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A note on the shape of quasi-homeomorphic compacta

The concept of the shape (see [1]) has been introduced in order to compare compacta from the point of view of their global topological properties. There exist some other notions (as the notion of the X-likeness, [4], p. 146, and the notion of the quasi-homeomorphism [3], p. 252) which play a similar role. The aim of this note is to study some relations between those notions.

§ 1. Basic definitions. We denote by Q the Hilbert cube, that is the subset of the real Hilbert space E^ω consisting of all points $(x_1, x_2, \ldots)$ satisfying the condition

$$0 \leqslant x_n \leqslant \frac{1}{n} \quad \text{for every } n = 1, 2, \ldots,$$

E^n denote the subset of E^ω consisting of all points of the form $(x_1, x_2, \ldots \ldots, x_n, 0, \ldots)$.

Let X, Y be two compacta lying in the Hilbert cube Q. A sequence of maps $f_k\colon Q \to Q$ is said to be a *fundamental sequence* (compare [2], p. 225) from X to Y (notation: $\underline{f} = \{f_k, X, Y\}$, or $\underline{f}\colon X \to Y$), if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \quad \text{in } V \text{ for almost all } k.$$

Two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{f}' = \{f'_k, X, Y\}$ are said to be *homotopic* (notation: $\underline{f} \simeq \underline{f}'$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq f'_k/U \quad \text{in } V \text{ for almost all } k.$$

If X, Y, Z are compacta lying in Q and $\underline{f} = \{f_k, X, Y\}$, $\underline{g} = \{g_k, Y, Z\}$ are fundamental sequences, then $\{g_k f_k, X, Z\}$ is a fundamental sequence which one denotes by $\underline{g}\,\underline{f}$. The fundamental sequence $\{f_k, X, X\}$, where $f_k = i$ (i denotes the identity map of Q onto itself) for every $k = 1, 2, \ldots$, is said to be the *fundamental identity sequence for X*; one denotes it by $\underline{i}_X$.

If there exist two fundamental sequences

$$\underline{f} = \{f_k, X, Y\} \quad \text{and} \quad \underline{g} = \{g_k, Y, X\}$$

such that $\underline{f}\underline{g} \simeq \underline{i}_Y$, then we say that X *fundamentally dominates* Y (compare [2], p. 233). Then we write $X \underset{F}{\geqslant} Y$. If there exist two fundamental sequences $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that both homotopies $\underline{f}\underline{g} \simeq \underline{i}_Y$ and $\underline{g}\underline{f} \simeq \underline{i}_X$ hold, then X and Y are said to be *fundamentally equivalent* (notation: $X \underset{F}{\simeq} Y$).

Since both relations $\underset{F}{\geqslant}$ and $\underset{F}{\simeq}$ are topological (compare [2], p. 234), one can extend them onto arbitrary compacta as follows:

$X \underset{F}{\geqslant} Y$ means that there exist two compacta $X', Y' \subset Q$ homeomorphic to X and Y respectively and such that $X' \underset{F}{\geqslant} Y'$. On the same way one extends the relation $\underset{F}{\simeq}$ onto arbitrary compacta.

Now one defines the *shape* $\mathrm{Sh}(X)$ *of a compactum* X as the collection of all compacta Y satisfying the condition $X \underset{F}{\simeq} Y$. The relation $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ means that $X \underset{F}{\geqslant} Y$.

We say that a space Y is *X-like* (compare [4], p. 146) if for every $\varepsilon > 0$ there is a map g of Y onto X such that

(1.1) $\delta[g^{-1}(x)] < \varepsilon$ for every point $x \in X$.

One sees easily that if X, Y, Z are compacta and if Y is X-like and Z is Y-like, then Z is X-like.

Two compacta X and Y are said to be *quasi-homeomorphic* (see [3], p. 252) if Y is X-like and X is Y-like. It is clear that quasi-homeomorphism is an equivalence relation.

§ 2. X-likeness and the shape. Let us prove the following

(2.1) THEOREM. *Let X, Y be two compacta. If $Y \in \mathrm{ANR}$ and Y is X-like, then* $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$.

Proof. Since the notions of the X-likeness and of the shape are topological, we may assume that X and Y are subsets of the Hilbert cube Q. Since $Y \in ANR$, there is a neighborhood V of Y (in Q) and a retraction

$$r\colon V \to Y.$$

Let a be a positive number so small that

(2.2) $\varrho(y, Y) < 3a$ implies $y \in V$ for every point $y \in Q$.

It is clear, that we can assign to this number a and to the given retraction r a positive number $\varepsilon < a$ so small that

(2.3) $\varrho(y, Y) < \varepsilon$ implies $\varrho(y, r(y)) < a$ for every point $y \in Q$.

Since Y is X-like, there exists a map

$$g\colon\ Y\to X$$

such that $g(Y) = X$ and that condition (1.1) is satisfied. To the number ε we can assign a number $\eta > 0$ such that

(2.4) If $A\subset X$ and $\delta(A) < \eta$, then $\delta[g^{-1}(A)] < \varepsilon$.

Now let us consider a finite cover of X by open (in X) sets $G_0, G_1, \ldots$ $\ldots, G_m$ with diameters less than $\frac{1}{3}\eta$. We may assume that none of those sets is contained in the union of the others. Thus we can select a point

$$a_i \in G_i - \bigcup_{j\neq i} G_j \quad \text{for every } i = 0, 1, \ldots, m.$$

Let us assign to every point a_i a point $b_i \in g^{-1}(a_i)$ and let us set

$$F_i = X - G_i \quad \text{for } i = 0, 1, \ldots, m.$$

Consider the function

$$f\colon\ X\to Q$$

given by the formula

$$f(x) = \lambda_0(x)\cdot b_0 + \lambda_1(x)\cdot b_1 + \ldots + \lambda_m(x)\cdot b_m \quad \text{for every point } x\in X,$$

where

$$\lambda_i(x) = \frac{\varrho(x, F_i)}{\varrho(x, F_0)+\varrho(x, F_1)+\ldots+\varrho(x, F_m)}.$$

One sees easily that f is a map (that it is continuous) and if $x\in G_i$, then $f(x)$ belongs to the convex hull H_i of the set consisting of all points b_{i_ν} such that $G_{i_\nu}\cap G_i \neq \emptyset$. Since $\delta(G_i) < \frac{1}{3}\eta$, we infer that $\delta(\bigcup_\nu G_{i_\nu}) < \eta$ and consequently the diameter of the set H_i is less than $\varepsilon < 3\alpha$. It follows by (2.2) that $H_i\subset V$. Thus the formula

$$\hat{f} = rf$$

defines a map $\hat{f}\colon X\to Y$ such that

$$\hat{f}g(b_i) = \hat{f}(a_i) = rf(a_i) = r(b_i) = b_i.$$

If $y\in g^{-1}(G_i)$, then

$$\varrho\big(\hat{f}g(y), y\big) = \varrho\big(rfg(y), y\big) \leqslant \varrho\big(rfg(y), fg(y)\big) + \varrho\big(fg(y), b_i\big) + \varrho(b_i, y).$$

But (2.3) implies that $\varrho\big(rfg(y), fg(y)\big) < \alpha$. Moreover, $g(y)\in G_i$, hence $fg(y)$ belongs to the set H_i. Since $b_i\in H_i$ and $\delta(H_i) < \varepsilon$, we infer that $\varrho\big(fg(y), b_i\big) < \varepsilon$. Finally $\varrho(b_i, y) < \varepsilon$, because $g(b_i) = a_i\in G_i$ and $g(y)\in G_i$, hence both points b_i and y belong to $g^{-1}(G_i)$, which implies that $\varrho(b_i, y) < \varepsilon$.

Thus we have shown that

$$\varrho\big(\hat{f}g(y), y\big) < 3\varepsilon < 3a \quad \text{for every point } y \in Y.$$

It follows by (2.2) that all points of the form

$$t\cdot\hat{f}g(y)+(1-t)\cdot y \quad , \text{with } 0 \leqslant t \leqslant 1,$$

belong to V. Consequently, setting

$$\varphi(y, t) = r[t\cdot\hat{f}g(y)+(1-t)\cdot y] \quad \text{for } (y, t) \in Y\times\langle 0, 1\rangle,$$

we get a homotopy φ: $Y\times\langle 0, 1\rangle \to Y$ joining the identity map i_Y with the map $\hat{f}g$. Hence $\hat{f}g \simeq i_Y$.

The result, we have obtained, may be formulated as follows:

(2.5) *If X is a compactum and if a compact* ANR*-set Y is X-like, then there exist two maps $\hat{f}$: $X \to Y$ and g: $Y \to X$ such that $\hat{f}g$ is homotopic to the identity map i_Y.*

It follows that the fundamental sequences $\underline{\hat{f}}$: $X \to Y$ and $\underline{g}$: $Y \to X$, generated (compare [2], p. 227) by the maps $\hat{f}$ and g respectively, satisfy the condition $\underline{\hat{f}}\underline{g} \simeq i_Y$. Hence $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ and the proof of Theorem (2.1) is finished.

Let us observe that proposition (2.5) fails if we omit the hypothesis $Y \in$ ANR. In fact, one sees easily that if X denotes the interval $\langle 0, 1\rangle$, then the closure Y of the diagram of the function $y = \sin\dfrac{1}{x}$ with $0 < x < 1$ is X-like. However, no map of the form $\hat{f}g$, where $\hat{f}$: $X \to Y$ and g: $Y \to X$ is homotopic to the identity map i_Y.

§ 3. Fundamental domination and components. Let $\square(X)$ denote the set of all components of a space X. Let us prove the following

(3.1) THEOREM. *If X, Y are two compacta lying in the Hilbert cube Q and if $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, X\}$ are two fundamental sequences such that $\underline{g}\underline{f} \simeq \underline{i}_X$, then there exist two functions*

$$\Lambda\colon \square(X) \to \square(Y) \quad \textit{and} \quad \Lambda'\colon \square(Y) \to \square(X)$$

satisfying the following conditions:

1° *$\Lambda'\Lambda(X_0) = X_0$ for every $X_0 \in \square(X)$.*

2° *If $X_0 \in \square(X)$, then $\underline{\hat{f}} = \{f_k, X_0, \Lambda(X_0)\}$ and $\underline{\hat{g}} = \{g_k, \Lambda(X_0), X_0\}$ are fundamental sequences such that $\underline{\hat{g}}\underline{\hat{f}} \simeq \underline{i}_{X_0}$.*

3° *If $Y_0, Y_1, \ldots \in \square(Y)$ and $\overline{\lim\limits_{n=\infty}} Y_n \subset Y_0$, then $\overline{\lim\limits_{n=\infty}} \Lambda'(Y_n) \subset \Lambda'(Y_0)$.*

Proof. Let a be a point of a component X_0 of X. Then

$$\lim_{k=\infty} \varrho\big(f_k(a), Y\big) = 0,$$

and we infer that there is an increasing sequence $\{k_n\}$ of indices such that the sequence $f_{k_1}(a), f_{k_2}(a), \ldots$ converges to a point $b \in Y$. Let Y_0 denote the component of Y containing b. Then for every neighborhood V_0 of Y_0 there is a neighborhood V of Y such that the component of V containing Y_0 lies in V_0. Since $\underline{f}$ is a fundamental sequence, there exists a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \quad \text{in } V \text{ for almost all } k.$$

If U_0 denotes the component of U containing X_0, one infers easily that

$$f_k/U_0 \simeq f_{k+1}/U_0 \quad \text{in } V_0 \text{ for almost all } k.$$

Thus

(3.2) $\{f_k, X_0, Y_0\}$ is a fundamental sequence.

It is clear that for every component X_0 of X there is only one component Y_0 of Y satisfying (3.2). Setting

$$\Lambda(X_0) = Y_0,$$

we get a function $\Lambda\colon \square(X) \to \square(Y)$.

By an analogous argument one infers that there exists a function Λ' assigning to every component Y_0 of Y a component $X_0' = \Lambda'(Y_0)$ of X such that

(3.3) $\{g_k, Y_0, X_0'\}$ is a fundamental sequence.

Consider now the component $X_0' = \Lambda'(Y_0) = \Lambda'\Lambda(X_0)$ of the compactum X. Then $\{g_k, Y_0, X_0'\}$ is a fundamental sequence and we infer that for every neighborhood U_0' of X_0' there is a neighborhood V_0 of Y_0 such that

$$g_k(V_0) \subset U_0' \quad \text{for almost all } k.$$

Moreover, (3.2) implies that $f_k(X_0) \subset V_0$ for almost all k. Hence

(3.4) $g_k f_k(X_0) \subset U_0'$ for almost all k.

On the other hand, for every neighborhood U_0 of X_0 there is a neighborhood $\hat{U}$ of X such that the component $\hat{U}_0$ of $\hat{U}$ containing X_0 lies in U_0. Since $\underline{g}\underline{f} \simeq \underline{i}_X$ and $i(X_0) = X_0 \subset \hat{U}_0$, we infer that

(3.5) $g_k f_k(X_0) \subset \hat{U}_0 \subset U_0$ for almost all k.

It follows by (3.4) and (3.5) that every neighborhood U_0' of the component X_0' intersects every neighborhood U_0 of the component X_0. Hence $X_0 = X_0' = \Lambda'(Y_0') = \Lambda'\Lambda(X_0)$, that is condition 1° is satisfied.

Moreover, the relations $Y_0 = \Lambda(X_0)$ and $X_0 = \Lambda'\Lambda(X_0)$ imply that (3.2) and (3.3) may be rewritten in the following form:

$$\underline{\hat{f}} = \{f_k, X_0, \Lambda(X_0)\} \quad \text{and} \quad \underline{\hat{g}} = \{g_k, \Lambda(X_0), X_0\}$$

are fundamental sequences.

The homotopy $\underline{gf} \simeq \underline{i}_X$ implies that there exists a neighborhood U' of X such that

(3.6) $g_k f_k / U' \simeq i / U'$ in $\hat{U}$ for almost all k.

Let $\hat{U}_0'$ denote the component of U' containing X_0. It follows by (3.6) that $g_k f_k(X) \subset \hat{U}$ for almost all k and we infer by (3.5) that

$$g_k f_k / \hat{U}_0' \simeq i / \hat{U}_0' \quad \text{in } U_0 \text{ for almost all } k.$$

Thus we have shown that

$$\hat{g}\hat{f} = \{g_k f_k, X_0, X_0\} \simeq \underline{i}_{X_0},$$

i.e. condition 2° is satisfied.

In order to prove 3°, let us consider a sequence $Y_0, Y_1, \ldots$ of components of the set Y with $\overline{\lim_{n=\infty}} Y_n \subset Y_0$. Let U be a neighborhood of the component $\Lambda'(Y_0)$. Then there exists an open neighborhood U_0 of $\Lambda'(Y_0)$ such that $U_0 \subset U$ and that

(3.7) $\overline{X \cap U_0} \cap (X - U_0) = \emptyset$.

It follows that there is a neighborhood V_0 of Y_0 such that

$$g_k(V_0) \subset U_0 \quad \text{for almost all } k.$$

Since $\overline{\lim_{n=\infty}} Y_n \subset Y_0$, we infer that there exists an index n_0 such that

$$Y_n \subset V_0 \quad \text{for every } n \geqslant n_0.$$

It follows that for $n \geqslant n_0$

$$g_k(Y_n) \subset U_0 \quad \text{for almost all } k.$$

Since every neighborhood of the set $\Lambda'(Y_0)$ contains $g_k(Y_n)$ for almost all k, we infer that $\Lambda'(Y_n) \subset U_0 \subset U$ for every $n \geqslant n_0$. Hence

$$\overline{\lim_{n=\infty}} \Lambda'(Y_n) \subset \Lambda'(Y_0),$$

that is condition 3° is satisfied and the proof of Theorem (3.1) is finished.

(3.8) Corollary. *If X, Y are two compacta and if* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, *then there exists a one-to-one correspondence Λ between the sets of components $\square(X)$ and $\square(Y)$ such that the corresponding components have the same shape and if $X_0, X_1, \ldots \in \square(X)$, then $\overline{\lim_{n=\infty}} X_n \subset X_0$ if and only if $\overline{\lim_{n=\infty}} \Lambda(X_n) \subset \Lambda(X_0)$.*

§ 4. Two quasi-homeomorphic compacta. Now let us construct two quasi-homeomorphic compacta for which we shall prove later (in § 5) that their shapes are different.

Let A_a, where $0 \leqslant a \leqslant \frac{1}{9}$, denote the circle given in the set $E^3 \cap Q$ by the equations:

$$(x_1 - \tfrac{1}{2})^2 + (x_2 - \tfrac{1}{4})^2 = \tfrac{1}{25}, \qquad x_3 = \tfrac{1}{6} + a,$$

and let B_a denote (for $0 \leqslant a \leqslant \frac{1}{9}$) the simple closed curve given in $E^3 \cap Q$ by the parametric equations:

$$x_a(t) = \big(x_{a,1}(t), x_{a,2}(t), x_{a,3}(t)\big), \quad \text{for } 0 \leqslant t < 2\pi,$$

where

$$x_{a,1}(t) = \tfrac{1}{2} + \tfrac{1}{5}\cos 2t + \tfrac{1}{15}a \cdot \cos 2t \cdot \cos t,$$

$$x_{a,2}(t) = \tfrac{1}{4} + \tfrac{1}{5}\sin 2t + \tfrac{1}{15}a \cdot \sin 2t \cdot \cos t,$$

$$x_{a,3}(t) = \tfrac{1}{6} + a + \tfrac{1}{15}a \cdot \sin t.$$

It is easy to see that the map h_a assigning to every point $(\cos t, \sin t, 0, 0, \ldots)$ (where $0 \leqslant t < 2\pi$) of the unit-circle $S^1 \subset E^2$ with center 0, the point $x_a(t)$ is a homeomorphism

$$h_a: S^1 \to B_a \quad \text{for } 0 < a \leqslant \tfrac{1}{9} \quad .$$

and

$$h_0: S^1 \to A_0$$

maps S^1 onto the circle A_0 with the degree 2.

Let us denote by p_a the projection of B_a onto A_a given by the formula

$$p_a\big(x_a(t)\big) = (\tfrac{1}{2} + \tfrac{1}{5}\cos 2t, \tfrac{1}{4} + \tfrac{1}{5}\sin 2t, \tfrac{1}{6} + a) \quad \text{for } 0 \leqslant t < 2\pi.$$

Consider the sets:

$$A = A_0 \cup \bigcup_{n=2}^{\infty} A_{3^{-n}}, \quad B = A_0 \cup \bigcup_{n=2}^{\infty} B_{3^{-n}}.$$

It is clear, that A and B are 1-dimensional compacta lying in the Hilbert cube Q and that A has as its components the sets A_0 and $A_{3^{-n}}$ for $n = 2, 3, \ldots$ Moreover, for every $0 \leqslant t < 2\pi$ the coordinate $x_{a,3}(t)$ of the point $x_a(t) \in B_a$ satisfies the inequality

$$\tfrac{1}{6} + a - \tfrac{1}{15}a \leqslant x_{a,3}(t) \leqslant \tfrac{1}{6} + a + \tfrac{1}{15}a.$$

It follows that

$$\tfrac{1}{6} + \tfrac{14}{15}3^{-n} \leqslant x_{3^{-n},3}(t) \quad \text{and} \quad \tfrac{1}{6} + \tfrac{16}{15}3^{-(n+1)} \geqslant x_{3^{-(n+1)},3}(t),$$

and since $\frac{14}{15}3^{-n} > \frac{16}{15}3^{-(n+1)}$, we infer that

$$B_{3^{-n}} \cap B_{3^{-(n+k)}} = \emptyset \quad \text{for } n = 2, 3, \ldots \text{ and } k = 1, 2, \ldots$$

It is also clear, that $B_{3^{-n}} \cap A_0 = \emptyset$, and consequently A_0 and $B_{3^{-n}}$, $n = 1, 2, \ldots$, are components of the compactum B.

Consider now a sequence of disjoint 3-dimensional cubs $Q_1, Q_2, \ldots$ lying in the set $E^3 \cap Q - (A \cup B)$ and converging to a point a belonging to the set $Q - (A \cup B) - \big(\bigcup_{i=1}^{\infty} Q_i\big)$. Let Y_n denote a subset of Q_n homeomorphic with B. Setting

$$(4.1) \qquad X = A \cup \bigcup_{n=1}^{\infty} Y_n \cup (a),$$

$$(4.2) \qquad Y = B \cup \bigcup_{n=1}^{\infty} Y_n \cup (a),$$

we get two compacta X and Y lying in $E^3 \cap Q$. Let us prove that they are quasi-homeomorphic.

It is clear, that for every positive ε there exists an index m so great that the set $\bigcup_{n=1}^{m} A_{3-n}$ is a retract of the set A by a retraction r_ε satisfying the condition

$$\varrho\big(x, r_\varepsilon(x)\big) < \tfrac{1}{2}\varepsilon \qquad \text{for every point } x \in A.$$

Moreover, there exists a homeomorphism h_m of the set $\bigcup_{n=1}^{m} A_{3-n}$ onto the set $\bigcup_{n=1}^{m} B_{3-n}$. Setting $\varphi = h_m r_\varepsilon$, we get a map of the set A onto the set $\bigcup_{n=1}^{m} B_{3-n}$ satisfying the condition

$$\delta[\varphi^{-1}(y)] < \varepsilon \qquad \text{for every point } y \in \bigcup_{n=1}^{m} B_{3-n}.$$

Moreover, there exists a homeomorphism ψ of the set $X-A$ onto the set $Y - \bigcup_{n=1}^{m} B_{3-n}$. Setting

$$f_\varepsilon(x) = \varphi(x) \qquad \text{for every point } x \in A,$$

$$f_\varepsilon(x) = \psi(x) \qquad \text{for every point } x \in X-A,$$

we get a map f_ε of X onto Y satisfying the condition

$$\delta[f_\varepsilon^{-1}(y)] < \varepsilon \qquad \text{for every point } y \in Y.$$

In order to define a map g_ε of Y onto X satisfying the condition

$$(4.3) \qquad \delta[g_\varepsilon^{-1}(x)] < \varepsilon \text{ for every point } x \in X,$$

consider a natural m so great that for every $n > m$ the projection p_{3-n} of the set B_{3-n} onto the set A_{3-n} satisfies the condition

$$\delta[p_{3-n}^{-1}(X)] < \varepsilon \qquad \text{for every point } x \in A_{3-n}.$$

Now let us consider a homeomorphism $\bar{h}_m$ of the set $\bigcup_{n=1}^{m} B_{3-n}$ onto the set $\bigcup_{n=1}^{m} A_{3-n}$. It suffices to set

$$g_\varepsilon(y) = p_{3-n}(y) \qquad \text{for every point } y \in B_{3-n} \text{ and } n = m+1, m+2, \ldots,$$

$$g_\varepsilon(y) = \bar{h}_m(y) \qquad \text{for every point } y \in \bigcup_{n=1}^{m} B_{3-n},$$

$$g_\varepsilon(y) = y \qquad \text{for every point } y \in Y - \bigcup_{n=1}^{\infty} B_{3-n},$$

in order to obtain a map g_ε of Y onto X satisfying the condition (4.3). Thus X and Y are quasi-homeomorphic.

§ 5. X and Y have different shapes. We shall show more, actually that the relation $\mathrm{Sh}(Y) \geqslant \mathrm{Sh}(X)$ does not hold.

Otherwise, there exist two fundamental sequences

$$\underline{f} = \{f_k, X, Y\}, \quad \underline{g} = \{g_k, Y, X\}$$

such that $\underline{g}\underline{f} \simeq \underline{i}_X$. It follows, by Theorem (3.1), that there exist two functions

$$\Lambda\colon \square(X) \to \square(Y) \quad \text{and} \quad \Lambda'\colon \square(Y) \to \square(X)$$

such that $\Lambda'\Lambda(X_0) = X_0$ for every $X_0 \in \square(X)$ and that the maps f_k and g_k constitute two fundamental sequences

$$\{f_k, X_0, \Lambda(X_0)\}, \quad \{g_k, \Lambda(X_0), X_0\},$$

satisfying the relation

$$\{g_k f_k, X_0, X_0\} \simeq i_{X_0}.$$

Let us denote by p the projection of Q onto $E^3 \cap Q$ given by the formula

$$p(x_1, x_2, x_3, x_4, \ldots) = (x_1, x_2, x_3, 0, \ldots).$$

It is easy to see that setting $f_k' = pf_k$ and $g_k' = pg_k$ for $k = 1, 2, \ldots$, one gets two fundamental sequences $\{f_k', X_0, \Lambda(X_0)\}$, $\{g_k', \Lambda(X_0), X_0\}$ homotopic to $\{f_k, X_0, \Lambda(X_0)\}$ and $\{g_k, \Lambda(X_0), X_0\}$ respectively, hence satisfying the relation $\{g_k' f_k', X_0, X_0\} \simeq \underline{i}_{X_0}$. It follows that we can assume that the maps $f_k\colon Q \to Q$ and $g_k\colon Q \to Q$ satisfy the condition

(5.1) $\quad f_k(Q) \subset E^3 \cap Q,\ g_k(Q) \subset E^3 \cap Q$ for every $k = 1, 2, \ldots$

Consider the component A_0 of the set X. Then $A_0 = \lim\limits_{n=\infty} A_{3^{-n}}$. Since $\Lambda'(\square(Y)) = \square(X)$, there exists, for every $n = 1, 2, \ldots$, a component $\hat{Y}_n$ of Y such that $\Lambda'(\hat{Y}_n) = A_{3^{-n}}$. Since Y is compact, we infer that there exist a component $\hat{Y}_0$ of Y and a sequence of indices $n_1 < n_2 < \ldots$ such that

$$\overline{\lim_{j=\infty}} \hat{Y}_{n_j} \subset \hat{Y}_0 = \Lambda(A_0).$$

It is clear that all components $\hat{Y}_n$ are different, and consequently the component $\hat{Y}_0$ of Y is not isolated (that is every neighborhood of it contains points belonging to components of Y different from Y_0). Moreover, $\hat{Y}_0 = \Lambda'(A_0)$ implies that $\hat{Y}_0 \underset{F}{\geqslant} A_0$ and consequently $\hat{Y}_0 \neq (a)$. Now let us observe that for every two non-isolated components Y', Y'' of Y, different from (a), there exists a homeomorphism mapping Y onto itself and mapping Y' onto Y''. It follows, that in the sequel we can restrict ourselves to the case when $\hat{Y}_0 = A_0$, i.e. to the case when $A_0 = \Lambda'(A_0)$.

Since $\overline{\lim_{j=\infty}} \hat{Y}_{n_j} \subset \hat{Y}_0 = A_0$, we infer that for almost all k the set $\hat{Y}_{n_k}$ belongs to the sequence $B_{3-1}, B_{3-2}, \ldots$ Replacing the sequence $\{n_j\}$ by one suitably selected of its subsequences, we can assume that $\hat{Y}_{n_j} = B_{3-m_j}$, where $\lim_{j=\infty} m_j = \infty$.

Consider now a toroidal neighborhood V of the circle A_0 in the set $E^3 \cap Q$. Then $B_{3-m_j} \subset V$ for almost all j, and we infer by (5.1) that for almost all k the map f_k/A_{3-n_j} is homotopic in V to a map of the set A_{3-n_j} vith values in the set B_{3-m_j}.

Now let us observe that the construction of the curve B_{3-n} implies that every map f of a circle S into a subset of $B_{3-n} \subset V$ is homotopic in V with a map of S into A_0 of an even degree. In particular, f_k/A_{3-n_j} for almost all k is homotopic in V with a map of A_{3-n_j} into A_0 of an even degree. It follows that $g_k f_k/A_{3-n_j}$ can not be homotopic in V (for almost all k) to the identity map of the set A_{3-n_j}, contrary to the relation

$$\{g_k f_k, A_{3-n_j}, A_{3-n_j}\} \simeq \underline{i}_{A_{3-n_j}}.$$

Thus the supposition that $\mathrm{Sh}(Y) \geqslant \mathrm{Sh}(X)$ leads to a contradiction and our proof is finished.

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