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A characterization of Šilov boundary in function algebras*

0. Let A be a commutative complex Banach algebra with unit e. It was proved by Arens [1] that an element $x \in A$ is permanently singular (i.e. *singular in any superalgebra of* A) if and only if x is a topological divisor of zero. Consequently, if M is a maximal ideal in A and M belongs to the Šilov boundary $\Gamma(A)$ of A, then M consists entirely of topological divisors of zero.

In this paper we make an observation that if A is a function algebra and $M \in \Gamma(A)$, then all elements of M are topological divisors of zero which are realised by the same Moore-Smith sequence (x_σ) with $\|x_\sigma\| = 1$, i.e., $\lim ax_\sigma = 0$ for each $a \in M$. On the other hand, if M is a maximal ideal in A and there is a Moore-Smith sequence satisfying above conditions for all $x \in M$, then $M \in \Gamma(A)$. This is the characterization of Šilov boundary in a function algebra A announced in the title. We give also a subsequent characterization of non-removable ideals in a function algebra A and some examples showing that the results cannot be refined for function algebras.

1. We now recall some concepts and give necessary definitions used in the sequel. The proofs of the statements can be found, e.g., in [5]. From now on A will stand for a function algebra, i.e., a uniformly closed subalgebra of the algebra $C(\Omega)$ of all complex valued continuous functions on a compact Hausdorff space Ω. Without loss of generality we can assume that A contains constant functions and Ω is the maximal ideal space for A, i.e., the functions in A separate between points of the space Ω and every multiplicative linear functional on A is of the form

$$f_\omega(x) = x(\omega),$$

* This work was done during the author's stay in Budapest and Szeged in December 1968.

where ω is a fixed point in Ω. We put

$$M_\omega = \{x \in A : f_\omega(x) = 0\};$$

it is a maximal ideal in A and every maximal ideal of A is of this form. We denote by Γ (or $\Gamma(A)$) the Šilov boundary of A, i.e., a uniquely determined closed subset of Ω which is minimal among all closed subsets $F \subset \Omega$ with the property that $\sup_F |x(\omega)| = \sup_\Omega |x(\omega)|$ for each $x \in A$. A point $\omega_0 \in \Omega$ belongs to the Šilov boundary Γ if and only if for each neighbourhood U of ω_0 in Ω there is a function $x \in A$ such that

$$\sup_U |x(\omega)| > \sup_{\Omega \setminus U} |x(\omega)|. \tag{1}$$

By a *superalgebra* of A we mean an arbitrary complex Banach algebra B with unit element (which is not necessarily a function algebra) such that there is in B a (closed) subalgebra with unit, isomorphic to A. In this case we write $A \subset B$. An ideal I in A is said to be *non-removable* (cf. [2]) if for every superalgebra $B \supset A$ the ideal J generated by I in B is a proper ideal. Since

$$J = \Big\{\sum_{i=1}^{k} a_i z_i \in B : a_i \in I,\ z_i \in B\Big\},$$

I is a removable ideal if and only if there exist a superalgebra $B \supset A$ and elements $z_1, \ldots, z_k \in B$, $a_1, \ldots, a_k \in I$ such that

$$\sum_{i=1}^{k} a_i z_i = e. \tag{2}$$

If M is a maximal ideal in Γ (i.e., $M = M_\omega$ for $\omega \in \Gamma$, in such a situation we shall write $M \in \Gamma$), then it is a non-removable ideal in A. More generally, an ideal $I \subset A$ is a non-removable ideal if $I \subset M \in \Gamma$. On the other hand, from the imbedding $A \subset C(\Gamma)$ it follows that the only non-removable ideals in A are those which are contained in maximal ideals belonging to Γ.

An element $a \in A$ is said to be a *topological divisor* of zero if

$$\delta(a) = \inf_{\substack{x \in A \\ \|x\|=1}} \|ax\| = 0. \tag{3}$$

It was shown in [1] that for a non-invertible $x \in A$ the principal ideal xA is non-removable if and only if x is a topological divisor of zero.

A directed set Σ is a partially ordered set with partial order relation $\leqslant$ such that for any $\sigma_1, \sigma_2 \in \Sigma$ there is a $\sigma \in \Sigma$ such that $\sigma \geqslant \sigma_1$ and $\sigma \geqslant \sigma_2$. A Moore-Smith sequence of elements of A is a map $\Sigma \to A$ written as (x_σ), $\sigma \in \Sigma$, $x_\sigma \in A$. It is said to be convergent to $x_0 \in A$ if for each $\varepsilon > 0$ there is a $\sigma_\varepsilon \in \Sigma$ such that $\|x_\sigma - x_0\| \leqslant \varepsilon$ for all $\sigma \geqslant \sigma_\varepsilon$.

We need a generalization of the function δ defined by (3). Let S be a non-void subset in A. Denote by $\mathscr{F}(S)$ the collection of all finite subsets of S; its elements will be denoted by $\tilde{a} = (a_1, \ldots, a_k)$, $a_i \in S$. We put now

$$\delta(\tilde{a}) = \inf_{\substack{x \in A \\ \|x\|=1}} \sum_{i=1}^{k} \|a_i x\| \tag{4}$$

(such an expression is considered also in [3]).

2. PROPOSITION 1. *If $\omega_0 \in \Gamma$, then there exists a Moore-Smith sequence $(x_\sigma) \subset A$, $\sigma \in \Sigma$, such that*

$$\|x_\sigma\| = 1 \tag{5}$$

for every $\sigma \in \Sigma$ and

$$\lim_{\sigma} a x_\sigma = 0 \tag{6}$$

for every $a \in M_{\omega_0}$.

Proof. Denote by Σ the set of all pairs $\sigma = (U, n)$, where U is a neighbourhood of ω_0 and n is a positive integer. We write $\sigma_1 \geqslant \sigma_2$ if $U_1 \subset U_2$ for corresponding neighbourhoods, and $n_1 \geqslant n_2$ for corresponding integers. In order to construct the desired sequence (x_σ) we observe that for a neighbourhood U of ω_0 there is a function $x \in A$ satisfying (1). Assuming $\|x\| = 1$ and taking suitable power x^N we have

$$\sup_{\Omega \setminus U} |x^N(\omega)| < \varepsilon$$

for a prescribed $\varepsilon > 0$. So for $\sigma = (U, n)$ we can find a function $x_\sigma \in A$ satisfying (5) and such that

$$\sup_{\Omega \setminus U} |x_\sigma(\omega)| < \frac{1}{n} \tag{7}$$

holds.

For an element $a \in M_{\omega_0}$ we have $a(\omega_0) = 0$ and so for a given $\varepsilon > 0$ there is a neighbourhood U_ε of ω_0 such that

$$\sup_{U} |a(\omega)| \leqslant \varepsilon. \tag{8}$$

If we take an integer n_ε in such a way that

$$\varepsilon \geqslant \frac{\|a\|}{n_\varepsilon} \tag{9}$$

and if we put $\sigma_\varepsilon = (U_\varepsilon, n_\varepsilon)$, we see that for any $\sigma \geqslant \sigma_\varepsilon$, $\sigma = (U, n)$ we have, by (8)

$$\sup_{U} |a(\omega) x_\sigma(\omega)| \leqslant \varepsilon \|x_\sigma\| \leqslant \varepsilon \tag{10}$$

and by (7) and (9)

$$\sup_{\Omega \setminus U_4} |a(\omega) x_\sigma(\omega)| \leqslant \|a\| \frac{1}{n} \leqslant \varepsilon. \tag{11}$$

Relation (6) is now a consequence of (10) and (11).

COROLLARY 1. *For $M \in \Gamma(A)$ and for $a_1, \ldots, a_k \in M$ there exists a sequence (x_n) indexed by natural numbers such that $\|x_n\| = 1$ and*

$$\lim_n a_i x_n = 0$$

for $i = 1, 2, \ldots, k$,

COROLLARY 2. *If A is a separable algebra, then one can take in Proposition 1 the set N of natural numbers instead of Σ.*

Proof. In this case Ω is metrisable so we can put $\sigma_n = (U_n, n)$, where U_n is a ball in Ω with center ω_0 and radius $1/n$, and then apply the proof of Proposition 1.

COROLLARY 3. *If $M \in \Gamma(A)$, then $\delta(\tilde{a}) = 0$ for every $\tilde{a} \in \mathscr{F}(M)$, where δ is given by formula* (3).

We show now an example indicating that in Proposition 1 we cannot replace a Moore-Smith sequence by an ordinary sequence indexed by natural numbers. Let Ω be a non-denumerable product of the unit interval, e.g. $\Omega = [0, 1]^c$. The elements of Ω are of the form $\omega = (\eta_t)$, $t \in [0, 1]$, and $0 \leqslant \eta_t \leqslant 1$. The number η_t will be called the *t-th coordinate of* ω. Let A be a subalgebra of $C(\Omega)$ consisting of functions depending upon at most denumerable number of coordinates, i.e., constant with respect to the remaining ones. This is easy to see that A is a uniformly closed subalgebra of $C(\Omega)$ which contains constants and together with a function it contains its complex conjugate. Thus, by Stone-Weierstrass theorem, $A = C(\Omega)$ and so $\Gamma = \Omega$. It is sufficient to show that for a fixed $\omega_0 \in \Omega$ and for an arbitrary sequence $(x_n) \subset C(\Omega)$, $\|x_n\| = 1, n = 1, 2, \ldots$, there is a function a in $C(\Omega)$ such that $a(\omega_0) = 0$, and $\|ax_n\| = 1$ for $n = 1, 2, \ldots$ We know that each function x_n depends on a denumerable number of coordinates, so there is a coordinate η_{t_0} such that every x_n is constant with respect to this coordinate. We put $a(\omega) = (\eta_{t_0} - \eta_{t_0}^0) \cdot \alpha$, where $\eta_{t_0}^0$ is t_0-th coordinate of ω_0, and α is choosen in such a way that $\|a\| = 1$. So $a(\omega_0) = 0$ and there is a value η_{t_0}' such that if the t_0-th coordinate of ω has this value, then $|a(\omega)| = 1$. We have then

$$\|ax_n\| = \sup_\omega |a(\omega) x_n(\omega)| = \sup_{\eta_{t_0} = \eta'_{t_0}} |a(\omega) x_n(\omega)| = \sup_{\eta_{t_0} = \eta'_{t_0}} |x_n(\omega)| = 1$$

and the desired conclusion follows.

PROPOSITION 2. *If M is a maximal ideal in A, i.e., $M = M_\omega$ for some $\omega \in \Omega$, and if there is a Moore-Smith sequence $(x_\sigma) \subset A$, $\sigma \in \Sigma$, which satisfies* (5), *such that for each $x \in M$ holds relation* (6), *then $M \in \Gamma(A)$.*

Proof. Since A is a function algebra it is sufficient to show that M is a non-removable ideal in A. But if we assume that M is a removable ideal we have some elements $a_1, \ldots, a_k \in M$, and $z_1, \ldots, z_k$ in a certain extension B of A such that (2) holds. Multiplying both sides of (2) by x_σ we have

$$1 = \|x_\sigma\| = \Big\|\sum_{i=1}^{k} a_i x_\sigma z_i\Big\| \leqslant (\max_{i\leqslant k}\|z_i\|)\sum_{i=1}^{k}\|a_i x_\sigma\|;$$

in view of (6), this leads to a contradiction.

COROLLARY 4. *If M is a maximal ideal in A and $\delta(\tilde{a}) = 0$ for every $\tilde{a} \in \mathscr{F}(M)$, then $M \in \Gamma(A)$.*

Proof. Denote by Σ the set of all pairs $\sigma = (\tilde{a}, n)$, where $\tilde{a} \in \mathscr{F}(M)$ and n is a positive integer. We write $\sigma_1 \geqslant \sigma_2$ if for corresponding subsets of $\mathscr{F}(M)$ it is $\tilde{a}_1 \supset \tilde{a}_2$ and for corresponding integers it is $n_1 \geqslant n_2$. By the definition of the function δ and by assumptions of the corollary, for every $\sigma \in \Sigma$ we can find a function $x_\sigma \in A$ with $\|x_\sigma\| = 1$ such that

$$\|ax_\sigma\| \leqslant 1/n$$

for each $a \in \tilde{a}$, where $\sigma = (\tilde{a}, n)$. This is easy to see that for every $a \in M$ it is $\lim_\sigma ax = 0$, so Proposition 2 implies the desired conclusion.

In Proposition 2 it is essential that all elements in M are topological divisors of zero with the same sequence (x_σ). Here we give an example of a function algebra on a certain Ω for which every maximal ideal consists entirely of topological divisors of zero and for which $\Gamma \neq \Omega$. To this end we put

$$\Omega = \{(\lambda_1, \lambda_2) \in C^2\colon\ |\lambda_1|^2 + |\lambda_2|^2 \leqslant 1\}$$

and take as A the algebra of all functions in $C(\Omega)$ which are holomorphic in the interior of Ω. It is known that Ω is the maximal ideal space for A while Γ is its topological boundary in C^2. On the other hand, every function in A which is zero in some point of Ω must be zero on some point in the boundary of Ω (cf. e.g. [4]), so it must be a topological divisor of zero in A.

We can formulate now our main result, which is a consequence of Propositions 1 and 2.

THEOREM 1. *A maximal ideal M of a function algebra A belongs to the Šilov boundary $\Gamma(A)$ if and only if there exists a Moore-Smith sequence $(x_\sigma) \subset A$, $\sigma \in \Sigma$, such that $\|x_\sigma\| = 1$ for each $\sigma \in \Sigma$ and*

$$\lim_\sigma ax_\sigma = 0$$

for each $a \in M$.

From Corollaries 3 and 4 we get the following version of this theorem:

THEOREM 1'. *A maximal ideal M of a function algebra A belongs to the Šilov boundary $\Gamma(A)$ if and only if the function δ, given by* (3), *equals identically zero on $\mathscr{F}(M)$.*

Since an ideal I of a function algebra A is a non-removable ideal if and only if it is contained in some maximal ideal belonging to the Šilov boundary $\Gamma(A)$, Theorem 1' implies the following

THEOREM 2. *Let A be a function algebra and let I be an ideal in A. $I \neq A$. Then I is a non-removable ideal if and only if*

$$\delta(\tilde{a}) = 0$$

for each $\tilde{a} \in \mathscr{F}(I)$.

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