infty} a_n \frac{n^{-pz}}{1-n^{-qz}}\right\} = \frac{1}{cp(\ln k)}\sum_{r=1}^{\infty} a_{k^r}/r. \tag{5.5}$$

Proof. In the limit on the left-hand side of (5.5) let $pz = t$ and $pz' = t'$. Then this limit is

$$\frac{1}{p}\lim_{t\to t'}\left\{(t-t')\sum_{n=2}^{\infty} a_n \frac{n^{-t}}{1-n^{-ct}}\right\}. \tag{5.6}$$

Under the hypothesis the P-Q series converges absolutely for $R(z) > 0$ and hence may be replaced by its equivalent Dirichlet series. Further for $R(t-t') > 0$

$$t-t' = 1\Big/\sum_{m=1}^{\infty}(t-t'+1)^{-m}$$

so that limit (5.6) can be written

$$\frac{1}{p}\lim_{t\to t'}\left\{\sum_{m=2}^{\infty} h_m m^{-t}\Big/\sum_{m=1}^{\infty}(t-t'+1)^{-m}\right\}.$$

If in this last limit we set $t-t' = r$ and $h_1 = 0$, the conditions of the theorem of Knopp ([3], p. 122) referred to above are fulfilled so that we can consider the limit

$$\frac{1}{p}\lim_{s\to\infty}\left\{\frac{1}{[\ln s]}\sum_{m=2}^{s} h_m m^{-t}\right\}.$$

It is sufficient then to show that for any $\varepsilon > 0$ there exists an N such that

$$\left|\frac{1}{p[\ln s]}\sum_{m=2}^{s} h_m m^{-t} - \frac{1}{pc(\ln k)}\sum_{m=1}^{\infty} a_{k^m}/m\right| < \varepsilon$$

for all $s > N$. Now this absolute value is

$$\leqslant \frac{1}{p(\ln k)}\left|\frac{\ln k}{[\ln s]}\sum_{m=2}^{s} h_m m^{-t'} - \frac{1}{c}\sum_{m=1}^{n} a_{k^m}/m\right| + \frac{\varepsilon}{2}$$

for all n greater than some integer N'. The theorem follows from the fact that for n sufficiently large

$$\left|\frac{\ln k}{[\ln k^n]}\sum_{m=2}^{k^n} h_m m^{-t'} - \frac{1}{c}\sum_{m=1}^{n} a_{k^m}/m\right|$$

and

$$\left|\frac{\ln k}{[\ln k^n]}\sum_{m=k^n+1}^{d'} h_m m^{-t'}\right|$$

can be shown to be each less than $\varepsilon/4$ for any natural number d' such that $k^n+1 \leqslant d' \leqslant k^{n+1}-1$.

References

[1] M. C. Garvin, *A Generalized Lambert Series*, Amer. J. Math. 58 (1936), p. 507-513.

[2] E. S. Kennedy, *Exponential Analogues of the Lambert Series*, bidem 63 (1941), p. 443-460.

[3] K. Knopp, *Grenzwerte von Dirichletshen Reihen bei der Annaherung an die Konvergenzgrenze*, Journal für Mathematik, 138 (1910), p. 109-132.