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On the existence of solutions of differential-integral equations with a lagging argument

§ 1. Introduction. In the paper [2] Bielecki and Maksym deal with a system of differential equations with continuous "memory" extended to $-\infty$, written in an integro-differential form

$$\begin{cases} y_k(t) = q_k(t) \quad \text{for } t\in(-\infty, t_0],\ k = 1, 2, \ldots, n, \\ y_k'(t) = g_k(t) + \displaystyle\sum_{i=1}^{n} \int_{0}^{\infty} f_{ki}\big(t, y_1(t-s), \ldots, y_n(t-s)\big) d_s r_{ki}(t, s) \\ \qquad\qquad\qquad\qquad\qquad \text{for } t\in[t_0, T),\ k = 1, 2, \ldots, n. \end{cases}$$

The existence of solutions was proved under certain continuity and Lipschitz type conditions. Latter on, Błaż [3] dropped the Lipschitz condition. This paper pushnes this program. We prove a Carathéodory type theorem concerning this equation, in which case the continuity condition may be weakened at the expense that the equation is to be satisfied almost everywhere. Further on, in § 4, we give a theorem concerning the continuation of the solution. In this paper we shall consider the differential-integral equation

$$(1)\qquad \begin{cases} y(t) = q(t) & \text{for } t\leqslant t_0, \\ y'(t) = \displaystyle\int_{0}^{\infty} f\big(t, y(t-s)\big) d_s r(t, s) + g(t) & \text{for almost every } t\in[t_0, T], \end{cases}$$

where f, r, q and g are given functions satisfying the following assumptions H:

Assumptions H.

I. The function $f\colon P \to R$, where $P = \{(t, y)\colon t_0 \leqslant t \leqslant T,\ a \leqslant y \leqslant b\}$, is continuous for fixed t and measurable for fixed $y\in S$, where the set S is dense in $[a, b]$.

II. There exist a Lebesgue-integrable function $m\colon [t_0, T] \to R$ such that

$$|f(t, y)| \leqslant m(t) \quad \text{for } (t, y) \in P.$$

III. The function $r\colon Q \to R$, where $Q = \{(t, s)\colon t_0 \leqslant t \leqslant T, s \geqslant 0\}$, satisfies the following conditions:

(i) $r(t, 0) = 0$ for every $t \in [t_0, T]$.

(ii) There is a finite number V such that

$$\overset{\infty}{\underset{s=0}{\Upsilon}} r(t, s) \leqslant V \quad \text{for } t \in [t_0, T].$$

(iii) For every $\eta > 0$ there is a finite number $K > 0$ such that

$$\overset{\infty}{\underset{s=K}{\Upsilon}} r(t, s) < \eta \quad \text{for } t \in [t_0, T].$$

(iv) For every $k > 0$ and $u \in [t_0, T]$

$$\lim_{t \to u} \int_0^k |r(t, s) - r(u, s)|\, ds = 0, \quad \text{where } t \in [t_0, T].$$

IV. The function $g\colon [t_0, T] \to R$ is continuous on $[t_0, T]$ and the function $q\colon (-\infty, t_0] \to R$ — continuous on its domain — satisfies the conditions:

$$\inf_{t<t_0} q(t) \geqslant a, \quad \sup_{t<t_0} q(t) \leqslant b, \quad q(t_0) \in (a, b).$$

By a solution of (1) we mean a function $y(t)$ which is continuous for $t \leqslant t_0$, absolutely continuous for $t_0 \leqslant t \leqslant T$ and satisfies coditions (1). It is understood that integral (1) is the Stieltjes-Riemann integral.

Let us observe that assumption (I) implies measurability in t of f for every fixed $y \in [a, b]$. Indeed, given any $y \in [a, b]$ there exists a sequence $\{y_n\}$ such that $y_n \to y$. By continuity of f in y, $f(t, y_n) \to f(t, y)$ for each t; therefore for fixed t the function $f(t, y)$ is the limit of a sequence of measurable functions whence it is also measurable. § 2 contains an approximation theorem of A. Alexiewicz and W. Orlicz and a lemma of A. Bielecki and M. Maksym. In § 3, we give the proof of a Carathéodory type theorem, finally, in § 4 we consider the problem of continuation of solutions.

§ 2. The approximation theorem.

THEOREM 1 (Alexiewicz-Orlicz). *Let the set S be dense in $[\alpha, \beta]$ and let the function $s(t)$ be measurable. If the function $f(t, u)$ defined for $a \leqslant t \leqslant b$, $\alpha \leqslant u \leqslant \beta$ is continuous for fixed t and measurable for fixed $u \in S$ and $|f(t, u)| \leqslant s(t)$, then there exist continuous functions $f_n(t, u)$ such that*

$|f_n(t, u)| \leqslant s(t)$ *and* $\lim_{n\to\infty} \max_{a\leqslant u\leqslant \beta} |f_n(t, u)-f(t, u)| = 0$ *for almost every* $t\in[a, b]$.

The proof of this theorem — see [1].

LEMMA 2 (Bielecki-Maksym). *Suppose* $P(t, s)$ *is continuous and bounded for* $t\in[A, B]$ *and* $s\in[0, \infty)$. *If the function* $r(t, s)$ *satisfies assumptions* H, III, *then the function*

$$G(t) = \int_0^\infty P(t, s)d_s r(t, s)$$

is continuous in interval $[A, B]$ (see [2]).

Lemma 2 implies

COROLLARY 3. *Suppose that the function* $F(t, y)$ *is defined and continuous in the rectangle* $P = \{(t, y)\colon t_0\leqslant t\leqslant T,\ a\leqslant y\leqslant b\}$. *Let the funcs tion* $r(t, s)$ *satisfy assumptions* H, III. *If the function* $h(t)$, *continuou- and bounded in the interval* $(-\infty, T]$, *satisfies the conditions*: $\inf_{t<t_0} h(t)\geqslant a$, $\sup_{t<t_0} h(t)\leqslant b$, $h(t_0)\in(a, b)$ *and* $a\leqslant h(t)\leqslant b$ *for* $t\in(t_0, T]$, *then the function*

$$H(t) = \int_0^\infty F\big(t, h(t-s)\big)d_s r(t, s)$$

is continuous in the interval $[t_0, T]$.

§ 3. Local existence of solutions.

THEOREM 4. *Suppose that assumptions* H *are fulfilled. Then there exist at least one absolutely continuous solution of* (1) *in the interval* $[t_0, t_0+\Delta]$, *where* $\Delta>0$ *satisfies the conditions*: $t_0+\Delta\leqslant T$ *and*

$$\int_{t_0}^{t_0+\Delta} \big(Vm(t)+|g(t)|\big)dt \leqslant \min\big(b-q(t_0), q(t_0)-a\big).$$

We precede the proof of Theorem 4 by the

LEMMA 5. *If the function* F *is continuous in the rectangle* P *and* r, g, q *satisfies assumptions* H, III, IV, *then there is a number* $\Delta>0$ *such that at least one solution of the equation*

$$(2)\qquad \begin{cases} z(t) = q(t) & \text{for } t\leqslant t_0,\\ z'(t) = \int_0^\infty F\big(t, z(t-s)\big)d_s r(t, s)+g(t) & \text{for } t\in[t_0, T] \end{cases}$$

exists for $t\in(-\infty, t_0+\Delta]$.

Proof. In virtue of Corollary 3 equation (2) is equivalent to the following integral equation:

$$(3)\qquad z(t)=\begin{cases} q(t) & \text{for } t\leqslant t_0,\\ q(t_0)+\int\limits_{t_0}^{t}\Big\{\int\limits_{0}^{\infty}F\big(u,z(u-s)\big)d_sr(u,s)+g(u)\Big\}du & \text{for } t\in[t_0,T].\end{cases}$$

Let $\varDelta>0$ be chosen such that $t_0+\varDelta\leqslant T$. Let us denote by B the space of continuous and bounded functions $z=z(t)$ defined on the halfaxis $(-\infty,t_0+\varDelta]$. It is Banach space under the norm

$$\|z\|=\sup_{t\leqslant t_0+\varDelta}|z(t)|.$$

Moreover, let us denote by K the set of all functions $z\in B$ which satisfy the following conditions:

$$z(t)=q(t)\qquad \text{for } t\leqslant t_0,$$
$$\left|z(t)-\frac{a+b}{2}\right|\leqslant\frac{b-a}{2}\qquad \text{for } t_0\leqslant t\leqslant t_0+\varDelta.$$

It is easy to verify that the set K is a non-empty, closed and convex subset of B.

For any $z\in K$ consider the mapping T, defined by

$$(4)\qquad (Tz)(t)=\begin{cases} q(t) & \text{for } t\leqslant t_0\\ q(t_0)+\int\limits_{t_0}^{t}\Big\{\int\limits_{0}^{\infty}F\big(u,z(u-s)\big)d_sr(u,s)+g(u)\Big\}du & \\ & \text{for } t_0\leqslant t\leqslant t_0+\varDelta.\end{cases}$$

We must show that T is defined for all functions $z\in K$, $Tz\in K$ and that $T(K)$ is a compact set. For $t\leqslant t_0$ the mapping is well defined and $(Tz)(t)=q(t)$. It follows from Corollary 3 and (4) that $(Tz)(t)$ is a continuous functions for $t_0\leqslant t\leqslant t_0+\varDelta$. Therefore $Tz\in B$ and $(Tz)(t)=q(t)$ for $t\leqslant t_0$. For any $z\in K$ and $t_0\leqslant t\leqslant t_0+\varDelta$ we have

$$\begin{aligned}\left|(Tz)(t)-\frac{a+b}{2}\right| &\leqslant\left|\int\limits_{t_0}^{t}\Big\{\int\limits_{0}^{\infty}F\big(u,z(u-s)\big)d_sr(u,s)+g(u)\Big\}\,du\right|+\left|q(t_0)-\frac{a+b}{2}\right|\\ &\leqslant\int\limits_{t_0}^{t_0+\varDelta}(MV+Q)\,dt+\left|q(t_0)-\frac{a+b}{2}\right|\\ &\leqslant\varDelta(MV+Q)+\left|q(t_0)-\frac{a+b}{2}\right|,\end{aligned}$$

where

$$M = \max_{P} |F(t, y)|, \qquad Q = \max_{[t_0, T]} |g(t)| \qquad \text{and} \qquad \mathop{\Upsilon}\limits_{s=0}^{\infty} r(t, s) \leqslant V.$$

Taking now Δ such that

$$\Delta (MV+Q) \leqslant \min\big(b-q(t_0), q(t_0)-a\big)$$

we obtain

$$\left|(Tz)(t) - \frac{a+b}{2}\right| \leqslant \frac{b-a}{2}$$

for $t_0 \leqslant t \leqslant t_0+\Delta$. Then $Tz \in K$ for all functions $z \in K$. The mapping T: $K \to K$ is continuous. Indeed, since the function F is uniformly continuous in the rectangle $P_1 = \{(t, y)\colon t_0 \leqslant t \leqslant t_0+\Delta;\ a \leqslant y \leqslant b\}$, for any $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$|F(t, z) - F(t, z_0)| \leqslant \frac{\varepsilon}{V}$$

whenever $a \leqslant z \leqslant b$, $a \leqslant z_0 \leqslant b$, $|z-z_0| < \delta$ and $t_0 \leqslant t \leqslant t_0+\Delta$. Let $z, z_0 \in K$ and $\|z-z_0\| < \delta$, where

$$\|z-z_0\| = \max_{[t_0, t_0+\Delta]} |z(t) - z_0(t)|.$$

Since $a \leqslant z(t-s) \leqslant b$, $a \leqslant z_0(t-s) \leqslant b$ and $|z(t-s) - z_0(t-s)| \leqslant \delta$ for $s \geqslant 0$ and $t_0 \leqslant t \leqslant t_0+\Delta$, the conditions $z, z_0 \in K$ and $\|z-z_0\| < \delta$ imply that

$$\big|F\big(t, z(t-s)\big) - F\big(t, z_0(t-s)\big)\big| \leqslant \frac{\varepsilon}{V},$$

for an arbitrary $\varepsilon > 0$, $t_0 \leqslant t \leqslant t_0+\Delta$ and $s \geqslant 0$. Hence by (4) we have

$$|(Tz)(t) - (Tz_0)(t)|$$
$$\leqslant \int_{t_0}^{t} \Big\{\sup_{s \geqslant 0} \big|F\big(u, z(u-s)\big) - F\big(u, z_0(u-s)\big)\big| \mathop{\Upsilon}\limits_{s=0}^{\infty} r(u, s)\Big\} du \leqslant \varepsilon$$

for every $t \in [t_0, t_0+\Delta]$ whenever $\|z-z_0\| < \delta$ with $z, z_0 \in K$, which proves that the mapping T is continuous.

We shall show that $T(K)$ is a compact set. For $t, t_1, t_2 \in [t_0, t_0+\Delta]$ and $z \in K$ we have

$$|(Tz)(t_1) - (Tz)(t_2)| = \left|\int_{t_1}^{t_2} \Big\{\int_0^{\infty} F\big(t, z(t-s)\big) d_s r(t, s) + g(t)\Big\} dt\right|$$
$$\leqslant (MV+Q)|t_2 - t_1|,$$

and

$$|(Tz)(t)| \leqslant |q(t_0)| + \Delta(MV+Q).$$

Hence it follows, by the Ascoli-Arzèla theorem, that $T(K)$ is compact. Now, Schauder's fixed point theorem (see [4]) asserts the existence of at least one function $z \in K$ such that $Tz = z$. Thus the proof is completed.

LEMMA 6. *Suppose that assumptions* H, I-III, *are fulfilled. If the function* $h(t)$ — *defined, continuous and bounded in interval* $(-\infty, T]$ — *satisfies the conditions*: $h(t_0) \in (a, b)$, $\inf_{t<t_0} h(t) \geqslant a$, $\sup_{t<t_0} h(t) \leqslant b$ *for* $t_0 < t \leqslant T$, *then the function* W *defined by*

$$W(t) = \int_0^\infty f\big(t, h(t-s)\big) d_s r(t, s)$$

is Lebesgue integrable in $[t_0, T]$.

Proof. We shall show that W is a measurable function. It is enough to show that there exists a sequence $W_n(t)$ of continuous functions in the interval $[t_0, T]$ such that $|W_n(t) - W(t)|$ converges to 0 for almost any $t \in [t_0, T]$. By virtue of Theorem 1, and our assumptions, there exist a sequence $\{f_n(t, y)\}$ of continuous functions such that

$$(5) \qquad |f_n(t, y)| \leqslant m(t) \quad \text{for } (t, y) \in P$$

and

$$(6) \qquad \lim_{n\to\infty} \max_{a \leqslant y \leqslant b} |f_n(t, y) - f(t, y)| = 0 \quad \text{for almost every } t \in [t_0, T].$$

Let us consider the functions W_n defined by

$$W_n(t) = \int_0^\infty f_n\big(t, h(t-s)\big) d_s r(t, s)$$

for $t_0 \leqslant t \leqslant T$, $n = 1, 2, \ldots$ In virtue of Corollary 3 the functions W_n are continuous in the interval $[t_0, T]$. Since

$$\begin{aligned} |W_n(t) - W(t)| &= \Big|\int_0^\infty \big[f_n\big(t, h(t-s)\big) - f\big(t, h(t-s)\big)\big] d_s r(t, s)\Big| \\ &\leqslant \sup_{s \geqslant 0} \big|f_n\big(t, h(t-s)\big) - f\big(t, h(t-s)\big)\big| V \end{aligned}$$

we have, by (6), for almost every $t \in [t_0, T]$

$$\lim_{n\to\infty} |W_n(t) - W(t)| = 0.$$

It remains to show that there exists Lebesgue integrable function $K(t)$ such that

$$|W(t)| \leqslant K(t) \quad \text{for almost every } t \in [t_0, T].$$

From our assumptions concerning the function h it follows that $(t, h(t-s)) \in P$ for $t_0 \leqslant t \leqslant T$ and $s \geqslant 0$. Therefore $|f(t, h(t-s))| \leqslant m(t)$ for $t_0 \leqslant t \leqslant T$ and $s \geqslant 0$. Hence

$$|W(t)| \leqslant m(t) V.$$

The function $m(t) V$ is Lebesgue integrable in the interval $[t_0, T]$. Thus the proof is completed.

LEMMA 7. *Differential-integral equation* (1) *is equivalent to the integral equation*

$$y(t) = \begin{cases} q(t) & \textit{for } t \leqslant t_0, \\ q(t_0) + \int\limits_{t_0}^{t} \Big\{ \int\limits_{0}^{\infty} f(u, y(u-s)) d_s r(u, s) + g(u) \Big\} du & \\ & \textit{for } t_0 \leqslant t \leqslant T. \end{cases} \tag{7}$$

Proof. Let the function $\tilde{y}(t)$ be a solution of equation (1). We have

$$\begin{cases} \tilde{y}(t) = q(t) \quad \text{for } t \leqslant t_0, \\ \tilde{y}'(t) = \int\limits_{0}^{\infty} f(t, \tilde{y}(t-s)) d_s r(t, s) + g(t) \\ \qquad\qquad \text{for almost every } t_0 \leqslant t \leqslant T. \end{cases} \tag{8}$$

Then integrating the second of equalities in (8), taking into account that $\tilde{y}(t) = q(t)$ for $t \leqslant t_0$, we obtain

$$\tilde{y}(t) = \begin{cases} q(t) & \text{for } t \leqslant t_0, \\ q(t_0) + \int\limits_{t_0}^{t} \Big\{ \int\limits_{0}^{\infty} f(u, y(u-s)) d_s r(u, s) + g(u) \Big\} du & \\ & \text{for } t_0 \leqslant t \leqslant T. \end{cases} \tag{9}$$

Therefore $\tilde{y}(t)$ is a solution of (7). Let $\tilde{y}(t)$ be a solution of integral equation (7). Since by Lemma 6 the function

$$J(t) = \int\limits_{0}^{\infty} f(t, \tilde{y}(t-s)) d_s r(t, s) + g(t)$$

is Lebesgue integrable in the interval $[t_0, T]$, it follows by (9) that the function $\tilde{y}(t)$ is continuous in the interval $[t_0, T]$. Differentiating the second condition in (9) by $\tilde{y}(t) = q(t)$ we obtain identity (8). This proves the lemma.

Proof of Theorem 4. By Lemma 7, we can deal with integral equation (7) instead of differential equation (1). By Theorem 1, there exists a sequence of continuous functions $\{f_n(t, y)\}$ satisfying (5) and (6).

Let us consider the differential equation

$$(10)\qquad \begin{cases} y_n(t) = q(t) & \text{for } t \leqslant t_0, \\ y_n'(t) = \int\limits_0^\infty f_n\big(t, y_n(t-s)\big)d_s r(t,s) + g(t) & \text{for } t_0 \leqslant t \leqslant T \end{cases}$$

for $n = 1, 2, \ldots$ By Lemma 5, there exist for $n = 1, 2, \ldots$, at least one solution $y_n(t)$ of (10) such that $y_n(t) \in C^0(-\infty, t_0+\Delta]$ and $y_n(t) \in C^1[t_0, t_0+\Delta]$, where $\Delta > 0$ is such that $t_0+\Delta \leqslant T$ and

$$\int\limits_{t_0}^{t_0+\Delta} \big(m(t)V + |g(t)|\big)dt \leqslant \min\big(b - q(t_0), q(t_0) - a\big).$$

Then for $n = 1, 2, \ldots$, we have

$$(11)\qquad y_n(t) = \begin{cases} q(t) & \text{for } t \leqslant t_0, \\ q(t_0) + \int\limits_{t_0}^{t} \Big\{\int\limits_0^\infty f_n\big(u, y_n(u-s)\big)d_s r(u,s) + g(u)\Big\}\,du & \\ \qquad\qquad\qquad\qquad \text{for } t_0 \leqslant t \leqslant t_0+\Delta. \end{cases}$$

Since $\big(t, y_n(t-s)\big) \in P$ for $t_0 \leqslant t \leqslant t_0+\Delta$ and $s \geqslant 0$, by (5) we must have

$$\big|f_n\big(t, y_n(t-s)\big)\big| \leqslant m(t) \quad \text{for } s \geqslant 0 \text{ and } t_0 \leqslant t \leqslant t_0+\Delta.$$

Therefore we obtain from (11)

$$|y_n(t)| \leqslant \int\limits_{t_0}^{t_0+\Delta} \big(Vm(t) + |g(t)|\big)dt + |q(t_0)| \leqslant |q(t_0)| + \min\big(b - q(t_0), q(t_0) - a\big)$$

and

$$|y_n(t_1) - y_n(t_2)| \leqslant \Big|\int\limits_{t_1}^{t_2} \big(Vm(t) + |g(t)|\big)dt\Big|$$

for $t_1, t_2 \in [t_0, t_0+\Delta]$, $n = 1, 2, \ldots$

The functions $y_n(t)$ are equicontinuous and uniformly bounded. Hence, by Arzèla's theorem, there exists a subsequence $\{y_{n_k}(t)\}$ converging uniformly in the interval $(-\infty, t_0+\Delta]$ to a function $\tilde{y}(t)$. It remains to show that $\tilde{y}(t)$ is a solution of equation (7).

For $t \in [t_0, t_0+\Delta]$ we have

$$(12)\qquad \tilde{y}(t) - q(t_0) - \int\limits_{t_0}^{t} \Big\{\int\limits_0^\infty f\big(u, \tilde{y}(u-s)\big)\, d_s r(u,s) + g(u)\Big\}\,du = \sum_{i=1}^{3} J_i(t),$$

where

$$J_1(t) = \tilde{y}(t) - y_{n_k}(t),$$

$$J_2(t) = \int_{t_0}^{t} \Big\{ \int_0^\infty \big[f_{n_k}\big(u, y_{n_k}(u-s)\big) - f\big(u, y_{n_k}(u-s)\big) \big] d_s r(u, s) \Big\} du,$$

$$J_3(t) = \int_{t_0}^{t} \Big\{ \int_0^\infty \big[f\big(u, y_{n_k}(u-s)\big) - f\big(u, \tilde{y}(u-s)\big) \big] d_s r(u, s) \Big\} du.$$

Since

$$\tilde{y}(t) = \lim_{k\to\infty} \operatorname*{unf}_{[t_0, t_0+\Delta]} y_{n_k}(t),$$

it follows that

$$\lim_{k\to\infty} \operatorname*{unf}_{[t_0, t_0+\Delta]} J_1(t) = 0.$$

Let

$$H_{n_k}(t, y_{n_k}) = f_{n_k}(t, y_{n_k}) - f(t, y_{n_k}).$$

The functions $H_{n_k}(t, y_{n_k})$ are continuous with respect to y_{n_k}, and measurable for fixed $y_{n_k} \in S$. Moreover,

$$|H_{n_k}(t, y_{n_k})| \leqslant 2m(t) \quad \text{for } (t, y_{n_k}) \in P$$

and

$$\lim_{k\to\infty} \operatorname*{unf}_{s\geqslant 0} H_{n_k}\big(t, y_{n_k}(s)\big) = 0 \quad \text{for almost every } t \in [t_0, t_0+\Delta].$$

Thus the functions

$$K_{n_k}(t) = \int_0^\infty H_{n_k}\big(t, y_{n_k}(t-s)\big)\, d_s r(t, s)$$

are measurable in the interval $[t_0, t_0+\Delta]$ and such that

$$|K_{n_k}(t)| \leqslant p(t) \quad \text{for } t_0 \leqslant t \leqslant t_0+\Delta,$$

moreover, $p(t) = Vm(t)$ is Lebesgue integrable. Since

$$0 \leqslant |K_{n_k}(t)| \leqslant \sup_{s\geqslant 0} \big| H_{n_k}\big(t, y_{n_k}(t-s)\big) \big| V,$$

it follows that

$$\lim_{k\to\infty} K_{n_k}(t) = 0 \quad \text{for almost every } t \in [t_0, t_0+\Delta].$$

Passing to the limit in (12) with $k \to \infty$ we get

$$\tilde{y}(t) - q(t_0) - \int_{t_0}^{t} \Big\{ \int_0^\infty f\big(u, \tilde{y}(u-s)\big) d_s r(u, s) + g(u) \Big\} du = 0$$

for $t \in [t_0, t_0+\Delta]$, which completes the proof.

Remark 1. In particular, for $r(t,s) = e\big(s-\sigma(t)\big)$, where

$$e(u) = \begin{cases} 0 & \text{for } -\infty < u \leqslant 0, \\ 1 & \text{for } 0 < u < +\infty, \end{cases}$$

the second condition from (1) takes on the form

$$y'(t) = f\Big(t, y\big(t-\sigma(t)\big)\Big).$$

Taking $\sigma(t) = 0$, we obtain the equation of the form

$$y'(t) = f\big(t, y(t)\big)$$

which was considered by the authors of paper [1].

§ 4. Continuation of solutions. Suppose that $L_1\colon [t_0, T] \to R$, $L_2\colon [t_0, T] \to R$ are given functions such that

(i) L_1 and L_2 are continuous in $[t_0, T]$,

(ii) L_1 is decreasing in $[t_0, T]$,

(iii) L_2 is increasing in $[t_0, T]$,

(iv) $L_1(t) < L_2(t)$ for $t_0 \leqslant t \leqslant T$.

Let

$$D = \{(t,y)\colon t_0 \leqslant t \leqslant T;\ L_1(t) \leqslant y \leqslant L_2(t)\}$$

and

$$a = L_1(t_0), \quad b = L_2(t_0).$$

We prove the following

THEOREM 8. *Let the functions* r, g, q *satisfy assumptions* H, III, IV. *If the function* $f\colon D \to R$ *is continuous for fixed* $t \in [t_0, T]$, *measurable for fixed* $y \in S_t$ *and* $|f(t,y)| \leqslant m(t)$ *for* $(t,y) \in D$, *where the set* S_t *is dense in* $[L_1(t), L_2(t)]$ *for* $t_0 \leqslant t \leqslant T$ *and* $m\colon [t_0, T] \to R$ *is a Lebesgue integrable function, then there exists at least one absolutely continuous solution* $y(t)$ *in the interval* $[t_0, T]$.

Proof. Let P be the rectangle defined by

$$P = \{(t,y)\colon t_0 \leqslant t \leqslant T;\ a \leqslant y \leqslant b\},$$

where

$$a = L_1(t_0), \quad b = L_2(t_0).$$

By Theorem 4, there is a number $\Delta_1 > 0$ such that $t_0 + \Delta_1 \leqslant T$ and such that equation (1) has at least one absolutely continuous solution $\tilde{y}(t)$ in the interval $[t_0, t_0+\Delta_1]$; moreover, $a \leqslant \tilde{y}(t) \leqslant b$ for $t_0 \leqslant t \leqslant t_0 + \Delta_1$. If $t_0 + \Delta_1 = T$, then the proof is completed. Let $t_0 + \Delta_1 < T$. Since $\big(t_0 + \Delta_1, y(t_0+\Delta_1)\big) \in D$, there exists a rectangle $\tilde{P} = \{(t,y)\colon \tilde{t} \leqslant t \leqslant \tilde{\tilde{t}};\ \tilde{a} \leqslant y \leqslant \tilde{b}\}$ with the centre $\big(t_0+\Delta_1, y(t_0+\Delta_1)\big)$. Let $P_1 = \{(t,y)\colon t_0 + \Delta_1 \leqslant t \leqslant T;$

$a_1 \leqslant y \leqslant b_1\}$, where $a_1 = \min(a, \tilde{a})$, $b_1 = \max(b, \tilde{b})$. Obviously $P_1 \subset D$. Consider now the function defined by

$$\varphi(t) = \begin{cases} q(t) & \text{for } -\infty < t \leqslant t_0, \\ \tilde{y}(t) & \text{for } t_0 < t \leqslant t_0 + \Delta_1. \end{cases}$$

This function is bounded, continuous and satisfies the conditions $\varphi(t_0+\Delta_1) \in (a_1, b_1)$, $\inf\limits_{t<t_0+\Delta_1} \varphi(t) \geqslant a_1$, $\sup\limits_{t<t_0+\Delta_1} \varphi(t) \leqslant b_1$. Therefore, by Theorem 4, there is a number $\Delta_2 > 0$ such that the equation

$$(13) \qquad \begin{cases} y(t) = \varphi(t) & \text{for } t \in t_0 + \Delta_1, \\ y'(t) = \int\limits_0^\infty f\big(t, y(t-s)\big) d_s r(t, s) + g(t) & \\ & \text{for almost every } t_0 + \Delta_1 \leqslant t \leqslant T \end{cases}$$

has at least one absolutely continuous solution $\hat{y}(t)$ in the interval $[t_0 + \Delta_1, t_0 + \Delta_2]$, where $t_0 + \Delta_2 \leqslant T$. Moreover, $a_1 \leqslant y(t) \leqslant b_1$ for $t_0 + \Delta_1 \leqslant t \leqslant t_0 + \Delta_2$. If $t_0 + \Delta_2 = T$, then the proof is completed. In this way the solution $\tilde{y}(t)$ of (1) may be continued to the whole halfaxis $(-\infty, T]$. Indeed, suppose $\tilde{y}(t)$ does not have a continuation to $(-\infty, T]$, and for definiteness assume $\tilde{y}(t)$ has a continuation $Y(t)$ existing up to $\tau < T$, but cannot be continued behind. In a similar way as above, it is easily verified that there is a number $\Delta > 0$ such that $Y(t)$ may be continued to $(-\infty, \tau + \Delta]$, where $\tau + \Delta \leqslant T$. This is a contradiction, proving that a continuation of $\tilde{y}(t)$ exists on $(-\infty, T]$. It remains to show that the continuation of $\tilde{y}(t)$ is absolutely continuous in $[t_0, T]$. We shall verify only that the solution $\hat{y}(t)$ of (13) is absolutely continuous in the interval $[t_0, t_0 + \Delta_2]$.

For $t \in (-\infty, t_0 + \Delta_2]$ we have

$$\hat{y}(t) = \begin{cases} q(t) & \text{for } t \leqslant t_0, \\ \tilde{y}(t) & \text{for } t_0 \leqslant t \leqslant t_0 + \Delta_1, \\ \tilde{y}(t_0 + \Delta_1) + \int\limits_{t_0+\Delta_1}^{t} \Big\{ \int\limits_0^\infty f\big(u, \hat{y}(u-s)\big) d_s r(u, s) + g(u) \Big\} du & \\ & \text{for } t_0 + \Delta_1 \leqslant t \leqslant t_0 + \Delta_2. \end{cases}$$

Hence

$$\hat{y}(t) = \begin{cases} \tilde{y}(t) & \text{for } t \leqslant t_0 + \Delta_1, \\ \tilde{y}(t_0 + \Delta_1) + \int\limits_{t_0+\Delta_1}^{t} \Big\{ \int\limits_0^\infty f\big(u, \hat{y}(u-s)\big) d_s r(u, s) + g(u) \Big\} du & \\ & \text{for } t_0 + \Delta_1 \leqslant t \leqslant t_0 + \Delta_2. \end{cases}$$

Since $\tilde{y}(t) = \hat{y}(t)$ for $t \in (-\infty, t_0 + \Delta_1]$, it follows that

$$\tilde{y}(t) = \begin{cases} q(t) & \text{for } t \leqslant t_0, \\ q(t_0) + \int\limits_{t_0}^{t} \Big\{ \int\limits_{0}^{\infty} f\big(u, \hat{y}(u-s)\big) d_s r(u, s) + g(u) \Big\} du & \\ \qquad\qquad\qquad \text{for } t_0 \leqslant t \leqslant t_0 + \Delta_1. \end{cases}$$

Hence

$$\hat{y}(t) = \begin{cases} q(t) & \text{for } t \leqslant t_0, \\ q(t_0) + \int\limits_{t_0}^{t} \Big\{ \int\limits_{0}^{\infty} f\big(u, \hat{y}(u-s)\big) d_s r(u, s) + g(u) \Big\} du & \\ \qquad\qquad\qquad \text{for } t_0 \leqslant t \leqslant t_0 + \Delta_2 \end{cases}$$

which completes the proof.

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