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The Banach–Mazur functor and related functors

The purpose of this paper is to find the adjoint functors of some functors appearing in functional analysis.

If X is a topological space, $\mathscr{C}(X)$ will denote the space of bounded scalar-valued continuous functions on X with the supremum norm; the scalar field is either the field $\boldsymbol{R}$ of reals or the field $\boldsymbol{C}$ of complex numbers. If $\varphi\colon X \longrightarrow Y$ is a continuous map,

$$\mathscr{C}(\varphi)\colon \mathscr{C}(Y) \longrightarrow \mathscr{C}(X)$$

will denote the induced linear operator defined as $\mathscr{C}\varphi \cdot g = g \circ \varphi$ for g in $\mathscr{C}(Y)$. If F is a Banach space,

$$\bigcirc^* F = \{\xi \in F^*\colon \|\xi\| \leqslant 1\}$$

will denote the unit ball of F^* provided with the *weak topology. If $f \in F$, $\varkappa_F f$ will denote the canonical image of f in $\mathscr{C}(\bigcirc^* F)$ defined as $\varkappa_F f . \xi = \xi(f)$ or ξ in $\bigcirc^* F$. It is well known that the canonical map

$$\varkappa_F\colon F \longrightarrow \mathscr{C}(\bigcirc^* F)$$

is a linear isometrical injection; it will be called the *Banach–Mazur embedding* of F. It has the following property (announced in [11]):

For every compact space X and every linear contraction $a\colon F \longrightarrow \mathscr{C}(X)$ there is a unique continuous map $\varphi\colon X \longrightarrow \bigcirc^* F$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\varkappa_F"] \arrow[dr, "a"'] & \mathscr{C}(\bigcirc^* F) \arrow[d, dashed, "\mathscr{C}(\varphi)"] \\ & \mathscr{C}(X) \end{tikzcd}$$

This theorem can be generalized if compact spaces are replaced by compact spaces with base points. If $x_0 \in X$, let

(1) $$\mathscr{C}_0(X \| x_0) = \{f \in \mathscr{C}(X) \colon f(x_0) = 0\};$$

if $\varphi \colon X \longrightarrow Y$ is a continuous map such that $\varphi(x_0) = y_0$, let $\mathscr{C}_0(\varphi)$ denote the restriction of $\mathscr{C}(\varphi)$ to $\mathscr{C}(Y \| y_0)$.

PROPOSITION 1. *Let x_0 be any point of a compact space X and let $\alpha \colon F \longrightarrow \mathscr{C}_0(X \| x_0)$ be any linear contraction. Then there exists a unique continuous map $\varphi \colon X \longrightarrow \bigcirc^* F$ such that $\varphi(x_0) = 0$ and the diagram*

(2) $$\begin{tikzcd} F \arrow[r, "\varkappa_F"] \arrow[dr, "\alpha"'] & \mathscr{C}_0(\bigcirc^* F \| 0) \arrow[d, dashed, "\mathscr{C}_0(\varphi)"] \\ & \mathscr{C}_0(X \| x_0) \end{tikzcd}$$

is commutativ.

Proof. If $\gamma \colon F \longrightarrow G$ is any linear contraction, let

$$\bigcirc^* \gamma \colon \bigcirc^* G \longrightarrow \bigcirc^* F$$

denote the restriction of the conjugate map $\gamma^* \colon G^* \longrightarrow F^*$ to the unit ball $\bigcirc^* G$; it is continuous (with respect to *weak topologies). Let us consider the diagram

$$\begin{tikzcd} F \arrow[r, "\varkappa_F"] \arrow[d, "\alpha"'] & \mathscr{C}_0(\bigcirc^* F \| 0) \arrow[d, "\mathscr{C}_0(\bigcirc^* \alpha)"] \\ \mathscr{C}_0(X \| x_0) & \mathscr{C}_0(\bigcirc^* \mathscr{C}_0(X \| x_0) \| 0) \arrow[l, "\mathscr{C}_0(\delta)"] \end{tikzcd}$$

where $\delta \colon X \longrightarrow \bigcirc^* \mathscr{C}_0(X \| x_0)$ is the canonical map which assigns to each point x in X the functional $\delta_x(g) = g(x)$ for g in $\mathscr{C}_0(X \| x_0)$. It is well known that δ is continuous; moreover, $\delta_{x_0} = 0$. We claim that the map $\varphi = \bigcirc^* \alpha \circ \delta$ has the desired property.

It is clear that $\varphi(x_0) = 0$. In order to verify the commutativity of diagram (2) let us consider any f in F. Then $\mathscr{C}_0 \varphi . \varkappa_F f$ is a function on X whose value at a point x of X is

$$\begin{aligned} (\mathscr{C}_0 \varphi . \varkappa_F f) x &= \mathscr{C}_0(\bigcirc^* \alpha \circ \delta) . \varkappa_F f . x = \varkappa_F f . \bigcirc^* \alpha . \delta_x \\ &= \varkappa_F f . (\delta_x \circ \alpha) = \delta_x(\alpha f) = \alpha f . x . \end{aligned}$$

Thus, $\mathscr{C}_0\varphi.\varkappa_F f = af$, i.e., $\mathscr{C}_0\varphi.\varkappa_F = a$. In order to prove that φ is the unique map satisfying this condition, let us suppose that $\psi\colon X \to \bigcirc^* F$ is a continuous map such that $\psi(x_0) = 0$ and $\mathscr{C}_0\psi.\varkappa_F = a$. Let $x \in X$ and $f \in F$. Then $\psi(x) \in \bigcirc^* F$ and

$$\psi x.f = \varkappa_F f.\psi x = (\mathscr{C}_0\psi.\varkappa_F f)x = af.x = \delta_x(af) = (\bigcirc^* a.\delta_x)f.$$

Therefore $\psi = \bigcirc^* a \circ \delta$. This concludes the proof.

DEFINITION. The *Banach–Mazur functor* $\mathscr{C}\bigcirc^*$: $\mathbf{Ban}_1 \to \mathbf{Bcf}$ is the composite of the two functors

$$\text{(3)} \qquad \mathbf{Ban}_1 \xrightarrow{\bigcirc^*} \mathbf{Comp} \quad \text{and} \quad \mathbf{Comp} \xrightarrow{\mathscr{C}} \mathbf{Bcf},$$

where $\mathbf{Ban}_1$ denotes the category of Banach spaces and linear contractions, $\mathbf{Comp}$ denotes the category of compact spaces and continuous maps, $\mathbf{Bcf}$ denotes the category of spaces of the type $\mathscr{C}(X)$ and operators of the type $\mathscr{C}(\varphi)$.

In other words, the Banach–Mazur functor assigns to each Banach space F the corresponding space $\mathscr{C}(\bigcirc^* F)$ and to each linear contraction $a\colon F \to G$ the induced map $\mathscr{C}(\bigcirc^* a)$ from $\mathscr{C}(\bigcirc^* F)$ to $\mathscr{C}(\bigcirc^* G)$. It is clear that each of the functors (3) is contravariant; consequently, the Banach–Mazur functor is covariant. Moreover, for each linear contraction $a\colon F \to G$, $\mathscr{C}(\bigcirc^* a)$ is the unique $\mathbf{Bcf}$-morphism from $\mathscr{C}(\bigcirc^* F)$ to $\mathscr{C}(\bigcirc^* G)$ such that the diagram

$$\begin{tikzcd}
F \arrow[r, "\varkappa_F"] \arrow[d, "a"'] & \mathscr{C}(\bigcirc^* F) \arrow[d, dashed, "\mathscr{C}(\bigcirc^* a)"] \\
G \arrow[r, "\varkappa_G"'] & \mathscr{C}(\bigcirc^* G)
\end{tikzcd}$$

is commutative. Therefore the Banach–Mazur functor is a left adjoint ([5], p. 80; [13], 12.1.1) of the forgetful functor $\square$: $\mathbf{Bcf} \to \mathbf{Ban}_1$ and the Banach–Mazur embedding yields the corresponding canonical natural transformation.

We shall now deal with the 10 categories and 34 functors exhibited below (Fig. 1). Let us explain the notation and terminology. $\mathbf{Ens}$ denotes the category of sets and (all) maps; $\mathbf{Compconv}$ denotes the category of compact convex sets (subsets of locally convex Hausdorff spaces) and continuous affine maps (affine = preserving convex combinations); $\mathbf{Comp}_\bullet$ denotes the category of compact spaces with base points and base-point-preserving continuous maps; $\mathbf{Compsaks}$ denotes the subcategory of $\mathbf{Compconv}$ consisting of compact Saks spaces and center-preserving continuous affine maps (by a compact Saks space we mean a set of the form

$$K = \{b \in B\colon \ \|b\| \leqslant 1\},$$

where $(B, \|\ \|)$ is a Banach space provided with a coarser locally convex Hausdorff topology τ such that (K, τ) is compact, cf., e.g., [7], [15]).

An object of **Bf** means the space $l_\infty(S)$ of all bounded scalar-valued functions on a set S; a **Bf**-morphism from $l_\infty(S)$ to $l_\infty(T)$ is a map of the form $l_\infty(\varphi)$, where $\varphi\colon T \to S$ is any map and $l_\infty\varphi . f = f \circ \varphi$ for f in $l_\infty(S)$.

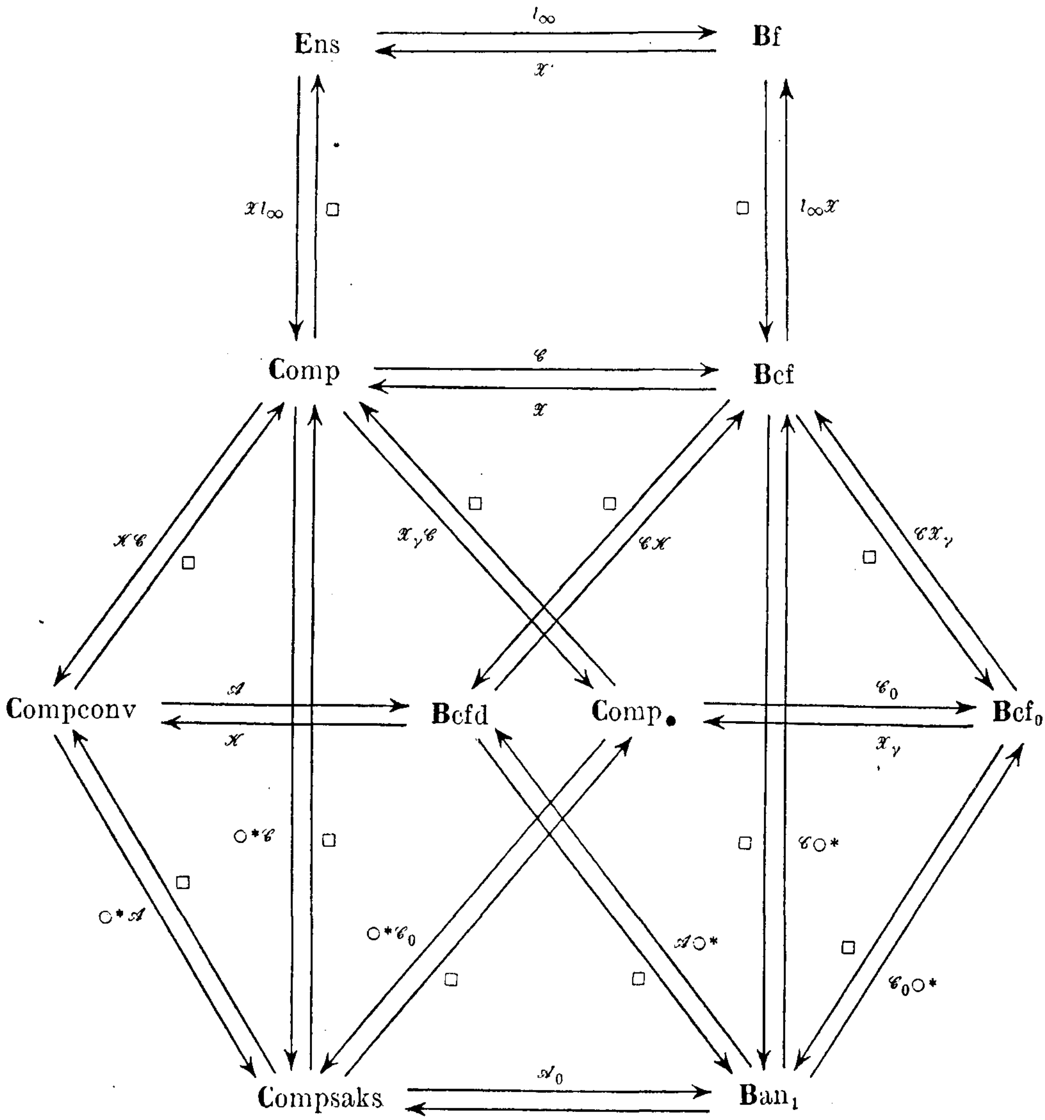

Fig. 1

A non-zero object of **Bcfd** is a closed subspace F of a space $\mathscr{C}(X)$ satisfying the condition $1_X \in F$, where 1_X denotes the constant function 1 on X; a zero object is $\{0\}$. A **Bcfd**-morphism from a subspace F of $\mathscr{C}(X)$ to a non-zero subspace G of $\mathscr{C}(Y)$ is a linear operator $\Phi\colon F \to G$ such that $\|\Phi\| = 1$ and $\Phi(1_X) = 1_Y$; a **Bcfd**-morphism from F to $\{0\}$ is the zero map; there is no **Bcfd**-morphism from $\{0\}$ to $F \neq \{0\}$.

An object of $\mathbf{Bcf}_0$ means a space of the form

$$\mathscr{C}_0(X\|A) = \{f \in \mathscr{C}(X)\colon x \in A \Rightarrow f(x) = 0\},$$

where A is a subset of X; if $B \subset Y$, then a $\mathbf{Bcf}_0$-morphism from $\mathscr{C}_0(Y\|B)$ to $\mathscr{C}_0(X\|A)$ is a map of the form $\mathscr{C}_0(\varphi)$, where $\varphi\colon X \to Y$ is a continuous map such that $\varphi(A) \subset B$ and $\mathscr{C}_0(\varphi)$ is the restriction of $\mathscr{C}(\varphi)$ to $\mathscr{C}_0(Y\|B)$. It is clear that any space $\mathscr{C}(X)$ and any space $\mathscr{C}(X\|x_0)$ are objects of $\mathbf{Bcf}_0$.

Let us note that each of 10 categories in Fig. 1 is complete in the sense of Freyd [5] provided that the empty compact space is regarded as an object in **Comp**, the empty set is regarded as an object of **Compconv**, and the one-point compact Saks space (the unit ball of a zero Banach space) is regarded as an object of **Compsaks**; with these conventions, $l_\infty(\emptyset)$, $\mathscr{C}(\emptyset)$, $\mathscr{C}_0(\emptyset\|\emptyset)$, and $\mathscr{A}(\emptyset)$ are spaces consisting of the single element 0 only.

The 10 functors drawn horizontally in Fig. 1 are contravariant. There are 5 functors directed to the right: l_∞, $\mathscr{C}$, $\mathscr{C}_0$, $\mathscr{A}$, $\mathscr{A}_0$. The functors l_∞, $\mathscr{C}$ and $\mathscr{C}_0$ were defined above. If K is a compact convex set, $\mathscr{A}(K)$ is the subspace of $\mathscr{C}(K)$ consisting of continuous affine functions on K. If K is a compact convex set with a distinguished point x_0, then $\mathscr{A}_0(K)$ is the set of all f in $\mathscr{A}(K)$ such that $f(x_0) = 0$; in particular, if K is a compact Saks space, the distinguished point is the center of K. If K and K' are compact convex sets and $\varphi\colon K \to K'$ is a continuous affine map, then $\mathscr{A}(\varphi)$ is the map from $\mathscr{A}(K')$ to $\mathscr{A}(K)$ defined as $\mathscr{A}\varphi.h = h\circ\varphi$ for h in $\mathscr{A}(K')$; if K and K' are compact Saks spaces and $\varphi\colon K \to K'$ is a continuous affine map such that $\varphi(0) = 0$, then $\mathscr{A}_0(\varphi)$ is the restriction of $\mathscr{A}(\varphi)$ to $\mathscr{A}_0(K')$.

There are 5 contravariant functors directed to the left in Fig. 1: $\mathscr{X}'$, $\mathscr{X}$, $\mathscr{X}_\gamma$, $\mathscr{K}$, $\bigcirc^*$. If F is an object of **Bcf** or $\mathbf{Bcf}_0$, $\mathscr{X}(F)$ denotes the set of non-zero multiplicative linear functionals on F with the *weak topology. The Gelfand functor $\mathscr{X}$ assigns to each **Bcf**-morphism $\Phi\colon F \to G$ the map

$$\mathscr{X}(\Phi)\colon \mathscr{X}(G) \to \mathscr{X}(F)$$

defined as the restriction of Φ^* to $\mathscr{X}(G)$. If F is an object of $\mathbf{Bcf}_0$, $\mathscr{X}_\gamma(F)$ will denote the set $\mathscr{X}(F)$ augmented with the zero functional, which is the base point of $\mathscr{X}_\gamma(F)$. Thus, $\mathscr{X}_\gamma(F)$ is a one-point compactification of $\mathscr{X}(F)$; if F has a unit, then $\mathscr{X}(F)$ is compact and $\mathscr{X}_\gamma(F)$ is homeomorphic to the space $\mathscr{X}(F)+\mathbf{1}$ obtained by adjoining an isolated point to $\mathscr{X}(F)$. If

$\Phi\colon F \to G$ is a $\mathbf{Bcf}_0$-morphism, then $\mathscr{X}_\gamma(\Phi)$ is the restriction of Φ^* to $\mathscr{X}_\gamma(G)$. If F is a space of the type $l_\infty(S)$, then $\mathscr{X}'(F)$ will denote the set of isolated points of $\mathscr{X}(F)$; $\mathscr{X}'(\Phi)$ is defined in an analogous way.

If F is a **Bcfd**-object, $F \subset \mathscr{C}(X)$ and $1_X \in F$, then $\mathscr{K}(F)$ is the set of states

$$\mathscr{K}(F) = \{\xi \in F^*\colon \|\xi\| = \xi(1_X) = 1\}$$

which is convex and *weakly compact (cf. [2], [13], [16]); if $\Phi\colon F \to G$ is a **Bcfd**-morphism, then $\mathscr{K}(\Phi)$ is the restriction of Φ^* to $\mathscr{K}(G)$. The symbol $\bigcirc^*$ will now denote a modification of the first of functors (3), namely $\bigcirc^* F$ is the unit ball in F^* regarded as a Saks space (compact in the *weak topology).

The non-horizontal functors in Fig. 1 are covariant. There are 6 forgetful functors directed upwards: the forgetful functor $\square$: **Comp** $\to$ **Ens** (the underlying-set functor), the functor $\square$: $\mathbf{Comp}_\bullet \to \mathbf{Comp}$ "forgetting" the base points, the functor $\square$: **Compsaks** $\to \mathbf{Comp}_\bullet$ assigning to each compact Saks space its underlying compact space with the base point 0, etc. They are faithful and are not one-to-one on objects.

There are 6 embedding functors directed downwards: the functor $\square$: **Bf** $\to$ **Bcf** assigning to each space $l_\infty(S)$ the space $\mathscr{C}(S)$, S being regarded as discrete; the inclusion functor $\square$: **Bcf** $\to \mathbf{Bcf}_0$, etc.

The remaining 12 functors are: 6 functors directed downwards:

$$\mathscr{X}l_\infty,\ \mathscr{K}\mathscr{C},\ \bigcirc^*\mathscr{C},\ \mathscr{X}_\gamma\mathscr{C},\ \bigcirc^*\mathscr{A},\ \bigcirc^*\mathscr{C}_0 \tag{4}$$

and 6 functors directed upwards:

$$l_\infty\mathscr{X},\ \mathscr{C}\mathscr{K},\ \mathscr{C}\bigcirc^*,\ \mathscr{C}\mathscr{X}_\gamma,\ \mathscr{A}\bigcirc^*,\ \mathscr{C}_0\bigcirc^*; \tag{5}$$

actually, these functors should be written as $\mathscr{X}\circ\square\circ l_\infty$, $\mathscr{K}\circ\square\circ\mathscr{C}, \ldots$ $\ldots, l_\infty\circ\square\circ\mathscr{X}, \ldots$, but we feel free to simplify the notation by omitting the forgetful functors here.

Some of these functors are well known: $\mathscr{X}l_\infty(S)$ is the Stone–Čech compactification of S provided with the discrete topology; $\mathscr{K}\mathscr{C}(X)$ is a Choquet simplex whose extreme boundary is homeomorphic to X; $\mathscr{C}\mathscr{X}_\gamma$ is naturally equivalent to the functor of adding a unit to the Banach algebra $\mathscr{C}_0(X\|A)$.

The terms: left adjoint, right adjoint, adjoint on the left and adjoint on the right will be understood in the sense of Freyd [5].

Proposition 2. *The horizontal functors in Fig. 1 are pairwise adjoint on the left and on the right simultaneously. Moreover, each of the compositions $l_\infty\mathscr{X}'$, $\mathscr{X}'l_\infty$, $\mathscr{C}\mathscr{X}$, $\mathscr{X}\mathscr{C}, \ldots$ is naturally equivalent to the corresponding identity functor.*

This proposition, establishing quasi-dualities between the categories **Ens** and **Bf**, **Comp** and **Bcf**, $\mathbf{Comp}_\bullet$ and $\mathbf{Bcf}_0$, **Compconv** and **Bcfd**, **Compsaks** and $\mathbf{Ban}_1$, is well known (cf. [1], [2], [4], [6], [8], [9], [10], [13], [14]).

THEOREM. *Each of the functors* (4) *and each of the functors* (5) *is a left adjoint of the corresponding functor in Fig.* 1 *marked with* $\square$.

Proof. $\mathscr{X} l_\infty$ is a left adjoint of $\square$: **Comp** $\to$ **Ens**; it is a well-known property of the Stone-Čech compactification.

$\mathscr{K}\mathscr{C}$ is a left adjoint of $\square$: **Compconv** $\to$ **Comp**; see [2], [3], [9], [10], [13].

$\mathscr{X}_\gamma \mathscr{C}$ is a left adjoint of $\square$: **Comp$_\bullet$** $\to$ **Comp** because it is naturally equivalent to the functor $? + \mathbf{1}$ (adding the isolated base point to the space, cf. [11], [13]).

The proofs of left adjointness of $\bigcirc^*\mathscr{C}$, $\bigcirc^*\mathscr{C}_0$ and $\bigcirc^*\mathscr{A}$ are analogous to proofs of some properties of the Stone-Čech functor β or the simplex functor $\mathscr{S} = \mathscr{K}\mathscr{C}$ (cf. [13]); therefore we leave them to the reader.

The left adjointness of $\mathscr{C}\bigcirc^*$ and $\mathscr{C}_0\bigcirc^*$ have been established above; a proof of left adjointness of $\mathscr{A}\bigcirc^*$ can be found in [10], p. 289; a proof of left adjointness of $\mathscr{C}\mathscr{K}$ can be found in [13], 23.3.4. A similar kind of technique is used in the proofs of left adjointness of $\mathscr{C}\mathscr{X}_\gamma$ and $l_\infty\mathscr{X}$; therefore these proofs are omitted as well.

The diagram in Fig. 1 is partially commutative in the following sense:

PROPOSITION 3. *For each of the squares*

$$
\begin{tikzcd}
\mathbf{Ens} \arrow[r, dash] \arrow[d, dash] & \mathbf{Bf} \arrow[d, dash] & \mathbf{Comp} \arrow[r, dash] \arrow[d, dash] & \mathbf{Bcf} \arrow[d, dash] \\
\mathbf{Comp} \arrow[r, dash] & \mathbf{Bcf} & \mathbf{Compsaks} \arrow[r, dash] & \mathbf{Ban}_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Comp} \arrow[r, dash] \arrow[d, dash] & \mathbf{Bcf} \arrow[d, dash] & \mathbf{Comp} \arrow[r, dash] \arrow[d, dash] & \mathbf{Bcf} \arrow[d, dash] \\
\mathbf{Compconv} \arrow[r, dash] & \mathbf{Bcfd} & \mathbf{Comp}_\bullet \arrow[r, dash] & \mathbf{Bcf}_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{Compconv} \arrow[r, dash] \arrow[d, dash] & \mathbf{Bcfd} \arrow[d, dash] & \mathbf{Comp}_\bullet \arrow[r, dash] \arrow[d, dash] & \mathbf{Bcf}_0 \arrow[d, dash] \\
\mathbf{Compsaks} \arrow[r, dash] & \mathbf{Ban}_1 & \mathbf{Compsaks} \arrow[r, dash] & \mathbf{Ban}_1
\end{tikzcd}
$$

and for each of the following choices of directions of arrows:

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd},
\qquad
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l] \arrow[d] \\
\cdot & \cdot \arrow[l]
\end{tikzcd},
\qquad
\begin{tikzcd}
\cdot & \cdot \arrow[l] \\
\cdot \arrow[u] & \cdot \arrow[l] \arrow[u]
\end{tikzcd},
\qquad
\begin{tikzcd}
\cdot \arrow[r] & \cdot \\
\cdot \arrow[u] \arrow[r] & \cdot \arrow[u]
\end{tikzcd},
$$

(a) (b) (c) (d)

the compositions of the corresponding functors in Fig. 1 are naturally equivalent.

Proof. Note, e.g., that $\square\circ\mathscr{X}\circ\square\circ l_\infty(S) = \beta S \neq S$. If directions of arrows are chosen as in (a), then we have a diagram of the form

$$\begin{tikzcd}
\mathfrak{A}_1 \arrow[r, shift left, "\Gamma_1"] \arrow[d, "\Lambda_2\circ\square\circ\Gamma_1"'] & \mathfrak{B}_1 \arrow[l, shift left, "\Lambda_1"] \arrow[d, "\square"] \\
\mathfrak{A}_2 \arrow[r, shift left, "\Gamma_2"] & \mathfrak{B}_2 \arrow[l, shift left, "\Lambda_2"]
\end{tikzcd}$$

and we are to prove that $\square\circ\Gamma_1$ is naturally equivalent to $\Gamma_2\circ\Lambda_2\circ\square\circ\Gamma_1$; since $\Gamma_2\circ\Lambda_2$ is naturally equivalent to the identity functor $\iota_{\mathfrak{B}_2}$, the compositions of these functors with $\square\circ\Gamma_1$ are naturally equivalent as well. In the cases (b), (c) and (d) the argument is similar.

Let us also note that the Banach–Mazur functor is naturally equivalent to each of the compositions

$$\mathbf{Ban}_1 \xrightarrow{\mathscr{C}_0\bigcirc^*} \mathbf{Bcf}_0 \xrightarrow{\mathscr{C}\mathscr{X}_\gamma} \mathbf{Bcf} \quad\text{and}\quad \mathbf{Ban}_1 \xrightarrow{\mathscr{A}\bigcirc^*} \mathbf{Bcfd} \xrightarrow{\mathscr{C}\mathscr{K}} \mathbf{Bcf}.$$

Similarly, the functor $\bigcirc^*\mathscr{C}$: $\mathbf{Comp} \to \mathbf{Compsaks}$ is naturally equivalent to each of the compositions

$$\mathbf{Comp} \xrightarrow{\mathscr{X}_\gamma\mathscr{C}} \mathbf{Comp}_\bullet \xrightarrow{\bigcirc^*\mathscr{C}_0} \mathbf{Compsaks}$$

and

$$\mathbf{Comp} \xrightarrow{\mathscr{K}\mathscr{C}} \mathbf{Compconv} \xrightarrow{\bigcirc^*\mathscr{A}} \mathbf{Compsaks}.$$

From Proposition 3 it follows that for each of the above mentioned 6 squares in Fig. 1 we get 6 diagonal contravariant functors (each of them is defined, up to natural equivalence, by one of two possible compositions). If we paste the upper square ($\mathbf{Ens}$, $\mathbf{Bf}$, $\mathbf{Comp}$, $\mathbf{Bcf}$) with one of the three adjacent squares, we get three more squares. Thus, in Fig. 1, we actually have 9 squares and (up to natural equivalence) 36 diagonal contravariant functors, e.g., the functors

$$(6) \qquad \mathscr{C}\colon \mathbf{Comp} \to \mathbf{Ban}_1 \quad\text{and}\quad \mathscr{C}_0\colon \mathbf{Comp}_\bullet \to \mathbf{Ban}_1.$$

The diagram in Fig. 1 may be regarded as a scheme showing 5 mutually dual theories in functional analysis and the canonical functors establishing some relations between those theories; from this point of view one may discuss, e.g., whether are one theory more general than the other. One may add further categories and functors to that diagram, e.g. one may consider the quasi-dual pair: compact 0-dimensional spaces and Boolean algebras (or those algebras $\mathscr{C}(X)$ in which the idempotents are linearly dense).

For each of 24 covariant functors in Fig. 1 one may ask whether they preserve equalizers (= difference kernels), coequalizers, products, coproducts (= sums), inverse (= projective) limits, direct (= inductive) limits and so on (cf. [11]). The positive answer to one half of those questions follows from Freyd's Adjoint Functor Theorem ([5], p. 81).

It is clear that each of the 10 contravariant horizontal functors in Fig. 1 sends equalizers to coequalizers, coequalizers to equalizers, products to coproducts, etc. Similar questions of preservation properties may be asked for each of the 36 contravariant functors mentioned after Proposition 3. Most of the answers to these questions either follow from the Adjoint Functor Theorem, or can be established directly. In some cases, however, the answers do not seem to be trivial and may be regarded as open. Let us mention that in [12] it is proved that the functors (6) send inverse limits (over upward filtering diagram schemes) to direct limits.

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