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Loss of uniqueness property in difference approximation of a Dirichlet problem

In this note we give an example of a Dirichlet problem for a (smooth) elliptic differential equation possessing a unique solution and admitting a difference approximation without uniqueness property. The differential equation is of the form

$$a(x, y)u_{xx}+b(x, y)u_{xy}+c(x, y)u_{yy}+d(x, y)u = 0 \tag{1}$$

and satisfies the assumption of elipticity

$$4a(x, y)c(x, y)-b(x, y)^2 > 0 \tag{2}$$

as well as uniqueness conditions

$$d(x, y) \leqslant 0, \qquad a(x, y) > 0. \tag{3}$$

Inequalities (2), (3) are classical conditions implying the uniqueness of solutions for any Dirichlet problem for (1).

Replacing derivatives in (1) by suitable difference quotients gives a difference equation possessing non-vanishing solution of homogeneous Dirichlet problem. This example shows that the translation of uniqueness theorems from differential to difference equations need not be straightforward.

We shall start with construction of a solution of the difference equation, then we define the difference equation and from it we obtain the differential equation.

Notations. We shall consider a set Z of nodal points $Z = \{(i, j)$: i, j integers, $0 \leqslant i \leqslant 3$, $|j| \leqslant n\}$, where integer $n > 1$ will be determined later (to be sufficiently large). On the set of interior nodal points $S = \{(i, j)$: $i = 1$ or $2, j$ integer $|j| < n\}$ the difference equation will be satisfied. Difference quotients corresponding to the derivatives u_{xx}, u_{xy}, u_{yy} will be denoted by $u_{11}^{ij}, u_{12}^{ij}, u_{22}^{ij}$

$$u_{11}^{ij} = u^{i+1,j} - 2u^{ij} + u^{i-1,j},$$
$$u_{12}^{ij} = \tfrac{1}{4}(u^{i+1,j+1} - u^{i-1,j+1} - u^{i+1,j-1} + u^{i-1,j-1}),$$
$$u_{22}^{ij} = u^{i,j+1} - 2u^{ij} + u^{i,j-1},$$

for $(i,j) \in S$.

Now we define a solution of difference equation. We define

$$u^{ij} = 0 \quad \text{for } (i,j) \in Z \setminus S,$$
$$u^{1j} = 1 - \frac{1}{n}|j| + (j^2 - n^2)r, \quad |j| < n,$$
$$u^{21} = \tfrac{4}{9} + r,$$
$$u^{2j} = r \quad \text{for } |j| < n,\ j \neq 1,$$

where $r = (n+1)^{-3}$.

Obviously

(4) $$0 < u^{ij} < 1 \quad \text{for } (i,j) \in S$$

and therefore

(5) $$|u_{11}^{ij}|,\ |u_{22}^{ij}| < 2 \quad \text{for } (i,j) \in S.$$

We have

(6) $$u_{22}^{1j} = 2r \quad \text{for } 0 < |j| < n,$$

(7) $$0 > u_{22}^{10} = -\frac{2}{n} + 2r > -\frac{2}{n},$$

(8) $$u_{12}^{10} = \tfrac{1}{9},$$

(9) $$u_{11}^{2j} = 1 - \frac{1}{n}|j| + (j^2 - n^2 - 2)r > 2r \quad \text{for } |j| < n, j \neq 1,$$

(10) $$u_{11}^{21} = \frac{1}{9} - \frac{1}{n} - (n^2+1)r > 2r \quad \text{for large } n.$$

Now we define coefficients of the difference equation. We put

$$a^{10} = 1, \quad b^{10} = -9u_{11}^{10} - 9nu_{22}^{10}, \quad c^{10} = n, \quad d^{10} = 0.$$

It is easy to see that for $(i,j) = (1,0)$ the difference equation

(11) $$a^{ij}u_{11}^{ij} + b^{ij}u_{12}^{ij} + c^{ij}u_{22}^{ij} + d^{ij}u^{ij} = 0$$

is satisfied and in virtue of (5) and (7) we have the inequality

(12) $$4a^{ij}c^{ij} - (b^{ij})^2 > 0,$$

for large n, as well as the inequalities

$$d^{ij} \leqslant 0, \tag{13}$$

$$a^{ij} > 0. \tag{14}$$

For $i = 1, 0 < |j| < n$ we put

$$a^{1j} = r, \quad b^{1j} = 0, \quad c^{1j} = 1, \quad d^{1j} = -(ru_{11}^{1j} + u_{22}^{1j})/u^{1j}.$$

Conditions (11), (12) and (14) are evident and (13) results from (5), (6) and (4).

For $i = 2, |j| < n$ we put

$$a^{2j} = 1, \quad b^{2j} = 0, \quad c^{2j} = r, \quad d^{2j} = -(u_{11}^{2j} + ru_{22}^{2j})/u^{2j}.$$

Similarly as in the previous case conditions (11), (12) and (14) are evident. Condition (13) results from (9) or (10), (5) and (4).

From now on we consider integer n to be sufficiently large for (12), to be satisfied, and fixed.

We have constructed the coefficients $a^{ij}, b^{ij}, c^{ij}, d^{ij}$ satisfying (12), (13) and (14) for $(i, j) \in S$. Equation (11) has the non-zero solution u^{ij} defined on Z and vanishing on $Z \setminus S$.

Now we define coefficients of (1) on nodal points. We put

$$a(i, j) = a^{ij}, \quad b(i, j) = b^{ij}, \quad c(i, j) = c^{ij}, \quad d(i, j) = d^{ij}$$

for $i = 1$ or 2, j being integer and $|j| < n$. Inequalities (2) and (3) result from (12), (13) and (14). It is easy to see that the definition of functions $a(x, y), b(x, y), c(x, y)$ and $d(x, y)$ can be extended in a smooth manner on the whole rectangle $R = \{(x, y)\colon 0 \leqslant x \leqslant 3, |y| \leqslant n\}$ in such a way that inequalities (2) and (3) are satisfied on R.

If we suitably replace derivatives in (1) by difference quotients $u_{11}^{ij}, u_{12}^{ij}, u_{22}^{ij}$ and function $u(x, y)$ as well as coefficients of (1) by their values at the nodal points $(x, y) \in R$, x, y being integers, we obtain back the difference equation (11), and therefore equation (1) has the claimed properties.

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