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On the dual semigroups of compact semigroups

Let S be a commutative compact semigroup (multiplication being jointly continuous). By a *semicharacter* of S we mean a continuous homomorphism γ of S into the complex unit disc, i.e., a complex-valued continuous function γ on S such that 1° $|\gamma(x)| \leqslant 1$ for all $x \in S$, 2° $\gamma(x_1 x_2) = \gamma(x_1)\gamma(x_2)$ for $x_1, x_2 \in S$. The set $\hat{S}$ of all semicharacters of S is a commutative semigroup under the ordinary pointwise product $\gamma_1\gamma_2(x) = \gamma_1(x)\gamma_2(x)$. The unit semicharacter $\gamma^1(x) \equiv 1$ and the zero semicharacter $\gamma^0(x) \equiv 0$ are the identity and the zero of $\hat{S}$, respectively. They are called the *trivial semicharacters*. The semigroup $\hat{S}$ with the uniform topology, i.e., the smallest topology in which all the sets

$$U(\gamma', \varepsilon) = \{\gamma \in \hat{S}\colon |\gamma(x) - \gamma'(x)| < \varepsilon \text{ for all } x \in S\}$$

are open, is a topological semigroup and is called the *dual semigroup* of S. If S has zero element 0 and identity e, then the semicharacters γ^0 and γ^1 are isolated points in S. Indeed,

$$U(\gamma^0, \tfrac{1}{2}) = \{\gamma \in \hat{S}\colon |\gamma(x)| < 1/2,\, x \in S\} = \{\gamma^0\}$$

since $\gamma \neq \gamma^0$ implies $\gamma(e) = 1$, and

$$U(\gamma^1, \tfrac{1}{2}) = \{\gamma \in \hat{S}\colon |\gamma(x) - 1| < 1/2,\, x \in S\} = \{\gamma^1\}$$

since $\gamma \neq \gamma^1$ implies $\gamma(0) = 0$.

In this paper we present some sufficient conditions for S to be locally compact.

THEOREM 1. *Let S be a compact commutative semigroup with identity e and zero 0, such that*

(i) *e has a basis of open connected neighborhoods,*

(ii) *for every open set $U \subset S$ and every $x \in S$, $x \neq 0$, the set Ux is also open.*

Then the dual semigroup $\hat{S}$ is locally compact.

Proof. Let $\gamma' \in \hat{S}$, $\gamma' \neq \gamma^0$, $\gamma' \neq \gamma^1$. We shall prove that the set $U_0 = U(\gamma', \frac{1}{2})$ is relatively compact in $\hat{S}$. Since all semicharacters are uniformly bounded and S is compact, it is sufficient to prove that U_0 is an equicontinuous set, i.e.,

$$\forall_{\varepsilon>0} \forall_{x'\in S} \exists_{N(x')} \forall_{\gamma\in U_0} \forall_{x\in N(x')} \; |\gamma(x)-\gamma(x')| < \varepsilon, \tag{1}$$

where $N(x')$ denotes an open neighborhood of x' (see, e.g., [1]). In order to prove (1) we shall distinguish three cases: (I) $x' = 0$, (II) $x' = e$, (III) $x' \neq 0$, $x' \neq e$.

(I) $x' = 0$. Let M be an open neighborhood of 0 in S such that $|\gamma'(x)| \leqslant 1/4$ for every $x \in M$. For every $\gamma \in U_0$ and every $x \in S$ we have $|\gamma(x)-\gamma'(x)| < 1/2$; therefore if $x \in M$, then $|\gamma(x)| < 1/2+|\gamma'(x)| < 3/4$.

Let $\varepsilon > 0$ and let k be a natural number such that $(3/4)^k < \varepsilon$. Denote $N = M^k = M\cdot M\cdot \ldots \cdot M$. In virtue of (ii) N is an open neighborhood of 0. If $x \in N$, then x is of the form $x_1\cdot x_2\cdot \ldots \cdot x_k$ with $x_i \in M$, $i = 1, 2, \ldots, k$, and for every $\gamma \in U_0$ we have

$$|\gamma(x)| = |\gamma(x_1)\cdot \ldots \cdot\gamma(x_k)| < (3/4)^k < \varepsilon;$$

this means that (1) is satisfied.

(II) $x' = e$. The proof consists of three steps.

(a) Write $M = \{|\gamma| : \gamma \in U_0\}$. Here $|\gamma|$ denotes the semicharacter $|\gamma(x)|$. We claim that the set M is equicontinuous at the point e. Suppose that this is not true, i.e.,

$$\exists_{\varepsilon_0>0} \forall_{N(e)} \exists_{\gamma_N\in U_0} \exists_{x_N\in N(e)} \; 1-|\gamma_N(x_N)| \geqslant \varepsilon_0.$$

Choose an integer k such that $(1-\varepsilon_0)^k < 1/4$. Let N be a neighborhood of e such that $1-|\gamma'(x^k)| < 1/4$ for $x \in N$, where x^k denotes the semigroup power $x\cdot \ldots \cdot x$. Consequently,

$$\big||\gamma_N(x_N^k)|-|\gamma'(x_N^k)|\big| \geqslant 1-|\gamma_N(x_N^k)|-\big(1-|\gamma'(x_N^k)|\big) \geqslant 1-(1-\varepsilon_0)^k-1/4 \geqslant 1/2.$$

On the other hand, $\gamma_N \in U_0$ and $\big||\gamma_N(x)|-|\gamma'(x)|\big| \leqslant |\gamma_N(x)-\gamma'(x)| < 1/2$ for all $x \in S$, a contradiction.

(b) Denote $A = \{\operatorname{Arg}\gamma : \gamma \in U_0\}$. The set A is equicontinuous at e. Indeed, suppose the contrary; then there exists an ε_0 such that $0 < \varepsilon_0 < \pi/2$ and

$$\forall_{N(e)} \exists_{\gamma_N\in U_0} \exists_{x_N\in N(e)} \; |\operatorname{Arg}\gamma_N(x_N)| > \varepsilon_0.$$

Choose ε_1 and an integer k such that $\varepsilon_1 > \varepsilon_0$ and $\pi/2 \leqslant k\varepsilon_0 < k\varepsilon_1 \leqslant \pi$. By (a) and by the continuity of γ' there exists an $N = N(e)$ such that $|\gamma(x)| > \sqrt[2k]{5/8}$ and $|\operatorname{Arg}\gamma'(x)| < \pi/6k$ for $x \in N$ and $\gamma \in U_0$. By (i) we may assume that N is connected. Then, since $\operatorname{Arg}\gamma_N(e) = 0$, there exists an

$y_N \in N$ such that $\varepsilon_0 < |\operatorname{Arg}\gamma_N(y_N)| < \varepsilon_1$. Thus

$$|\operatorname{Arg}\gamma_N(y_N^k)| = |\operatorname{Arg}\gamma_N^k(y_N)| = k|\operatorname{Arg}\gamma_N(y_N)| \geqslant k\varepsilon_0 > \pi/2.$$

On the other hand, from the inequality

$$|\gamma(x)-\gamma'(x)|^2 = \big(|\gamma(x)|-|\gamma'(x)|\big)^2+2|\gamma(x)|\,|\gamma'(x)|\big(1-\cos(\operatorname{Arg}\gamma(x)- \\ -\operatorname{Arg}\gamma'(x))\big) < 1/4$$

$(\gamma \in \bar{U}_0,\ x \in S)$, we obtain

$$|\operatorname{Arg}\gamma(y_N^k)-\operatorname{Arg}\gamma'(y_N^k)| < \arccos\frac{|\gamma(y_N)|^{2k}+|\gamma'(y_N)|^{2k}-1/4}{2|\gamma(y_N)|^k|\gamma'(y_N)|^k} < \pi/3.$$

Hence $|\operatorname{Arg}\gamma(y_N^k)| < \pi/3+|\operatorname{Arg}\gamma'(y_N^k)| < \pi/3+\pi/6 = \pi/2$, a contradiction.

(c) The set U is equicontinuous at e. Let $\varepsilon > 0$. In virtue of (a) and (b) there exist open neighborhoods $N_1(e)$ and $N_2(e)$ such that $\big(|\gamma(x)|-1\big)^2 < \varepsilon/2$ for $x \in N_1(e)$ and $1-\cos\operatorname{Arg}\gamma(x) < \varepsilon/4$ for $x \in N_2(e)$. Consequently for $x \in N_1(e) \cap N_2(e)$ and $\gamma \in U_0$ we have:

$$|\gamma(x)-1|^2 \leqslant \big(|\gamma(x)|-1\big)^2+2\big(1-\cos\operatorname{Arg}\gamma(x)\big) < \varepsilon.$$

(III) $x' \in S$, $x' \neq 0$, $x' \neq e$. Let $\varepsilon > 0$. By part (II) there exists a neighborhood W of e such that $|\gamma(x)-1| < \varepsilon$ for all $\gamma \in U_0$ and all $x \in W$. Let $N = x'W$. By (i) N is an open neighborhood of x'. If $x \in N$, then $x = x'y_x$ with $y_x \in W$ and $|\gamma(x)-\gamma(x')| = |\gamma(x')|\,|\gamma(y_x)-1| \leqslant |\gamma(y_x)-1| < \varepsilon$.

Thus, U_0 is equicontinuous at x'. This concludes the proof of Theorem 1.

Now, let $(S_\sigma)_{\sigma\in\Sigma}$ be a family of commutative compact semigroups with identities e_σ. The product $P = \prod_{\sigma\in\Sigma} S_\sigma$ is the Cartesian product of the spaces S_σ with the Tychonoff topology and with the componentwise multiplication: if $x' = (x'_\sigma)$, $x'' = (x''_\sigma)$, then $x'x'' = (x'_\sigma x''_\sigma)$. It is clear that P is a commutative compact semigroup with identity $e = (e_\sigma)$.

THEOREM 2. *If each $\hat{S}_\sigma$ is locally compact, then $\hat{P}$ is also locally compact.*

The proof is founded on the following essentially known lemma:

LEMMA. *A complex-valued function γ on P is a non-trivial semicharacter if and only if there exist a finite subset of indices $\{\sigma_1, \ldots, \sigma_n\} \subset \Sigma$ and non-trivial semicharacters $\gamma_{\sigma_1}, \ldots, \gamma_{\sigma_n}$ $(\gamma_{\sigma_i} \in \hat{S}_{\sigma_i})$ such that*

$$\gamma(x) = \prod_{i=1}^{n}\gamma_{\sigma_i}(x_{\sigma_i}) \quad \textit{for every } x = (x_\sigma) \textit{ in } P. \tag{2}$$

Moreover, this representation is unique.

Proof. It is obvious that (2) is a non-trivial semicharacter on P. Conversely, let $\gamma \in \hat{P}$, γ non-trivial. Let $\xi_\sigma\colon S_\sigma \to P$ be the canonical injection defined as $\xi_\sigma(x_\sigma) = (y_\tau)_{\tau\in\Sigma}$, where $y_\tau = e_\tau$ for $\tau \neq \sigma$ and $y_\sigma = x_\sigma$. Denote $\gamma_\sigma = \gamma_0 \xi_\sigma$. It is clear that $\gamma_\sigma \in \hat{S}_\sigma$. Let γ_σ^1 denote the unit semicharacter of S_σ. We claim that the set $\Sigma_0 = \{\sigma \in \Sigma: \gamma_\sigma \neq \gamma_\sigma^1\}$ is finite. Let V be a neighborhood of a point $x = (x_\sigma)$ in P. It follows from the definition of Tychonoff topology that for almost all σ the projection of V on S_σ is equal to S_σ. If Σ_0 were infinite, then there would exist an index σ_0 such that $\gamma_{\sigma_0} \neq \gamma_{\sigma_0}^1$ and $\xi_\sigma(s) \in V$ for all $s \in S_{\sigma_0}$. Hence $\gamma_{\sigma_0}(s) = \gamma(\xi_{\sigma_0}(s)) \in \gamma(V)$ for all $s \in S_{\sigma_0}$ and $\gamma(V)$ would contain the non-trivial subsemigroup $\gamma_{\sigma_0}(S_{\sigma_0})$ (containing 1 since S_{σ_0} has identity e) of the complex unit disc; this would contradict the continuity of the semicharacter γ. It is clear that $\gamma(x) = \prod_{\sigma\in\Sigma_0} \gamma_\sigma(x_\sigma)$.

Let us now suppose that some non-trivial semicharacter γ can be written as

$$\gamma(x) = \prod_{i=1}^{n} \gamma_{\sigma_i}(x_{\sigma_i}) \quad \text{and} \quad \gamma(x) = \prod_{j=1}^{k} \beta_{\tau_j}(x_{\tau_j}), \tag{3}$$

where $\gamma_{\sigma_i} \in \hat{S}_{\sigma_i}$, $\beta_{\tau_j} \in \hat{S}_{\tau_j}$ are non-trivial. If the set $\{\sigma_1, \ldots, \sigma_n\}$ were different from $\{\tau_1, \ldots, \tau_k\}$, e.g., $\sigma_i \notin \{\tau_1, \ldots, \tau_k\}$, then substituting $x_\sigma = e_\sigma$ for $\sigma \neq \sigma_i$ in (3) we would have $\gamma_{\sigma_i}(x_{\sigma_i}) = 1$, i.e., $\gamma_{\sigma_i} = \gamma_{\sigma_i}^1$. Thus $\{\sigma_1, \ldots, \sigma_n\} = \{\tau_1, \ldots, \tau_k\}$. If we again substitute $x_\sigma = e_\sigma$ for $\sigma \neq \sigma_i$, we get $\gamma_{\sigma_i}(x_{\sigma_i}) = \gamma(x) = \beta_{\sigma_i}(x_{\sigma_i})$; hence $\gamma_{\sigma_i} = \beta_{\sigma_i}$.

Proof of Theorem 2. Since P has identity, the semicharacter γ^0 is an isolated point in $\hat{P}$. P may lack zero, but if $\gamma \neq \gamma^1$ and for some x $|\gamma(x)| < 1$, then we can find an integer k such that $|1-\gamma(x^k)| = |1-(\gamma(x))^k| > 1/2$. If, on the other hand, $|\gamma(x)| = 1$ for all x, then $\gamma(P)$ is a non-trivial subgroup of the unit circle; this implies the existence of an x such that $|1-\gamma(x)| > 1/2$. Thus $U(\gamma^1, 1/2) = \{\gamma^1\}$ and γ^1 is also an isolated point in $\hat{P}$.

Let γ' be a non-trivial semicharacter of P. By the Lemma we have $\gamma'(x) = \prod_{i=1}^{k} \gamma'_{\sigma_i}(x_{\sigma_i})$, where $\gamma'_{\sigma_i} \in \hat{S}_{\sigma_i}$, γ'_{σ_i} non-trivial. Let $0 < \varepsilon < 1$. If $\gamma \in U(\gamma', \varepsilon)$ and $\gamma(x) = \prod_{j=1}^{l} \gamma_{\tau_j}(x_{\tau_j})$, then the sets $\{\sigma_1, \ldots, \sigma_k\}$ and $\{\tau_1, \ldots, \tau_l\}$ are identical. Indeed, if $\sigma_{i_0} \notin \{\tau_1, \ldots, \tau_l\}$, then, by non-triviality of $\gamma'_{\sigma_{i_0}}$ we would choose (likewise as in the previous part of the proof) an $x'_{\sigma_{i_0}} \in S_{\sigma_{i_0}}$ such that for $x = (x_\sigma)$, where $x_\sigma = e_\sigma$ if $\sigma \neq \sigma_{i_0}$ and $x_{\sigma_{i_0}} = x'_{\sigma_{i_0}}$, we would have

$$|\gamma(x) - \gamma'(x)| = \Big|\prod_{j=1}^{l} \gamma_{\tau_j}(x_{\tau_j}) - \prod_{i=1}^{k} \gamma'_{\sigma_i}(x_{\sigma_i})\Big| = |1 - \gamma'_{\sigma_{i_0}}(x'_{\sigma_{i_0}})| > \varepsilon.$$

Let us put again $x = (x_\sigma)$, where $x_\sigma = e_\sigma$ for all σ except some σ_i, $1 \leqslant i \leqslant k$, and x_{σ_i} arbitrary. We have then $|\gamma(x) - \gamma'(x)| = |\gamma_{\sigma_i}(x_{\sigma_i}) - \gamma'_{\sigma_i}(x_{\sigma_i})| < \varepsilon$ which means that $\gamma_{\sigma_i} \in U(\gamma'_{\sigma_i}, \varepsilon) \subset \hat{S}_{\sigma_i}$. Since the sets of the form $U(\gamma_{\sigma_i}, \varepsilon)$ constitute a basis of neighborhoods in $\hat{S}_{\sigma_i}$, then, changing possibly ε, we may assume that $U(\gamma'_{\sigma_i}, \varepsilon)$ is relatively compact in $\hat{S}_{\sigma_i}$, so the image of $U(\gamma', \varepsilon)$ under the natural projection onto $\hat{S}_{\sigma_i}$ is also relatively compact. Therefore $U(\gamma', \varepsilon)$ is relatively compact in $\hat{P}$.

References

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