ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XV (1971)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XV (1971)

Norman LEVINE (Ohio)

On the intersection of open sets containing the diagonal

1. Introduction. In a topological space $(X, \mathscr{T})$, we shall let $E = \bigcap\{O : \Delta \subseteq O \in \mathscr{T} \times \mathscr{T}\}$, Δ denoting the diagonal in $X \times X$. In particular, we will be most interested in the case in which E is an equivalence relation; when this occurs, we will call $(X, \mathscr{T})$ an *e-space* and $\mathscr{T}$ will be called an *e-topology* for X. In Section 2, we introduce the concept of a symmetric space (written henceforth as S-space) and characterize property S in several ways in terms of E. We give four characterizations of property e in Section 3 together with several sufficient conditions for this property. Section 4 is concerned with subspaces of e-spaces, Section 5 is concerned with products of e-spaces and Section 6 is devoted to relationships between regularity and property e. In Section 7, we explore the relationships between property e and normality and finally in Section 8, we study transfers of e-topologies.

Throughout this paper, we use $c(x)$ to denote $c(\{x\})$, c being the closure operator.

THEOREM. 1.1. *In a space* $(X, \mathscr{T})$, *E has the following properties*: (i) $\Delta \subseteq E$, (ii) $E = E^{-1}$ *and* (iii) $(x, y) \in E$ *iff* $c(x) \cap c(y) \neq \emptyset$.

Proof. (i) follows from the definition of E. (ii) follows from the fact that if $\Delta \subseteq O \in \mathscr{T} \times \mathscr{T}$, then $\Delta \subseteq O \cap O^{-1} \in \mathscr{T} \times \mathscr{T}$. Thus E is an intersection of symmetric open sets and hence is symmetric. Finally (iii), suppose that $(x, y) \notin E$. Then there exists a set O^* in $\mathscr{T} \times \mathscr{T}$ such that $(x, y) \notin O^*$ and $\Delta \subseteq O^*$. Then $(x, y) \in \mathscr{C}O^*$, a closed set and hence $c(x) \times c(y) \subseteq \mathscr{C}O^* \subseteq \mathscr{C}\Delta$. Since $c(x) \times c(y) \subseteq \mathscr{C}\Delta$, it follows that $c(x) \cap c(y) = \emptyset$. Conversely, suppose that $c(x) \cap c(y) = \emptyset$.

Then $\Delta \subseteq \mathscr{C}c(x) \times \mathscr{C}c(x) \cup \mathscr{C}c(y) \times \mathscr{C}c(y) \in \mathscr{T} \times \mathscr{T}$. But $(x, y) \notin \mathscr{C}c(x) \times \mathscr{C}c(x) \cup \mathscr{C}c(y) \times \mathscr{C}c(y)$ and thus $(x, y) \notin E$.

2. S-spaces. We will call a space $(X, \mathscr{T})$ an *S-space* iff $x \in c(y)$ implies that $y \in c(x)$.

THEOREM 2.1. *In the space* $(X, \mathscr{T})$, *the following are equivalent*: (i) $(X, \mathscr{T})$ *is an S-space,* (ii) $x \in O \in \mathscr{T}$ *implies that* $c(x) \subseteq O$, (iii) $E[F] = F$ *for each closed set F in X and* (iv) $E[x] = c(x)$ *for each x in X.*

Proof. (i) implies (ii). Let $x \in O \in \mathcal{T}$ and suppose that $c(x) \nsubseteq O$. Take $y \in c(x) \cap \mathscr{C}O$ and observe that $y \in c(x)$, but $x \notin c(y)$.

(ii) implies (iii). It suffices to show that $E[F] \subseteq F$ for each closed set F. Let $x \in E[F]$ and suppose that $x \notin F$. By (ii), $c(x) \subseteq \mathscr{C}F$. Now $(x, y) \in E$ for some y in F and by (iii) of Theorem 1.1, $c(x) \cap c(y) \neq \emptyset$. But $c(y) \subseteq F$ since F is closed and hence $c(x) \cap c(y) = \emptyset$, a contradiction.

(iii) implies (iv). $E[x] \subseteq E[c(x)] = c(x)$ by (iii). It suffices therefore to show that $c(x) \subseteq E[x]$. If $y \in c(x)$, then $c(y) \cap c(x) \neq \emptyset$ and by (iii) of Theorem 1.1, $(x, y) \in E$ and $y \in E[x]$.

(iv) implies (i). Suppose $x \in c(y)$, but $y \notin c(x)$. By (iv), $y \notin E[x]$ and hence by (iii) of Theorem 1.1, $c(x) \cap c(y) = \emptyset$. But $x \in c(x) \cap c(y)$, a contradiction.

3. *e*-spaces.

THEOREM. 3.1. *In* $(X, \mathcal{T})$, *the following are equivalent*: (i) $E \circ E = E$, (ii) $F_1 \cap F_2 = \emptyset$, F_i *closed*, x *in* X *implies that* $c(x) \cap F_1 = \emptyset$ *or* $c(x) \cap F_2 = \emptyset$, (iii) $X = O_1 \cup O_2$, O_i *open*, x *in* X *implies that* $c(x) \subseteq O_1$ *or* $c(x) \subseteq O_2$, (iv) $c(x) \cap c(y) \neq \emptyset \neq c(y) \cap c(z)$ *implies that* $c(x) \cap c(z) \neq \emptyset$, (v) $c(x) \cap c(y) = \emptyset$ *and* z *in* X *implies that* $c(x) \cup c(y) \nsubseteq c(z)$.

Proof. (i) implies (ii). Let F_1 and F_2 be disjoint closed sets and suppose $x \in X$. If $c(x) \cap F_1 \neq \emptyset$ and $c(x) \cap F_2 \neq \emptyset$, take $y \in c(x) \cap F_1$ and $z \in c(x) \cap F_2$. Then $c(y) \cap c(z) \subseteq F_1 \cap F_2 = \emptyset$. But $c(x) \cap c(y) \neq \emptyset$ and $c(z) \cap c(x) \neq \emptyset$ and by (iii) of Theorem 1.1, $(x, y) \in E$ and $(x, z) \in E$. Thus $(y, z) \in E$ and $c(y) \cap c(z) \neq \emptyset$, a contradiction.

(ii) implies (iii). Let $X = O_1 \cup O_2$, O_i being open and let $x \in X$. Then $\emptyset = \mathscr{C}O_1 \cap \mathscr{C}O_2$ and by (ii), we may assume that $c(x) \cap \mathscr{C}O_1 = \emptyset$. Then $c(x) \subseteq O_2$.

(iii) implies (iv). Let $c(x) \cap c(y) \neq \emptyset \neq c(y) \cap c(z)$ and suppose that $c(x) \cap c(z) = \emptyset$. Then $X = \mathscr{C}c(x) \cup \mathscr{C}c(z)$, $\mathscr{C}c(x)$ and $\mathscr{C}c(y)$ being open. However, $c(y) \nsubseteq \mathscr{C}c(x)$ and $c(y) \nsubseteq \mathscr{C}c(z)$, contrary to (iii).

(iv) implies (v). Let $c(x) \cap c(y) = \emptyset$ and suppose that $c(x) \cup c(y) \subseteq c(z)$. Then $c(x) \cap c(z) \neq \emptyset \neq c(y) \cap c(z)$ and by (iv), $c(x) \cap c(y) \neq \emptyset$, a contradiction.

(v) implies (i). Let $(x, y) \in E$ and $(y, z) \in E$. By (iii) of Theorem 1.1, $c(x) \cap c(y) \neq \emptyset \neq c(y) \cap c(z)$. Take $u \in c(x) \cap c(y)$ and $v \in c(y) \cap c(z)$. Then $c(u) \cap c(v) \subseteq c(x) \cap c(z)$. Since $c(u) \cap c(v) \subseteq c(y)$, it follows by (v) that $c(u) \cap c(v) \neq \emptyset$ and hence $c(x) \cap c(z) \neq \emptyset$. Thus $(x, z) \in E$.

THEOREM. 3.2. *The following are sufficient conditions for* $(X, \mathcal{T})$ *to be an e-space*: (i) X *is an S-space*, (ii) X *is* T_1, (iii) X *is regular*, (iv) E *is closed and* (v) X *is normal*.

Proof. (i) We use (iii) of Theorem 3.1. Suppose $X = O_1 \cup O_2$ and $x \in X$. Assume $x \in O_1$. By (ii) of Theorem 2.1, we obtain $c(x) \subseteq O_1$.

(ii) T_1 implies S.

(iii) Regular implies S.

(iv) Suppose $X = O_1 \cup O_2$ and $x \in X$. If $c(x) \nsubseteq O_1$ and $c(x) \nsubseteq O_2$, take $p \in c(x) \cap \mathscr{C}O_1$ and $q \in c(x) \cap \mathscr{C}O_2$. Since E is closed, it follows that $(p, q) \in c(x) \times c(x) \subseteq E \subseteq O_1 \times O_1 \cup O_2 \times O_2$. But $(p, q) \notin O_1 \times O_1 \cup O_2 \times O_2$, a contradiction.

(v) Let $X = O_1 \cup O_2$ and $x \in X$. Since X is normal, there exist closed sets F_1 and F_2 such that $X = F_1 \cup F_2$ and $F_i \subseteq O_i$. Assume $x \in F_1$. Then $c(x) \subseteq F_1 \subseteq O_1$ and X is an e-space by (iii) of Theorem 3.1.

Whenever $(X, \mathscr{T})$ is an e-space, $(X/E, \mathscr{T}/E)$ will denote the quotient space with the quotient topology. $P\colon X \to X/E$ is the canonical map $P(x) = E[x]$.

COROLLARY 3.3. *If* $(X, \mathscr{T})$ *is an* S*-space, then* $P : X \to X/E$ *is a closed map.*

Proof. By (i) of Theorem 3.2, X is an e-space and (iii) of Theorem 2.1, P is a closed map.

EXAMPLE 3.4. Let $X = \{a, b\}$ and $\mathscr{T} = \{\emptyset, \{a\}, X\}$. Then $(X, \mathscr{T})$ is an e-space which is not an S-space (and hence neither T_1 nor regular). See Theorem 3.2.

EXAMPLE 3.5. Let $(X, \mathscr{T})$ be an infinite space with the cofinite topology. Then X is an e-space since it is T_1. However, $E = \Delta$ and E is not closed. Furthermore, X is not normal. See Theorem 3.2.

4. Subspaces.

THEOREM 4.1. *Let* $(X, \mathscr{T})$ *be an e-space and suppose that* $(Y, \mathscr{U})$ *is a closed subspace. Then* $(Y, \mathscr{U})$ *is an e-space.*

Proof. If $y \in Y$ and c^* denotes the closure operator in the Y-space, then $c(y) = c^*(y)$. Apply (iv) of Theorem 3.1.

THEOREM 4.2. *Let* $X = \bigcup \{E_\gamma : \gamma \in \Gamma\}$, *where* E_γ *is a closed subset of* X. *Then* X *is an e-space iff* E_γ *is an e-space for each* γ *in* Γ.

Proof. The necessity follows from Theorem 4.1. We employ (iii) of Theorem 3.1 to show the sufficiency. Let $X = O_1 \cup O_2$, O_i being open and $x \in X$. Then $x \in E_\gamma$ for some γ in Γ and $E_\gamma = (E_\gamma \cap O_1) \cup (E_\gamma \cap O_2)$. Since E_γ is an e-space, $c_\gamma(x) \subseteq E_\gamma \cap O_1$ or $c_\gamma(x) \subseteq E_\gamma \cap O_2$. But $c_\gamma(x) = c(x)$ and hence $c(x) \subseteq O_1$ or $c(x) \subseteq O_2$.

5. Product spaces.

THEOREM 5.1. *Let* $(X, \mathscr{T}) = \bigtimes \{(X_\gamma, \mathscr{T}_\gamma) : \gamma \in \Gamma\}$. *Then* $(X, \mathscr{T})$ *is an e-space iff* $(X_\gamma, \mathscr{T}_\gamma)$ *is an e-space for each* $\gamma \in \Gamma$.

Proof. Suppose $(X, \mathscr{T})$ is an e-space and let $\gamma \in \Gamma$. We use (iii) of Theorem 3.1. Let $X_\gamma = O_\gamma \cup U_\gamma$, O_γ and U_γ being open in X_γ and let $x_\gamma \in X_\gamma$. Then $X = P_\gamma^{-1}[O_\gamma] \cup P_\gamma^{-1}[U_\gamma]$ and $P_\gamma(x) = x_\gamma$ for some x in X. We may assume that $c(x) \subseteq P_\gamma^{-1}[O_\gamma]$. Then $c_\gamma(x_\gamma) = P_\gamma[c(x)] \subseteq P_\gamma P_\gamma^{-1}[O_\gamma] = O_\gamma$.

Conversely, suppose that $(X_\gamma, \mathscr{T}_\gamma)$ is an e-space for each γ in Γ. We use (iv) of Theorem 3.1. Let $c(x) \cap c(y) \neq \emptyset \neq c(y) \cap c(z)$. Then for each γ in Γ, $c_\gamma(x(\gamma)) \cap c_\gamma(y(\gamma)) \neq \emptyset \neq c_\gamma(y(\gamma)) \cap c_\gamma(z(\gamma))$. It follows then that $c_\gamma(x(\gamma)) \cap c_\gamma(z(\gamma)) \neq \emptyset$ for each γ in Γ and hence $c(x) \cap c(z) \neq \emptyset$.

6. Regular spaces.

THEOREM 6.1. *If $(X, \mathscr{T})$ is regular, then E is closed.*

Proof. Suppose $(x, y) \notin E$. By (iii) of Theorem 1.1, $c(x) \cap c(y) = \emptyset$ and hence $x \notin c(y)$. Since $(X, \mathscr{T})$ is regular, there exist open sets O_1 and O_2 for which $x \in O_1$, $c(y) \subseteq O_2$ and $c(O_1) \cap c(O_2) = \emptyset$. Hence $(x, y) \in O_1 \times \times O_2 \subseteq \mathscr{C}E$ by (iii) of Theorem 1.1, and hence E is closed.

THEOREM. 6.2. *If $(X, \mathscr{T})$ is regular, then $P: X \to X/E$ is an open map.*

Proof. By (iii) of Theorem 3.2, $(X, \mathscr{T})$ is an e-space. It suffices to show that $E[O]$ is open whenever O is open. We will show in fact that $E[O] = O$. Let $x \in E[O]$. Then by (iii) of Theorem 1.1, $c(x) \cap c(y) \neq \emptyset$ for some $y \in O$. Since $(X, \mathscr{T})$ is regular, there exists an open set O^* for which $y \in O^* \subseteq c(O^*) \subseteq O$. If $x \notin c(O^*)$, there exists an open set $O^\#$ such that $x \in O^\# \subseteq c(O^\#) \subseteq \mathscr{C}c(O^*)$. But then $c(x) \subseteq \mathscr{C}cO^* \subseteq \mathscr{C}c(y)$ and $c(x) \cap c(y) = \emptyset$, a contradiction. Thus $x \in c(O^*) \subseteq O$ and $E[O] \subseteq O$.

COROLLARY 6.3. *If $(X, \mathscr{T})$ is regular, then $(X/E, \mathscr{T}/E)$ is T_2.*

Proof. Use Theorem 6.1, Theorem 6.2 and the fact that E closed, $P: X \to X/E$ open implies that $(X/E, \mathscr{T}/E)$ is T_2 (see [3], p. 98).

COROLLARY 6.4. *If $(X, \mathscr{T})$ is regular, then $(X/E, \mathscr{T}/E)$ is regular.*

Proof. Let F^* be closed in X/E and suppose $y^* \in X/E - F^*$. Take $x \in X$ such that $P(x) = y^*$. Then $x \notin P^{-1}[F^*]$ and since $(X, \mathscr{T})$ is regular, there exist open sets O_1 and O_2 for which $x \in O_1$, $P^{-1}[F^*] \subseteq O_2$ and $c(O_1) \cap \cap c(O_2) = \emptyset$. Then $y^* \in P[O_1]$ and $F^* \subseteq P[O_2]$ and $P[O_i]$ is open by Theorem 6.2. It suffices to show that $P[O_1] \cap P[O_2] = \emptyset$. Let $x_1 \in O_1$ and $x_2 \in O_2$. Then $c(x_1) \cap c(x_2) = \emptyset$ and $(x_1, x_2) \notin E$. Then $P(x_1) = E[x_1] \neq E[x_2] = P(x_2)$ and $P(x_1) \neq P(x_2)$.

COROLLARY 6.5. *$(X, \mathscr{T})$ is regular iff $(X, \mathscr{T})$ is an S-space and $(X/E, \mathscr{T}/E)$ is regular.*

Proof. The necessity follows from Corollary 6.4 and the fact that regularity implies property S. To show the sufficiency, let $x \notin F$, F being a closed set in X. Since $(X, \mathscr{T})$ is an S-space, $c(x) \cap F = \emptyset$ by (ii) of Theorem 2.1. It follows then that $P(x) \cap P[F] = \emptyset$ and that $P[F]$ is a closed set by Corollary 3.3. Since $(X/E, \mathscr{T}/E)$ is regular, there exist disjoint open sets O_1^* and O_2^* in X/E such that $P(x) \in O_1^*$ and $P[F] \subseteq O_2^*$. Then $x \in P^{-1}[O_1^*]$ and $F \subseteq P^{-1}[O_2^*]$ and $P^{-1}[O_1^*] \cap P^{-1}[O_2^*] = \emptyset$.

7. Normal spaces.

Lemma 7.1. *Let $(X, \mathscr{T})$ be an e-space and suppose that $(X/E, \mathscr{T}/E)$ is T_2. If $c(x) \cap c(y) = \emptyset$, there exist disjoint open sets O_1 and O_2 in X such that $c(x) \subseteq O_1$, $c(y) \subseteq O_2$.*

Proof. $c(x) \cap c(y) = \emptyset$ implies by (iii) of Theorem 1.1 that $(x, y) \notin E$ and hence $E[x] \neq E[y]$ in X/E. Since X/E is T_2, there exist disjoint open sets O_1^* and O_2^* in X/E such that $E[x] \in O_1^*$ and $E[y] \in O_2^*$. But $c(x) \subseteq E[x] \subseteq P^{-1}[O_1^*]$ and $c(y) \subseteq P^{-1}[O_2^*]$. Let $O_1 = P^{-1}[O_1^*]$ and $O_2 = P^{-1}[O_2^*]$.

Lemma 7.2. *Let $(X, \mathscr{T})$ be a compact e-space and suppose that $(X/E, \mathscr{T}/E)$ is T_2. If $c(x) \cap F = \emptyset$, F being a closed set, there exist disjoint open sets O_1 and O_2 in X such that $c(x) \subseteq O_1$ and $F \subseteq O_2$.*

Proof. $y \in F$ implies that $c(x) \cap c(y) = \emptyset$ and by the previous lemma, there exist disjoint open sets O_y and O_y' such that $c(x) \subseteq O_y$ and $c(y) \subseteq O_y'$. Since F is compact, $F \subseteq \bigcup \{O_{y_i}' : i = 1, \ldots, n\} = O_2$ and $c(x) \subseteq \bigcap \{O_{y_i} : i = 1, \ldots, n\} = O_1$.

Theorem 7.3. *Let $(X, \mathscr{T})$ be a compact e-space and suppose that $(X/E, \mathscr{T}/E)$ is T_2. Then $(X, \mathscr{T})$ is normal.*

Proof. Let $F_1 \cap F_2 = \emptyset$, F_i being closed sets. Then $x \in F_1$ implies that $c(x) \cap F_2 = \emptyset$. Applying Lemma 7.2, there exist disjoint open sets O_x and O_x' such that $c(x) \subseteq O_x$ and $F_2 \subseteq O_x'$. Since F_1 is compact, $F_1 \subseteq \bigcup \{O_{x_i} : i = 1, \ldots, n\}$ and $F_2 \subseteq \bigcap \{O_{x_i}' : i = 1, \ldots, n\}$.

Theorem. *$(X, \mathscr{T})$ is normal iff* (i) *$(X, \mathscr{T})$ is an e-space,* (ii) *$P : X \to X/E$ is a closed map and* (iii) *$(X/E, \mathscr{T}/E)$ is normal.*

Proof. *Necessity.* (i) follows from (v) of Theorem 3.2. To show (ii), it suffices to show that $E[F]$ is closed whenever F is closed. Let $x \notin E[F]$. Then by (iii) of Theorem 1.1, $c(x) \cap c(y) = \emptyset$ for all $y \in F$ and thus $c(x) \cap F = \emptyset$. Since $(X, \mathscr{T})$ is normal, there exist disjoint sets O_1 and O_2 such that $c(x) \subseteq O_1$ and $F \subseteq O_2$. It follows then that $x \in O_1 \subseteq \mathscr{C}E[F]$ and $E[F]$ is closed. (iii) follows from the fact that normality is invariant under closed, continuous surjections (see [3], p. 134, M).

Sufficiency. Let $F_1 \cap F_2 = \emptyset$, F_i being closed in X. Then $x \in F_1$ and $y \in F_2$ implies that $c(x) \cap c(y) = \emptyset$ and by (iii) of Theorem 1.1, $(x, y) \notin E$. Thus $E(x) \neq E(y)$ and $P(x) \neq P(y)$. Hence $P[F_1] \cap P[F_2] = \emptyset$. Since $P : X \to X/E$ is presumed to be a closed map, $P[F_i]$ is a closed set for each i. By the normality of X/E, there exist disjoint open sets O_1^* and O_2^* in X/E for which $P[F_1] \subseteq O_1^*$ and $P[F_2] \subseteq O_2^*$. Then $F_1 \subseteq P^{-1}[O_1^*]$ and $F_2 \subseteq P^{-1}[O_2^*]$.

Lemma 7.5. *Let $(X, \mathscr{T})$ be an e-space. Then $E[x] = \bigcup \{c(y) : c(y) \cap c(x) \neq \emptyset\}$ for each x in X.*

Proof. If $z \in E[x]$, then $c(z) \cap c(x) \neq \emptyset$ by (iii) Theorem 1.1 and

$z \in \bigcup \{c(y) : c(y) \cap c(x) \neq \emptyset\}$. Conversely, let $z \in c(y)$, where $c(y) \cap c(x) \neq \emptyset$. But $c(z) \cap c(y) \neq \emptyset$ and by (iv) of Theorem 3.1, $c(z) \cap c(x) \neq \emptyset$. Hence $z \in E[x]$ by (iii) of Theorem 1.1.

THEOREM 7.6. *Let* $(X, \mathscr{T})$ *be a finite topological space. Then* $(X, \mathscr{T})$ *is an e-space iff* $(X, \mathscr{T})$ *is normal.*

Proof. The sufficiency follows from (v) of Theorem 3.2. To show the necessity, by Theorem 7.3, it suffices to show that $(X/E, \mathscr{T}/E)$ is T_2. Since X/E is finite, it suffices to show that $(X/E, \mathscr{T}/E)$ is T_1 or that $\{E[x]\}$ is a closed set in X/E. But $E[x]$ by Lemma 7.5 is a finite union of closed sets in X and the theorem follows.

8. Transfers of *e*-spaces.

THEOREM 8.1. *Let* $f: (X, \mathscr{T}) \to (Y, \mathscr{U})$ *be a continuous, closed, surjection. If* $(X, \mathscr{T})$ *is an e-space, then* $(Y, \mathscr{U})$ *is an e-space.*

Proof. We employ (iii) of Theorem 3.1. Let $y \in Y$ and $Y = U_1 \cup U_2$, U_i being open in Y. Then $y = f(x)$ for some $x \in X$ and $X = f^{-1}[U_1] \cup f^{-1}[U_2]$. Since X is an *e*-space, we may assume that $c(x) \subseteq f^{-1}[U_1]$. Then $c(y) = c(f(x)) \subseteq c\big(f(c(x))\big) = f(c(x)) \subseteq ff^{-1}[U_1] = U_1$.

EXAMPLE 8.2. In Theorem 8.1, closed cannot be replaced by open. For let $X = [0, 1]$ the closed unit interval with the usual topology and let $Y = \{a, b, c\}$ with $\mathscr{U} = \big\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\big\}$. If $f(0) = b, f(1) = c$ and $f(x) = a$, when $0 < x < 1$, then $f: X \to Y$ is a continuous, open surjection and X is an *e*-space. But Y is a finite, non-normal space, and by Theorem 7.6, Y is not an *e*-space.

Since continuous open surjections are identifications, it follows that property *e* is not invariant under identifications. However,

THEOREM 8.3. *Let* $f: (X, \mathscr{T}) \to (Y, \mathscr{U})$ *be a surjection. Suppose that* $\mathscr{T} = \{f^{-1}[U] : U \in \mathscr{U}\}$, *that is,* $\mathscr{T}$ *is the weak topology for* X *determined by* f *and* $\mathscr{U}$. *Then* $(X, \mathscr{T})$ *is an e-space iff* $(Y, \mathscr{U})$ *is an e-space.*

Proof. *Necessity.* Let $y \in Y$, $Y = U_1 \cup U_2$, U_i being open. By (iii) of Theorem 3.1, it suffices to show that $c(y) \subseteq U_1$ or $c(y) \subseteq U_2$. Now $y = f(x)$ for some x in X and $X = f^{-1}[U_1] \cup f^{-1}[U_2]$. Since $(X, \mathscr{T})$ is an *e*-space, we may assume that $c(x) \subseteq f^{-1}[U_1]$. Thus $X = f^{-1}[U_1] \cup \mathscr{C}c(x)$. But $\mathscr{C}c(x)$ is open in X and hence $\mathscr{C}c(x) = f^{-1}[U^*]$ for some $U^* \in \mathscr{U}$. Thus

$$X = f^{-1}[U_1] \cup f^{-1}[U^*]$$

and

$$Y = f[X] = U_1 \cup U^*.$$

But $y \notin U^*$; for if $y \in U^*$, then $f(x) \in U^*$ and

$$x \in f^{-1}[U^*] = \mathscr{C}c(x),$$

a contradiction. Thus $y \in \mathscr{C}U^*$ and hence $c(y) \subseteq \mathscr{C}U^* \subseteq U_1$.

Sufficiency. Let $X = O_1 \cup O_2$ and $x \in X$. But $O_i = f^{-1}[U_i]$ for some open U_i in Y. Hence $Y = f(X) = f\big[f^{-1}[U_1] \cup f^{-1}[U_2]\big] = U_1 \cup U_2$ and $f(x) \in Y$. We may assume that $c\big(f(x)\big) \subseteq U_1$ since Y is an e-space. But $f\big(c(x)\big) \subseteq c\big(f(x)\big) \subseteq U_1$ and hence $c(x) \subseteq f^{-1}[U_1] = O_1$.

References

[1] Helen F. Cullen, *Introduction to general topology*, Boston 1968.
[2] James Dugundji, *Topology*, Allyn and Bacon, 1966.
[3] John L. Kelley, *General topology*, Princeton 1955.