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On the radius of starlikeness of some families of regular functions

1. Definitions and denotations. Preliminaries. Let a, β, M, δ and n be arbitrary fixed numbers: $a \in [0, 1)$, $\beta \in [0, 1)$, $M \in [1, \infty)$, $\delta \in [0, \infty)$, $n > 1$ — a natural number.

Denote by G the family of all functions of the form

$$(1) \qquad w = g(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

regular in the circle $K = \{z: |z| < 1\}$ and by S one of its subclasses, the class of schlicht functions and let $S^*(a)$ be the family of functions of form (1) starlike of order a, i.e. of functions satisfying the condition

$$(2) \qquad \operatorname{re} \frac{zg'(z)}{g(z)} > a \quad \text{for every } z \in K.$$

Obviously, $S^*(a) \subset S$, $S^*(0) \equiv S^*$, where S^* is the family of all starlike functions of form (1), i.e. all functions mapping the circle K into domains starlike with respect to the point $w = 0$. The notion of "starlikeness of order a" has been introduced by Robertson [4], who has associated it with a class of functions smaller than the class defined here as well as in some other papers (comp. e.g. [5]) by condition (2). Moreover, denote by H_n the family of all functions of the form

$$(3) \qquad h(z) = b_n z^n + b_{n+1} z^{n+1} + \ldots, \qquad b_n \neq 0,$$

regular in the circle K.

Let $\varphi(z)$ be a function of the form

$$(4) \qquad \varphi(z) = f(z) + h(z),$$

where $f(z)$ and $h(z)$ are functions of the families $S^*(a)$ and H_n respectively such that

$$(5) \qquad \left| \frac{(\varphi(z)/f(z)) - \beta}{1 - \beta} - M \right| < M \quad \text{for every } z \in K.$$

The family of all functions $\varphi(z)$ satisfying conditions (4) and (5) will be denoted by $L_{M,n}(\beta, \alpha)$, while $H_{M,n}(\beta, \alpha)$ stands for the family of functions $h(z)$ satisfying the condition

$$\left| \frac{1}{1-\beta} \cdot \frac{h(z)}{f(z)} + 1 - M \right| < M$$

for some function $f(z) \epsilon S^*(\alpha)$ and every $z \epsilon K$.

Finally, let for an integer δ, $g(z)$ be a function of the family G such that

$$(6) \qquad \left| \left(\frac{g(z)}{f(z)} \right)^{1+\delta} - M \right| < M$$

for some function $f(z) \epsilon S^*(\alpha)$ and every $z \epsilon K$; if, on the other hand, δ is not an integer, let $g(z)$ denote a function of the family G which takes the value 0 only at the point $z = 0$ and satisfies condition (6) for some function $f(z) \epsilon S^*(\alpha)$ and every $z \epsilon K$. In this case we mean by $(g(z)/f(z))^{1+\delta}$ this single-valued branche $L(z)$ of a multi-valued function $F(z) = (g(z)/f(z))^{1+\delta}$, for which $L(0) = 1$.

The family of all functions $g(z)$ defined above will be denoted by $\hat{G}_M^{(\delta)}(0, \alpha)$.

In the limit case $M = \infty$ condition (5) takes the form

$$(7) \qquad \operatorname{re} \frac{\varphi(z)}{f(z)} > \beta$$

for some function $f(z) \epsilon S^*(\alpha)$ and every $z \epsilon K$.

A function $\varphi(z)$ of form (1) regular in the circle K and satisfying condition (7) will be called *close-to-starlike* of order α and of type β in that circle. The notion of an close-to-starlike function has been introduced in the case $\alpha = \beta = 0$ by Reade [3]. Thus the family $L_{\infty,n}(\beta, \alpha)$ is a subclass of the family $G(\beta, \alpha)$ of all functions close-to-starlike of order α and of type β in the circle K. Since $L_{M,n}(\beta, \alpha) \subset L_{\infty,n}(\beta, \alpha)$, $L_{M,n}(\beta, \alpha) \subset G(\beta, \alpha)$. In the limit case $M = \infty$ condition (6) takes the form

$$\operatorname{re} \left(\frac{g(z)}{f(z)} \right)^{1+\delta} > 0.$$

Obviously $\hat{G}_M^{(\delta)}(0, \alpha) \subset \hat{G}_\infty^{(\delta)}(0, \alpha)$ and $\hat{G}_\infty^{(0)}(0, \alpha) \equiv G(0, \alpha)$. By the radius of starlikeness of an arbitrary subclass T of the family G we mean the upper bound ϱ of the radii of the circles $|z| \leqslant r$, $0 < r \leqslant 1$, in which every function of this family is schlicht and starlike.

Since a function of form (1) regular in a circle $|z| < r$ is schlicht and starlike in that circle if and only if

$$\operatorname{re} \frac{zg'(z)}{g(z)} > 0 \qquad \text{for every } z \epsilon K_r,$$

$K_r = \{z \colon |z| < r\}$, $0 < r \leqslant 1$, [4], the problem of finding the radius of starlikeness of the family T, if T is compact, is reduced to finding the greatest value of r, $0 < r \leqslant 1$, for which

$$\text{re} \frac{zg'(z)}{g(z)} \geqslant 0$$

for every $|z| \leqslant r$ and every function $g(z) \epsilon T$.

In this paper the radius of starlikeness of the family $L_{M,n}(\beta, a)$ for $\beta \neq 0$ is estimated and its exact value is given for $\beta = 0$. An analogous problem for the family $\hat{G}_M^{(\delta)}(0, a)$ has also been solved.

2. Auxiliary lemmas. Denote by $\mathscr{P}$ the family of all functions of the form

(8) $$P(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

regular in the circle K, satisfying the condition

$$\text{re} P(z) > 0 \quad \text{for every } z \epsilon K$$

and let $\mathscr{P}_k$ be the set of function of the family $\mathscr{P}$ of the form

(9) $$P(z) = 1 + p_k z^k + p_{k+1} z^{k+1} + \ldots, \quad k \geqslant 1.$$

A function $f(z)$ belongs to the family S^* if and only if

$$\frac{zf'(z)}{f(z)} \epsilon \mathscr{P}, \quad f'(0) = 1$$

(cf. [2]). Then let $\mathscr{P}(M)$ and $\mathscr{P}_k(M)$ be families of functions of form (8) or (9) respectively, regular in the circle K and satisfying the condition

(10) $$|P(z) - M| < M \quad \text{for every } z \epsilon K.$$

Condition (10) means that the values of the function $P(z)$ for every $z \epsilon K$ belong to the circle with the centre M and the radius M. The family $\mathscr{P}_k(M)$ is a subclass of the family $\mathscr{P}_k$. In the limit case $M = \infty$ we have $\mathscr{P}_k(\infty) \equiv \mathscr{P}_k$.

In the sequel we will need lemmas concerning functions of the family $\mathscr{P}_k(M)$.

LEMMA 1. *If a function $P(z)$ belongs to the family $\mathscr{P}_k(M)$, then it can be represented in the form*

(11) $$P(z) = \frac{1 + w(z)}{1 - mw(z)}, \quad m = 1 - \frac{1}{M},$$

where $w(z)$, $w(0) = 0$, is a function regular in the circle K satisfying the condition

(12) $$|w(z)| \leqslant |z|^k \quad \text{for every } z \epsilon K.$$

Proof. Let

(13) $$Q(z) = \frac{P(z)}{M} - 1$$

and

(14) $$w(z) = \frac{Q(z) - Q(0)}{1 - Q(0)Q(z)}$$

for every $z \epsilon K$. Thus $w(0) = 0$. From (10) follows that $|Q(z)| < 1$, and the homograph function

$$\varphi = \frac{\xi - a}{1 - \overline{a}\xi},$$

where a is an arbitrary point of the circle $|\xi| < 1$, maps this circle onto the circle $|\varphi| < 1$, by which the function $w(z)$ is regular for $z \epsilon K$ and satisfies the condition $|w(z)| < 1$ in the circle K. The function $w(z)$ can be expanded in the circle K in a power series of the form

$$w(z) = c_k z^k + c_{k+1} z^{k+1} + \ldots, \qquad c_k \neq 0,$$

thus the function

$$\psi(z) = \frac{w(z)}{z^k}, \qquad \psi(0) = c_k$$

is regular in the circle K and consequently bounded in the circle $|z| \leqslant r$, $0 < r < 1$. Similarly, as in the proof of the lemma of Schwarz, we prove that $|\psi(z)| \leqslant 1$. Hence we obtain inequality (12). From formulas (13) and (14) formula (11) follows immediately. Thus the lemma has been proved.

LEMMA 2. *If a function $P(z)$ belongs to the family $\mathscr{P}_k(M)$, then*

(15) $$\frac{1 - |z|^k}{1 + m|z|^k} \leqslant |P(z)| \leqslant \frac{1 + |z|^k}{1 - m|z|^k}, \qquad m = 1 - 1/M \geqslant 0.$$

Estimations (15) are sharp and equalities in (15) hold for functions of the form

(16) $$P(z) = \frac{1 + \varepsilon z^k}{1 - m\varepsilon z^k}, \qquad |\varepsilon| = 1.$$

Proof. By Lemma 1 and elementary properties of modulus inequalities we have

$$|P(z)| \leqslant \frac{1 + |w(z)|}{1 - m|w(z)|}.$$

Hence in view of (12) we obtain the second of inequalities (15). Similarly we prove the first inequality. Estimations (15) are sharp, as for functions

$$P^*(z) = \frac{1 - \varepsilon z^k}{1 + m\varepsilon z^k}, \quad P^{**}(z) = \frac{1 + \varepsilon z^k}{1 - m\varepsilon z^k}, \quad \varepsilon = \exp(-k\arg z_0)$$

of the family $\mathscr{P}_k(M)$ for every $|z_0| < 1$ we have

$$|P^*(z_0)| = \frac{1 - |z_0|^k}{1 + m|z_0|^k} \quad \text{and} \quad |P^{**}(z_0)| = \frac{1 + |z_0|^k}{1 - m|z_0|^k}.$$

LEMMA 3. *If a function $P(z)$ belongs to the family $\mathscr{P}_k(M)$, then*

$$(17) \qquad \left| \frac{zP'(z)}{P(z)|} \right| \leqslant \frac{k(1 + m)|z|^k}{1 - (1 - m)|z|^k - m|z|^{2k}}, \quad m = 1 - 1/M.$$

Estimation (17) is sharp and equality takes place in (17) for functions of form (16).

Proof. Differentiating function (11) with respect to z, we obtain

$$P'(z) = \frac{(1 + m)w'(z)}{[1 - m \cdot w(z)]^2},$$

whence

$$\frac{zP'(z)}{P(z)} = \frac{(1 + m)z \cdot w'(z)}{[1 + w(z)][1 - mw(z)]}.$$

Thus

$$(18) \qquad \left| \frac{zP'(z)}{P(z)} \right| \leqslant \frac{(1 + m)|z||w'(z)|}{1 - (1 - m)|w(z)| - m|w(z)|^2}.$$

For functions of the form

$$F(z) = A_0 + A_k z^k + \dots, \quad k \geqslant 1$$

regular in the circle K and satisfying the condition $|F(z)| < 1$ for $z \epsilon K$ the following sharp estimation of the modulus of the derivative holds [1]:

$$(19) \qquad |F'(z)| \leqslant \frac{k|z|^{k-1}}{1 - |z|^{2k}} \cdot (1 - |F(z)|^2).$$

Applying estimations (19) to the derivative $w'(z)$ in inequality (18) we obtain

$$\left| \frac{zP'(z)}{P(z)} \right| \leqslant \frac{k(1 + m)|z|^k}{1 - |z|^{2k}} \cdot \frac{1 - |w(z)|^2}{1 - (1 - m)|w(z)| - m|w(z)|^2},$$

whence in view of

$$\frac{1-|w(z)|^2}{1-(1-m)|w(z)|-m|w(z)|^2} \leqslant \frac{1-|z|^{2k}}{1-(1-m)|z|^k-m|z|^{2k}}$$

(17) follows.

Since for the function

$$P^*(z) = \frac{1+\varepsilon z^k}{1-m\varepsilon z^k}, \qquad \varepsilon = -\exp(-k\arg z_0)$$

of the family $\mathscr{P}_k(M)$ for every $|z_0| < 1$ we have

$$\left| \frac{z_0 P^{*\prime}(z_0)}{P^*(z_0)} \right| = \frac{k(1+m)|z_0|^k}{1-(1-m)|z_0|^k-m|z_0|^{2k}},$$

estimation (17) is sharp.

3. Theorems on the radius of starlikeness of the families $L_{M,n}(\beta, a)$ and $\hat{G}_M^{(\delta)}(0, a)$. From the considerations represented at the end of Section 1 of this paper it follows immediately, that the radius of starlikeness of an arbitrary subclass T of the family G equals to the smallest root, r_0, $0 < r_0 \leqslant 1$ of the equation

$$\Omega(r) = 0,$$

where

$$\Omega(r) = \min_{\substack{|z|=r \\ g(z)\epsilon T}} \operatorname{re} \frac{zg'(z)}{g(z)}.$$

Hence the radii of starlikeness of the families $L_{M,n}(\beta, a)$ and $\hat{G}_M^{(\delta)}(0, a)$ may be found by evaluating

$$\min_{\substack{|z|=r \\ \varphi(z)\epsilon L_{M,n}(\beta,a)}} \operatorname{re} \frac{z\varphi'(z)}{\varphi(z)}, \quad 0 < r \leqslant 1$$

and

$$\min_{\substack{|z|=r \\ g(z)\epsilon \hat{G}_M^{(\delta)}(0,a)}} \operatorname{re} \frac{zg'(z)}{g(z)}, \quad 0 < r \leqslant 1.$$

Assume

$$m = 1 - \frac{1}{M}, \qquad \lambda = \frac{\beta}{1-\beta}.$$

THEOREM 1. *The radius of starlikeness ϱ of the family $L_{M,n}(\beta, a)$ is greater than or equal to the smallest positive root r_0 of the equation*

$$(20) \quad q(r) \equiv (1-2a)(m-\lambda)\cdot r^{2n-1} - (m-\lambda)r^{2n-2} - [(1-2a)(m-1-2\lambda) + (n-1)(1+m)]r^n + [m-1-2\lambda-(n-1)(1+m)]r^{n-1} - (1-2a)(1+\lambda)r+1+\lambda = 0.$$

If $\beta = 0$, then $\varrho = r_0$.

Proof. (a) If $\varphi(z) \in L_{M,n}(\beta, \alpha)$, then

$$\frac{(\varphi(z)/f(z)) - \beta}{1 - \beta} = P(z)$$

for some function $f(z) \in S^*(\alpha)$ and $P(z) \in \mathscr{P}_{n-1}(M)$, thus

$$(21) \qquad \varphi(z) = f(z)[\beta + (1-\beta)P(z)].$$

Differentiating (21) we obtain

$$\varphi'(z) = f'(z)[\beta + (1-\beta)P(z)] + (1-\beta)f(z)P'(z),$$

in view of which

$$(22) \qquad \frac{z\varphi'(z)}{\varphi(z)} = \frac{zf'(z)}{f(z)} + \frac{zP'(z)}{P(z) + \lambda}.$$

Thus

$$\mathrm{re}\,\frac{z\varphi'(z)}{\varphi(z)} \geqslant \min_{f(z) \in S^*(\alpha)} \mathrm{re}\,\frac{zf'(z)}{f(z)} - \max_{P(z) \in \mathscr{P}_{n-1}(M)} \left| \frac{zP'(z)}{P(z) + \lambda} \right|.$$

Since $f(z) \in S^*(\alpha)$, then

$$(23) \qquad \mathrm{re}\,\frac{zf'(z)}{f(z)} \geqslant \frac{1 - (1-2\alpha)r}{1 + r}, \quad |z| = r,\ 0 \leqslant r < 1$$

(cf. [2]). Next, we have

$$\left| \frac{zP'(z)}{P(z) + \lambda} \right| = \frac{\left| \dfrac{zP'(z)}{P(z)} \right|}{\left| 1 + \dfrac{\lambda}{P(z)} \right|} \leqslant \frac{\left| \dfrac{zP'(z)}{P(z)} \right|}{1 + \lambda\,\mathrm{re}\,\dfrac{1}{P(z)}}.$$

By Lemmas 2 and 3 we obtain for $k = n-1$ the inequality

$$(24) \qquad \frac{\left| \dfrac{zP'(z)}{P(z)} \right|}{1 + \mathrm{re}\,\dfrac{1}{P(z)}} \leqslant \frac{(n-1)(1+m)|z|^{n-1}}{1 - (1-m)|z|^{n-1}m|z|^{2n-2}} \cdot \frac{1}{1 + \dfrac{1 - |z|^{n-1}}{1 + m|z|^{n-1}}}.$$

By (23) and (24) we have

$$(25) \quad \mathrm{re}\,\frac{z\varphi'(z)}{\varphi(z)} \geqslant \frac{1 - (1-2\alpha)r}{1 + r} - \frac{(n-1)(1+m)r^{n-1}}{[1 + \lambda + (m-\lambda)r^{n-1}](1 - r^{n-1})}, \quad |z| = r.$$

After performing some transformations we obtain

$$(26) \qquad \mathrm{re}\,\frac{z\varphi'(z)}{\varphi(z)} \geqslant \frac{q(r)}{(1+r)(1-r^{n-1})[1 + \lambda + (m-\lambda)r^{n-1}]}, \quad |z| = r,$$

where $q(r)$ is the polynomial appearing in (20).

The denominator of the right-hand side expression of inequality (26) is positive for $0 \leqslant r < 1$, $q(0) = 1 + \lambda > 0$ and $q(1) = -2(n-1) \times (1+m) < 0$, thus the smallest positive root of equation (20) r_0 lies between 0 and 1. Therefore inequality (26) takes place for $r = |z| < r_0$. Thus the radius od stralikeness of the family $L_{M,n}(\beta, a)$ is greater or equal to r_0, which was to be proved.

(b) Let $\beta = 0$. The functions

$$f^*(z) = \frac{z}{(1+z)^{2(1-a)}}, \qquad h^*(z) = \frac{-(1+m)z^n}{(1+mz^{n-1})(1+z)^{2(1-a)}}$$

as it can be easily verified, belong to the families $S^*(a)$ and $H_{M,n}(0, a)$ respectively, and therefore the function

$$\varphi^*(z) = f^*(z) + h^*(z) = \frac{z}{(1+z)^{2(1-a)}} \cdot \frac{1 - z^{n-1}}{1 + mz^{n-1}}$$

belongs to the family $L_{M,n}(0, a)$. Since

$$\frac{z\varphi^{*\prime}(z)}{\varphi^*(z)} = \frac{1 - (1-2a)z}{1+z} - \frac{(n-1)(1+m)z^{n-1}}{(1+mz^{n-1})(1-z^{n-1})},$$

we obtain by (26)

$$\mathrm{re}\,\frac{z\varphi^{*\prime}(z)}{\varphi^*(z)} = 0$$

for $z = r_0$. Thus the function $\varphi^*(z)$ is not starlike in the circle $|z| < r$ for $r > r_0$. Hence in view of $\varrho \geqslant r_0$, we obtain $\varrho = r_0$, which ends the proof of the theorem.

COROLLARY 1. *The radius of starlikeness of the family of functions* $\varphi(z)$ *of the form* $z + h(z)$, $h(z) \epsilon H_n$, *satisfying the condition*

$$\left| \frac{\varphi(z)}{z} - M \right| < M \quad \text{for every } z \epsilon K$$

is given by the formula

$$(27) \quad \varrho = \begin{cases} \left[\dfrac{2m - n \cdot (1+m) + \sqrt{[n(1+m) - 2m]^2 + 4m}}{2m} \right]^{\frac{1}{n-1}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad m \neq 0 \left(m = 1 - \dfrac{1}{M} \right), \\ \left(\dfrac{1}{n} \right)^{\frac{1}{n-1}} \qquad \text{if} \quad m = 0. \end{cases}$$

To prove this observe that the function $f(z) = z$ is starlike in the circle K of an arbitrary order, in view of which $\varphi(z) \epsilon L_{M,n}(0, a)$ for every $a \epsilon [0, 1)$. Since the function $\varphi(z)$ is of the form $z + h(z)$, $h(z) \epsilon H_{M,n}(0, a)$, the first term in sum (22) is to be replaced by 1 and in the second $\lambda = 0$. Thus inequality (25) takes in the considered case the form

$$\operatorname{re} \frac{z \cdot \varphi'(z)}{\varphi(z)} \geqslant 1 - \frac{(n-1)(1+m)r^{n-1}}{(1+mr^{n-1})(1-r^{n-1})} \cdot$$

Hence the radius of starlikeness R of the function family in question is greater or equal to the (positive) root ϱ of the equation

(28) $$mr^{2n-2} + [n(1+m) - 2m]r^{n-1} - 1 = 0,$$

expressed by formula (27). Thus $R \geqslant \varrho$. The function

$$h^*(z) = -\frac{(1+m)z^n}{1+mz^{n-1}}$$

belongs to the family $H_{M,n}(0, a)$ (a — is arbitrary) and the logarithmic derivative of the function $\varphi^*(z) = z + h^*(z)$

$$\frac{z \cdot \varphi^{*\prime}(z)}{\varphi^*(z)} = \frac{-mz^{2n-2} - [n(1+m) - 2m]z^{n-1} + 1}{(1-z^{n-1})(1+mz^{n-1})},$$

thus in view of (28)

$$\operatorname{re} \frac{z\varphi^{*\prime}(z)}{\varphi^*(z)} = 0$$

for $z = \varrho$. Thus the function $\varphi^*(z)$ is not starlike in the circle $|z| < r$ for $r > \varrho$. Since $R \geqslant \varrho$ we obtain $R = \varrho$, which ends the proof.

COROLLARY 2. *The function* $z + h(z)$, *where*

$$h(z) = b_n z^n + b_{n+1} z^{n+1} + \ldots, \qquad b_n \neq 0, n > 1,$$

is regular in the circle K *and satisfying the condition*

$$|h(z)| < |z| \qquad \text{for every } z \epsilon K,$$

is starlike in the circle

$$|z| < \left(\frac{1}{n}\right)^{\frac{1}{n-1}}.$$

COROLLARY 3. *The radius of starlikeness of the family of functions of the form*

(29) $$\varphi(z) = f(z) + h(z), \qquad f(z) \epsilon S^*(a), \overset{*}{h}(z) \epsilon H_n,$$

clase-to-starlike in the circle K is equal to the smallest positive root of the equation

$$(30) \quad (1-2a)r^{2n-1} - r^{2n-2} - 2(n-1)r^n - 2(n-1)r^{n-1} - (1-2a)r + 1 = 0.$$

In fact, if a function $\varphi(z)$ satisfies the above assumptions, then $M = \infty$. Passing to the limit in (20) as m tends to 1, we obtain equation (30).

COROLLARY 4. *The radius of starlikeness of a family of form (29), for which*

$$|h(z)| < |f(z)| \quad \text{for every } z \epsilon K,$$

equals the smallest positive root of the equation

$$(31) \quad [2(1-a)-n]r^n - nr^{n-1} - (1-2a)r + 1 = 0.$$

Indeed, if a function $\varphi(z)$ satisfies the given assumptions, then $M = 1$, thus $m = 0$. Substituting in (20) $m = 0$ and $\lambda = 0$, we obtain equation (31). In particular if $a = 0$ and $n = 2$ the radius of starlikeness $\varrho = \frac{1}{3}$.

THEOREM 2. *The radius of starlikeness ϱ of the family $\hat{G}_M^{(\delta)}(0, a)$ equals the smallest positive root r_0 of the equation*

$$(32) \quad s(r) \equiv m(1+\delta)(1-2a)r^3 - [3m + \delta(2m-1) + 2a(1+\delta)(1-m)]r^2 +$$
$$+ [2a - 3 + \delta(m-2+2a)]r + 1 + \delta = 0.$$

Proof. If $g(z) \epsilon \hat{G}_M^{(\delta)}(0, a)$, then

$$\left(\frac{g(z)}{f(z)} \right)^{1+\delta} = P(z)$$

for some function $f(z) \epsilon S^*(a)$ and $P(z) \epsilon \mathscr{P}(M)$, thus

$$(33) \qquad g(z) = f(z)\big(P(z)\big)^{1/(1+\delta)}.$$

By $\big(P(z)\big)^{1/(1+\delta)}$ we mean here this single-valued branch $L(z)$ of the multi-valued function $F(z) = \big(P(z)\big)^{1/(1+\delta)}$, which takes the value 1 at the point $z = 0$. Differentiating (33) we obtain

$$g'(z) = f'(z)\big(P(z)\big)^{1/(1+\delta)} + \frac{f(z)}{1+\delta}\big(P(z)\big)^{-\delta/(1+\delta)}P'(z),$$

by which

$$(34) \qquad \frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)} + \frac{zP'(z)}{(1+\delta)P(z)}.$$

Thus

$$\text{re}\,\frac{zg'(z)}{g(z)} \geqslant \min_{f(z)\epsilon S^*(a)} \text{re}\,\frac{zf'(z)}{f(z)} - \max_{P(z)\epsilon\mathscr{P}(M)} \frac{zP'(z)}{P(z)}, \quad |z| = r.$$

By (23) and Lemma 3 we obtain for $k = 1$ the inequality

$$\text{(35)} \qquad \text{re} \frac{zg'(z)}{g(z)} \geqslant \frac{1-(1-2a)r}{1+r} - \frac{(1+m)r}{(1+\delta)(1-r)(1+mr)} ,$$

$$|z| = r, \ \ 0 \leqslant r < 1.$$

After some transformations we obtain

$$\text{(36)} \qquad \text{re} \frac{zg'(z)}{g(z)} \geqslant \frac{s(r)}{(1+\delta)(1-r)(1+r)(1+mr)} , \qquad |z| = r,$$

where $s(r)$ is the polynomials appearing in (32).

The denominator of the right-hand side expression of inequality (36) is positive for $0 \leqslant r < 1$, $s(0) = 1 + \delta > 0$ and $s(1) = -2(1+m) < 0$, thus the smallest positive root r_0 of equation (32) lies between 0 and 1. Thus inequality (36) holds for $r = |z| < r_0$. Thereby, the radius of starlikeness ϱ of the family $\hat{G}_M^{(\delta)}(0, a)$ is greater than or equal to r_0. The functions

$$f^*(z) = \frac{z}{(1+z)^{2(1-a)}} , \qquad g^*(z) = \frac{z}{(1+z)^{2(1-a)}} \left(\frac{1-z}{1+mz} \right)^{1/(1+\delta)}$$

belong to the families $S^*(a)$ and $\hat{G}_M^{(\delta)}(0, a)$ respectively. Since

$$\frac{zg^{*'}(z)}{g^*(z)} = \frac{1-(1-2a)z}{1+z} - \frac{(1+m)z}{(1+\delta)(1-z)(1+mz)} ,$$

thus by (36) we have

$$\text{re} \frac{zg^{*'}(z)}{g^*(z)} = 0$$

for $z = r_0$. Therefore the function $g^*(z)$ is not starlike in the circle $|z| < r$ for $r > r_0$. Hence in view of $\varrho \geqslant r_0$ we obtain $\varrho = r_0$, which ends the proof of the theorem.

COROLLARY 1. *The radius of starlikeness of the family of functions* $g(z), g(z) \epsilon G,$ *satisfying the condition*

$$\left| \left(\frac{g(z)}{z} \right)^{1+\delta} - M \right| < M \quad \textit{for every } z \epsilon K,$$

is given by the formula

$$\text{(37)} \quad \varrho = \begin{cases} \dfrac{\delta(m-1)-2+\sqrt{[\delta(m-1)-2]^2+4m(1+\delta)^2}}{2m} , \\ \qquad\qquad\qquad\qquad \textit{if } m \neq 0 \left(m = 1 - \dfrac{1}{M} \right), \\ \dfrac{1+\delta}{2+\delta}, \qquad \textit{if } m = 0 . \end{cases}$$

Similarly as in the proof of Corollary 1 from Theorem 1, we find that in the considered case the first term in sum (34) is to be replaced by 1, basing on which we conclude that the radius of starlikeness R of the family in question is greater than or equal to the (positive) root ϱ of the equation

$$(38) \qquad m(1+\delta)r^2 - [\delta(m-1)-2]r - 1 - \delta = 0,$$

given by formula (37). Thus $R \geqslant \varrho$. The function

$$g^*(z) = z\left(\frac{1-z}{1+mz}\right)^{1/(1+\delta)}$$

belongs to the considered family and its logarithmic derivative

$$\frac{zg^{*\prime}(z)}{g^*(z)} = 1 - \frac{(1+m)z}{1+\delta(1-z)(1+mz)},$$

thus in view of (38)

$$\mathrm{re}\,\frac{zg^{*\prime}(z)}{g^*(z)} = 0$$

for $z = \varrho$. Thus the function $g^*(z)$ is not starlike in the circle $|z| < r$ for $r > \varrho$. Hence in view of $R \geqslant \varrho$ we obtain that $R = \varrho$, which ends the proof.

COROLLARY 2. *The radius of starlikeness of the family of functions of the form*

$$g(z) = z + b_2 z^2 + b_3 z^3 + \ldots$$

regular in the circle and satisfying the condition

$$\mathrm{re}\left(\frac{g(z)}{f(z)}\right)^{1+\delta} > 0 \quad \text{for every } z \epsilon K,$$

where $f(z) \epsilon S^*(a)$, is given by the formula

$$(39) \quad \varrho = \begin{cases} \dfrac{2+\delta-(1+\delta)a-\sqrt{(1+\delta)^2a^2+2(1+\delta)(1-a)+1}}{(1+\delta)(1-2a)}, & \text{if } a \neq \tfrac{1}{2}, \\[2mm] \dfrac{1+\delta}{3+\delta}, & \text{if } a = \tfrac{1}{2}. \end{cases}$$

In fact, if the function $g(z)$ satisfies the above given assumptions, then $M = \infty$. In this case we obtain from condition (35) the equation

$$(1+\delta)(1-2a)r^2 - 2[(1+\delta)(1-a)+1]r + 1 + \delta = 0,$$

the smallest positive root of which is given by formula (39).

COROLLARY 3. *The radius of starlikeness of the family* $\hat{G}_1^{(\delta)}(0, a)$ *is given by the formula*

$$
\varrho =
\begin{cases}
\dfrac{1+2(1+\delta)(1-a)-\sqrt{4(1+\delta)^2 a^2+4(1+\delta)(2-a)+1}}{2[(1+\delta)(1-2a)-1]}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (1+\delta)(1-2a) \neq 1, \\[2ex]
\dfrac{1+\delta}{3+\delta}, \qquad\qquad\qquad\qquad\qquad\quad \text{if } (1+\delta)(1-2a) = 1.
\end{cases}
$$

We obtain this corollary from Theorem 2 putting in it $m = 0$.

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