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Two criteria for the continuity of the equilibrium Riesz potentials

1. Introduction. Let $|x-y|$ denote the distance of two points x and y in the Euclidean space $\boldsymbol{R}^p$ $(p \geqslant 2)$. Put

$$(1.1) \qquad \omega_a(x, y) = \begin{cases} \exp\big(-A(p, a)|x-y|^{a-p}\big) & \text{if } 0 < a < p, \\ |x-y| & \text{if } a = p = 2, \end{cases}$$

$A(p, a)$ being a normalizing positive constant depending only on p and a but not on $|x-y|$. Potentials

$$(1.2) \qquad U_a^\mu(x) = \int k_a(x, y)\, d\mu(y), \qquad x \in \boldsymbol{R}^p,$$

with the kernel k_a given by

$$(1.3) \qquad k_a(x, y) = -\operatorname{Log} \omega_a(x, y),$$

are called the *Riesz potentials* or the *k_a-potentials* (detailed study of the k_a-potentials may be found in [3]).

Define $P(x)$ by

$$(1.4) \qquad P(x) = C\exp\Big(-\sum_{i=1}^{n} k_a(x, x_i)\Big) = C\prod_{i=1}^{n} \omega_a(x, x_i), \qquad x \in \boldsymbol{R}^p,$$

where C is a positive constant and $x_1, \ldots, x_n$ are arbitrary fixed points of $\boldsymbol{R}^p$. The number n in (1.4) will be called the *degree of P* and denoted by $\deg P$.

Let E be a subset of $\boldsymbol{R}^p$. Let $\mathscr{F}_a^M$ be an arbitrary family of functions P given by (1.4) such that

$$K(x; \mathscr{F}_a^M) = \sup\{P(x) \colon P \in \mathscr{F}_a^M\} < M, \qquad x \in E.$$

CONDITION $(L_{a,1}^M)$. *We say that E satisfies the condition $(L_{a,1}^M)$ at $a \in \boldsymbol{R}^p$, if for every family $\mathscr{F}_a^M$ and for every $\varepsilon > 0$ there exist two positive numbers δ and K_0 such that*

$$P(x) \leqslant K_0 \exp(\varepsilon \deg P), \qquad |x-a| < \delta,\ P \in \mathscr{F}_a^M.$$

CONDITION (L_a^M). *We say that E satisfies the condition (L_a^M) at $a \in R^p$, if for every $r > 0$ the set $E_r = \{x \in E: |x-a| \leqslant r\}$ satisfies the condition ($L_{a,1}^M$) at a.*

MAIN THEOREM. *Let $0 < a \leqslant 2$. Let E be a compact subset of R^p with positive k_a-capacity, $C_a(E) > 0$. Let a be a fixed point of E. Then the following conditions are equivalent:*

(a) *For every $r > 0$ the equilibrium k_a-potential $U_a^\lambda(x)$ of the set E_r is continuous at a.*

(b) *E satisfies (L_a^∞) at a.*

(c) *E satisfies (L_a^1) at a.*

Remark. If $a = p = 2$ (logarithmic potentials) Leja [4] proved that (a) $\Leftrightarrow$ (c). Bach [1] generalized the Leja's result to the case of the Newtonian potentials ($p \geqslant 3$, $a = 2$). It has been proved in [7] that (a) $\Leftrightarrow$ (b), if $a = p = 2$.

Let $\mathscr{F}_a$ denote the family of all the functions $P(x) = C\prod_{i=1}^{n}\omega_a(x, x_i)$ ($n = 0, 1, \ldots$), where $C > 0$, $x_i \in E$ ($i = 1, \ldots, n$) and $P(x) \leqslant 1$ for $x \in E$. Put

$$L_a(x) = \sup\{P(x)^{1/\deg P}: P \in \mathscr{F}_a\}, \quad x \in R^p, \tag{1.5}$$

$$L_a^*(x) = \limsup_{y\to x} L_a(y), \quad x \in R^p. \tag{1.6}$$

One of the basic tools in the proof of the Main Theorem is the following

LEMMA 4.4. *If $0 < a \leqslant 2$ and $C_a(E) > 0$, then*

$$\exp\left(-U_a^\lambda(x) + 1/C_a(E)\right) = \begin{cases} L_a(x), & x \in R^p - E, \\ L_a^*(x), & x \in E, \end{cases}$$

$U_a^\lambda(x)$ being the equilibrium k_a-potential of E.

In this note we also investigate some numbers sequences which converge to constants related to the k_a-capacity. In particular we show that if $2 < a < 3$, then the Čebyšev constant of the unit ball B in R^3 with respect to ω_a is strictly smaller than the transfinite diameter of B with respect to ω_a (see Section 3). This implies that the first footnote on p. 227 in [3] is true only under the additional assumption that $0 < a \leqslant 2$.

2. Some known properties of k_a-potentials. For the sake of convenience of the reader we shall recall some statements concerning the k_a-potentials, which are to be used below. We shall follow [3].

In the rest of this paper E will always denote a compact subset of R^p ($p \geqslant 2$). Given any (non-negative) measure μ with $\operatorname{supp}\mu \subset E$ we put

$$I_a(\mu) = -\iint \operatorname{Log}\omega_a(x, y)\,d\mu(x)\,d\mu(y).$$

Let

$$W_a(E) = \inf\{I_a(\mu)\colon \operatorname{supp}\mu \subset E,\ \mu(E) = 1\}.$$

If $W_a(E) < \infty$, there exists unique *minimizing measure* λ ($\operatorname{supp}\lambda \subset E$, $\lambda(E) = 1$) such that

$$0 < W_a(E) = I_a(\lambda).$$

The number $C_a(E) = 1/W_a(E)$ is called the *k_a-capacity* of E. The potential $U_a^\lambda(x)$ is called the *equilibrium potential* of E with respect to the kernel k_a.

Given $0 < a < p$, one says that a property holds on a set $S \subset R^p$ *nearly everywhere* (*n.e.*) if it holds everywhere in S except a subset with interior k_a-capacity zero.

(2.1) If $0 < a < p$, then

$$U_a^\lambda(x) = W_a \quad \text{n.e. in } \operatorname{supp}\lambda,$$
$$U_a^\lambda(x) \leqslant W_a \quad \text{for } x \in \operatorname{supp}\lambda \ \text{([3], p. 174).}$$

(2.2) If $0 < a \leqslant 2$, then

$$U_a^\lambda(x) = W_a \quad \text{n.e. in } E,$$
$$U_a^\lambda(x) \leqslant W_a \quad \text{for } x \in \boldsymbol{R}^p \ \text{(Maximum principle; [3], p. 174).}$$

(2.3) Let $\{E_n\}$ be a sequence of closed subsets of E such that $E_n \subset E_{n+1}$, $E = \bigcup E_n$. Then

$$\lim C_a(E_n) = C_a(E) \quad \text{([3], p. 193).}$$

(2.4) Let $\{\sigma_n\}$ be a sequence of measures such that $\sigma_n \to \sigma$ (weakly), $\sigma_n(E) = 1$ and $\operatorname{supp}\sigma_n \subset E$ $(n = 1, \ldots)$. If $\{U_a^{\sigma_n}(x)\}$ converges to a lower-semicontinuous function $U(x)$, $x \in R^p$, then

$$U(x) = U_a^\sigma(x), \quad x \in R^p.$$

This is a direct consequence of Theorem 3.8 and Remarks 1 and 2 on pp. 237-238 of [3].

(2.5) Let μ and ν be arbitrary measures. Let $I_a(\mu) < \infty$. Let $f(x) = U_a^\mu(x) + c$, $c = \text{const} \geqslant 0$. If $U_a^\mu(x) \leqslant f(x)$ μ-almost everywhere, then the same inequality holds everywhere in R^p (Domination principle; [3], p. 149).

(2.6) If $C_a(E) > 0$, then there exists a measure $\mu \not\equiv 0$ such that $\operatorname{supp}\mu \subset E$ and $U_a^\mu(x)$ is continuous in R^p ([3], p. 236).

3. k_a-capacity and related constants. Let $x^{(n)} = \{x_0, \ldots, x_n\}$ denote an arbitrary system of $n+1$ points of R^p. Put

$$V(x^{(n)}) = \prod_{0 \leqslant i < k \leqslant n} \omega_a(x_i, x_k).$$

We define an *n-th system of extremal points of E*,

$$\xi^{(n)} = \{\xi_{n0}, \ldots, \xi_{nn}\},$$

with respect to ω_a by the condition

$$V(\xi^{(n)}) \geqslant V(x^{(n)}), \quad x^{(n)} \subset E. \tag{3.1}$$

We shall always assume that the points of $\xi^{(n)}$ are enumerated in such a way that

$$\Delta_{n0} \leqslant \Delta_{nj}, \quad j = 1, \ldots, n, \tag{3.2}$$

where $\Delta_{nj} = \prod_{\substack{k=0 \\ k \neq j}}^{n} \omega_a(\xi_{nj}, \xi_{nk})$.

Put

$$\begin{aligned} \varrho_n &= \Big[\inf_{(x_i)} \max_{x \in E} \prod_{i=1}^{n} \omega_a(x, x_i)\Big]^{1/n}, \quad x_i \in \boldsymbol{R}^p, \\ \tilde{\varrho}_n &= \Big[\inf_{(x_i)} \max_{x \in E} \prod_{i=1}^{n} \omega_a(x, x_i)\Big]^{1/n}, \quad x_i \in E. \end{aligned} \tag{3.3}$$

It is known [6] that

$$\varrho_n \leqslant \tilde{\varrho}_n \leqslant \delta_n = \Delta_{n0}^{1/n} \leqslant v_n = [V(\xi^{(n)})]^{2/(n+1)n}, \quad v_{n+1} \leqslant v_n, \tag{3.4}$$

$$\varrho_{\mu+\nu}^{\mu+\nu} \leqslant \varrho_\mu^\mu \varrho_\nu^\nu, \quad \mu, \nu = 0, 1, \ldots, \tag{3.5}$$

$$\tilde{\varrho}_{\mu+\nu}^{\mu+\nu} \leqslant \tilde{\varrho}_\mu^\mu \tilde{\varrho}_\nu^\nu, \quad \mu, \nu = 0, 1, \ldots, \tag{3.6}$$

whence it follows that the sequences $\{\varrho_n\}$, $\{\tilde{\varrho}_n\}$ and $\{v_n\}$ are convergent. The corresponding limits ϱ_a, $\tilde{\varrho}_a$ and v_a are called the (*unconditional*) *Čebyšev constant*, the *conditional Čebyšev constant* and the *transfinite diameter* (or the *span*) of E with respect to ω_a, respectively.

It follows from (3.4), (3.5) and (3.6) that

$$\varrho_a \leqslant \varrho_n, \quad \tilde{\varrho}_a \leqslant \tilde{\varrho}_n, \quad v_a \leqslant v_n, \quad n = 1, 2, \ldots, \tag{3.7}$$

and

$$0 \leqslant \varrho_a \leqslant \tilde{\varrho}_a \leqslant \delta_a = \liminf \delta_n \leqslant \limsup \delta_n \leqslant v_a \leqslant M = \sup_{x, y \in E} \omega_a(x, y). \tag{3.8}$$

It is known ([3], p. 203) that

$$v_a(E) = \exp\big(-W_a(E)\big). \tag{3.9}$$

Let η_0 be a fixed point of E. The sequence $\{\eta_n\}$ defined inductively by the condition

$$\max_{x \in E} \prod_{i=0}^{n-1} \omega_a(x, \eta_i) = \prod_{i=0}^{n-1} \omega_a(\eta_n, \eta_i), \quad n = 1, 2, \ldots$$

was first considered by Leja [5] (for $a = p = 2$) and is called the *sequence of extremal points* of E with respect to ω_a. Put

$$(3.10)\qquad a_n = \Big[\prod_{i=0}^{n-1} \omega_a(\eta_n, \eta_i)\Big]^{1/n}, \qquad n = 1, 2, \ldots$$

One may easily check that

$$(3.11)\qquad \tilde{\varrho}_n \leqslant a_n, \qquad (a_1 a_2^2 \ldots a_n^n)^{2/n(n+1)} \leqslant v_n, \qquad n = 1, \ldots$$

Leja [5] (see also [8]) proved that if $\tilde{\varrho}_a(E) = v_a(E)$, then the sequence $\{a_n\}$ converges to $v_a(E)$.

Szybiak [9] proved that if the kernel k_a satisfies the maximum principle, then $\varrho_a(E) = v_a(E)$. Therefore in virtue of (2.2) we have the following:

LEMMA 3.1. *If $0 < a \leqslant 2$, then the sequences $\{\delta_n\}$ and $\{a_n\}$ are convergent and $\varrho_a = \tilde{\varrho}_a = \delta_a = a_a = v_a = \exp(-W_a)$. We put $\delta_a = \lim \delta_n$ and $a_a = \lim a_n$.*

Indeed, let $c_1, \ldots, c_n$ be points of R^p such that

$$\varrho_n = \Big[\max_{x \in E} \prod_{i=1}^{n} \omega_a(x, c_i)\Big]^{1/n}.$$

Then

$$U_a^{\lambda_n^*}(x) \geqslant \operatorname{Log}(1/\varrho_n), \qquad x \in E,\ n \geqslant 1,$$

where $\lambda_n^* = (1/n) \sum_{i=1}^{n} \varepsilon_{c_i}$ and ε_x denotes the Dirac measure concentrated at the point x. Integrating both sides of the last inequality with respect to λ and applying the maximum principle we get

$$W_a \geqslant \int U_a^{\lambda} d\lambda_n^* = \int U_a^{\lambda_n^*} d\lambda \geqslant -\operatorname{Log} \varrho_n, \qquad n = 1, \ldots$$

These inequalities along with (3.8) give the result.

However, if $2 < a < 3$ and $B = \{x \in R^3\colon |x| = 1\}$, then $\varrho_a(B) < v_a(B)$. Indeed, taking into consideration $P(x) = \prod_{i=1}^{n} \omega_a(x, x_i)$, where $x_i = 0$ $(i = 1, \ldots, n)$, we see that

$$\varrho_a(B) \leqslant \max_{x \in B} P(x)^{1/n} \leqslant \exp\big(-A(p, a)\big).$$

Next, one can easily check (see [3], p. 166 and p. 204) that

$$W_a(B) = A(p, a) 2^{a-2} \pi^{-1/2} \Gamma\Big(\frac{a-1}{2}\Big) \Gamma\Big(\frac{p}{2}\Big) \Big/ \Gamma\Big(\frac{a+p}{2} - 1\Big).$$

So, if $p = 3$, then $W_a(B) = A(3, a) 2^{a-2}/(a-1) < A(3, a)$ for $2 < a < 3$. Therefore $\varrho_a(B) < \exp\big(-W_a(B)\big)$.

By the way we would like to remark that the quantity $w(p, a)$ on p. 166 and on p. 204 of [3] should be replaced by $w(p, a)/\omega_p$, where ω_p is the surface area of the unit sphere $\{x \in R^p: |x| = 1\}$.

We shall now prove that

$$\lim \delta_n = v_a \quad \text{for } 0 < a < p. \tag{3.12}$$

In virtue of (3.4) and (3.9) this equation holds if $W_a = \infty$. Let $W_a(E) < \infty$. Given an arbitrary system $\xi^{(n)}$ of extremal points of E with respect to ω_a put

$$\lambda_n = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{\xi_{ni}}, \quad n = 1, \ldots \tag{3.13}$$

Then (see [3], p. 203) the sequence $\{\lambda_n\}$ converges weakly to the minimizing measure λ of E. It is an easy consequence of (3.1) that

$$U_a^{\lambda_n}(x) \geqslant -\mathrm{Log}\,\delta_n, \quad x \in E,\ n = 1, \ldots$$

According to (2.6) there is a measure μ such that $\mu(E) = 1$, $\mathrm{supp}\,\mu \subset \mathrm{supp}\,\lambda$ and $U_a^{\mu}(x)$ is continuous in R^p. Integrating both sides of the last inequality with respect to μ, we get

$$\int U_a^{\mu}(x)\,d\lambda_n(x) = \int U_a^{\lambda_n}(x)\,d\mu(x) \geqslant -\mathrm{Log}\,\delta_n, \quad n = 1, 2, \ldots$$

whence by (2.1) we get

$$-\mathrm{Log}\,v_a = W_a \geqslant -\liminf \mathrm{Log}\,\delta_n, \quad \text{i.e. } \liminf \delta_n \geqslant v_a.$$

So by (3.8) we get the result.

Given a sequence $\{\eta_n\}$ of extremal points of E with respect to ω_a, we define

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \varepsilon_{\eta_i}, \quad n = 1, 2, \ldots \tag{3.14}$$

Górski [2] proved that the sequence of measures $\{\mu_n\}$ tends weakly to λ, if $a = 2$, $p = 3$ and $W_a(E) < \infty$. His proof is based only on the maximum principle and on the uniqueness of the measure λ. So $\mu_n \to \lambda$ (weakly) for $0 < a \leqslant 2$.

4. Approximation of the equilibrium k_a-potential by k_a-potentials of atomic measures. We shall start with the following

LEMMA 4.1. *Let $W_a(E) < \infty$. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be defined by* (3.13) *and* (3.14), *respectively. Then*

(a) *the sequences $\{U_a^{\lambda_n}(x)\}$ $(0 < a < p)$ and $\{U_a^{\mu_n}(x)\}$ $(0 < a \leqslant 2)$ are convergent to the equilibrium potential $U_a^{\lambda}(x)$ uniformly on every compact subset of $R^p - E$;*

(b) $\liminf_{y \to x} [\liminf_{n \to \infty} U_\alpha^{\lambda_n}(y)] = U_\alpha^\lambda(x)$, $x \in R^p$ *and the same holds for the sequence* $\{U_\alpha^{\mu_n}\}$.

Indeed, the convergence of the sequences $\{U_\alpha^{\lambda_n}\}$ and $\{U_\alpha^{\mu_n}\}$ in $R^p - E$ is guaranteed by the weak convergence of the sequences $\{\lambda_n\}$ and $\{\mu_n\}$. Next, the sequences $\{U_\alpha^{\lambda_n}\}$ and $\{U_\alpha^{\mu_n}\}$ are uniformly bounded and equicontinuous on every compact subset of $R^p - E$. So (a) follows from the Ascoli theorem.

To get (b) it is enough to apply Remark 2 on p. 238 of [3].

LEMMA 4.2 *Let* $0 < \alpha \leqslant 2$ *and* $W_\alpha(E) < \infty$. *Let* $\{E_n\}$ *be a sequence of compact subsets of* E *such that* $E_n \subset E_{n+1}$ *and* $E = UE_n$. *Let* σ_n *denote the minimizing unit measure of* E_n *with respect to the kernel* k_α. *Then the sequence* $U_\alpha^{\sigma_n}(x) + W_\alpha(E_n)$ $(n = 1, \ldots)$ *is increasing and*

$$\lim [U_\alpha^{\sigma_n}(x) - W_\alpha(E_n)] = U_\alpha^\lambda(x) - W_\alpha(E), \quad x \in R^p.$$

Proof. By (2.3) $W_\alpha(E_n) \searrow W_\alpha(E)$, so we may assume that $W_\alpha(E_n) < \infty$. By (2.2),

$$U_\alpha^{\sigma_n}(x) \leqslant U_\alpha^\lambda(x) + W_\alpha(E_n) - W_\alpha(E), \quad \text{n.e. on } E_n.$$

By virtue of the domination principle the same inequality holds for all $x \in R^p$. The same reasoning shows that the sequence $\{U_\alpha^{\sigma_n}(x) - W_\alpha(E_n)\}$ is increasing. Therefore its limit, say $V(x)$, is lowersemicontinuous in R^p. Let $\{\sigma_{n_k}\}$ be an arbitrary subsequent of $\{\sigma_n\}$ which is weakly convergent to a measure σ. By (2.4)

$$U_\alpha^\sigma(x) \leqslant U_\alpha^\lambda(x) \quad \text{in } R^p.$$

Hence and by the uniqueness of the minimizing measure λ, we get $\sigma = \lambda$. Consequently $\sigma_n \to \lambda$ (weakly) and $V(x) = U_\alpha^\lambda(x) - W_\alpha(E)$, $x \in R^p$. The proof is concluded.

LEMMA 4.3. *Let* $0 < \alpha \leqslant 2$ *and* $W_\alpha(E) < \infty$. *If* $P(x)$ *is given by* (1.4) *and* $P(x) \leqslant M$ *on* E, *then*

$$(*) \qquad P(x) \leqslant M \exp\big(-nU_\alpha^\lambda(x) + nW_\alpha(E)\big), \quad x \in R^p,$$

where $U_\alpha^\lambda(x)$ *is the equilibrium potential of* E *with respect to* k_α.

Proof. Observe that (3.3), (3.7) and Lemma 3.1 imply that $\max_{x \in E} \big(P(x)/C\big)^{1/n} \geqslant \varrho_n \geqslant \exp\big(-W_\alpha(E)\big)$. Hence $(1/n)\operatorname{Log}(M/C) + W_\alpha(E) \geqslant 0$. Put $f(x) = (1/n)\operatorname{Log}\big(M/P(x)\big) + W_\alpha(E) = (1/n)\sum_{i=1}^{n} k_\alpha(x, x_i) + (1/n)\operatorname{Log}(M/C) + W_\alpha(E)$. Then $f(x) \geqslant W_\alpha(E)$, $x \in E$. Hence, by (2.2) and by the domination principle, we get the inequality $U_\alpha^\lambda(x) \leqslant f(x)$ for $x \in R^p$, which is equivalent to (*).

Remark 1. Observe that if $a = p = 2$, then (*) is identical with the Bernstein-Walsh inequality ([10], p. 77), proved in the quoted book only for sets E such that the unbounded component of $R^2 - E$ is regular with respect to the classical Dirichlet problem.

Remark 2. Let $W_a(E) < \infty$. The necessary condition that inequality (*) be true for every function $P(x)$ given by (1.4) and satisfying the inequality $P(x) \leqslant M$ on E is that $\varrho_a(E) = \exp\big(-W_a(E)\big)$.

Indeed, let $c_1, \ldots, c_n$ be points of R^p such that $\max\limits_{x\in E}\big[\prod\limits_{i=1}^{n}\omega_a(x, c_i)\big]^{1/n} = \varrho_n$. Put $P(x) = \prod\limits_{i=1}^{n}\omega_a(x, c_i)/\varrho_n^n$. Then $P(x) \leqslant 1$ on E. Suppose that

$$P(x)^{1/n} \leqslant \exp\big(W_a(E) - U_a^\lambda(x)\big), \qquad x \in R^p.$$

Then, letting $|x|$ tend to infinity, we get $1/\varrho_n \leqslant \exp W_a(E)$, whence $\varrho_n \geqslant \exp\big(-W_a(E)\big)$ $(n = 1, \ldots)$. Therefore by (3.8) and (3.9) we get the required equation.

Remark 2 implies the following

Remark 3. If $2 < a < 3$ and E is the unit ball in R^3, then there exists $P(x)$ with $P(x) \leqslant 1$ on E such that (*) does not hold.

Lemma 4.4. *If* $0 < a \leqslant 2$ *and* $W_a(E) < \infty$, *then*

$$\exp\big(-U_a^\lambda(x) + W_a(E)\big) = \begin{cases} L_a(x), & x \in R^p - E, \\ L_a^*(x), & x \in R^p, \end{cases}$$

where L_a *and* L_a^* *are given by* (1.5) *and* (1.6).

Indeed, by Lemma 4.3,

$$P(x)^{1/\deg P} \leqslant \exp\big(-U_a^\lambda(x) + W_a(E)\big), \qquad x \in R^p,\ P \in \mathscr{F}_a.$$

By condition (3.1), defining the extremal points ξ_{ni} $(i = 0, \ldots, n)$ of E with respect to ω_a, we have $\exp\big(-nU_a^{\lambda_n}(x) - \operatorname{Log}\Delta_{n0}\big) \in \mathscr{F}_a$. To conclude the proof it is enough to apply Lemmas 3.1 and 4.1.

5. Proof of the Main Theorem. The implication (b) $\Rightarrow$ (c) is obvious. The implication (c) $\Rightarrow$ (a) is a direct consequence of Lemma 4.4. What remains to prove is the implication (a) $\Rightarrow$ (b).

Given $r > 0$, let $\mathscr{F}_a^\infty$ be an arbitrary family of functions P defined by (1.4) such that $K(x) = K(x, \mathscr{F}_a^\infty) < \infty$ for $x \in E_r$. Put

$$F_n = \{x \in E_r\colon K(x) \leqslant n\}, \qquad n = 1, 2, \ldots$$

Since K is lowersemicontinuous and $K(x) < \infty$ in E_r, the set F_n is closed, $F_n \subset F_{n+1}$ and $E_r = UF_n$. So by Lemma 4.2

$$U_a^{\sigma_n}(x) - W_a(F_n) \nearrow U_a^\lambda(x) - W_a(E_r) \qquad \text{for } x \in R^p.$$

Hence, the limit function being continuous and equal to zero at a, for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0(\varepsilon)$ such that

$$U_a^{\sigma_n}(x) - W_a(F_n) \geqslant -\varepsilon, \qquad n \geqslant n_0(\varepsilon), \ |x-a| < \delta.$$

Therefore, in virtue of Lemma 4.3, we get

$$P(x) \leqslant n_0 \exp(\varepsilon \deg P), \qquad |x-a| < \delta, \ P \in \mathcal{F}_a^{\infty}.$$

The proof is concluded.

Remark. One may easily show that if $0 < M_i < \infty$ $(i = 1, 2)$, then the condition $(L_a^{M_1})$ is satisfied at a point $a \in R^p$ if and only if the condition $(L_a^{M_2})$ is satisfied at a.

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