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Summability of differentiated interpolating polynomials

Throughout this paper the function $f(t)$ is real, 2π-periodic, defined for all $t \in (-\infty, \infty)$. We consider the trigonometric interpolating polynomials of f:

$$I_n(z;f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos kz + b_k^{(n)} \sin kz),$$

with fundamental points

$$t_j^{(n)} = \frac{2\pi}{2n+1} j \quad (j = 0, \pm 1, \pm 2, \ldots).$$

Sections 1, 2 contain some theorems concerning the convergence of the derivatives of Cesàro and Riesz means of $I_n(z;f)$, respectively. The symbols $C_p, C_p(r), C_p(r, s, \ldots)$ $(p = 1, 2, \ldots)$ signify the suitable positive constants (absolute or depending on the indicated parameters, only).

1. Cesàro summability. Given an interval $\langle a, b\rangle$, let

$$t_{a-1}^{(n)} < a \leqslant t_a^{(n)} < t_{a+1}^{(n)} < \ldots < t_\beta^{(n)} < b \leqslant t_{\beta+1}^{(n)}.$$

Denote by $\varphi_n(t)$ the step function equal to $2\pi j/(2n+1)$ for $t \in \langle t_{j-1}^{(n)}, t_j^{(n)})$ $(j = 0, \pm 1, \pm 2, \ldots)$. We shall write

$$\int_a^b g(t)\, d\varphi_n(t) = \frac{2\pi}{2n+1} \sum_{j=a}^{\beta} g(t_j^{(n)})$$

for any function $g(t)$ defined in $\langle a, b\rangle$. Moreover, by the convention,

$$\int_a^a g(t)\, d\varphi_n(t) = 0.$$

If g is of period 2π, the integral

$$\int_a^{a+2\pi} g(t)\, d\varphi_n(t)$$

is independent of a. In particular, setting

$$I_{n,\nu}(z;f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{\nu} (a_k^{(n)} \cos kz + b_k^{(n)} \sin kz) \qquad (0 \leqslant \nu \leqslant n),$$

we have

$$I_{n,\nu}(z;f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_\nu(t-z)\, d\varphi_n(t) = \frac{1}{\pi} \int_{z-\pi}^{z+\pi} f(t) D_\nu(t-z)\, d\varphi_n(t),$$

where

$$D_\nu(t) = \frac{1}{2} + \sum_{k=1}^{\nu} \cos kt = \frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t}$$

(see [4], II, pp. 1-8).

Let A_k^a be as in [4], I, pp. 76-77. Write, for $a > 0$,

$$\sigma_n^a(z;f) = \sum_{\nu=0}^{n} (A_{n-\nu}^{a-1}/A_n^a) I_{n,\nu}(z;f)$$
$$= \tfrac{1}{2} a_0^{(n)} + \sum_{k=1}^{n} (A_{n-k}^{a}/A_n^a)(a_k^{(n)} \cos kz + b_k^{(n)} \sin kz).$$

Evidently, these Cesàro means can be represented in the form

$$\sigma_n^a(z;f) = \frac{1}{\pi} \int_{z-\pi}^{z+\pi} f(t) K_n^a(t-z)\, d\varphi_n(t),$$

with the kernel

$$K_n^a(t) = \frac{1}{A_n^a} \sum_{\nu=0}^{n} A_{n-\nu}^{a-1} D_\nu(t) = \frac{1}{A_n^a (2\sin\frac{1}{2}t)} \operatorname{Im} \sum_{\nu=0}^{n} A_{n-\nu}^{a-1} e^{i(\nu+\frac{1}{2})t}.$$

In the case $1/n \leqslant t \leqslant \pi$,

$$|K_n^a(t)| \leqslant \begin{cases} C_1(a)/n^a t^{a+1} & \text{if } 0 < a \leqslant 1, \\ C_1(a)/nt^2 & \text{if } a > 1. \end{cases}$$

Moreover, considering $a > 0$, we have

$$|K_n^a(t)| \leqslant 2n$$

for all t and $n = 1, 2, \ldots$ ([4], I, pp. 94-95). The above estimates ensure that

$$\lim_{n\to\infty} \sigma_n^a(x;f) = f(x) \qquad (a > 0)$$

at any point x of continuity of f bounded in $\langle -\pi, \pi\rangle$. This relation holds uniformly on $\langle a, b\rangle$ if the function f is continuous at every point of the last interval (cf. [1], pp. 177-179, [2], pp. 562-566).

We shall examine the convergence of the derivatives

$$\frac{d}{dz}\sigma_n^a(z;f) = \frac{1}{A_n^a}\sum_{\nu=1}^{n} A_{n-\nu}^{a-1}\frac{d}{dz}I_{n,\nu}(z;f)$$

$$= \frac{1}{\pi}\int_{z-\pi}^{z+\pi} f(t)\frac{\partial}{\partial z}K_n^a(t-z)\,d\varphi_n(t)$$

in three principial cases: $a>1$, $a=1$, $0<a<1$. For this purpose the following auxiliary result will be needed.

LEMMA 1. *Let the derivative $f'(x)$ be finite. Write*

$$F_x(t) = f(t)-f(x)-f'(x)\sin(t-x).$$

Then the condition

$$\lim_{n\to\infty}\frac{d}{dz}\sigma_n^a(z;F_x)/_{z=x} = 0 \qquad (a>0) \tag{1}$$

implies

$$\lim_{n\to\infty}\frac{d}{dz}\sigma_n^a(z;f)/_{z=x} = f'(x). \tag{2}$$

If relation (1) *holds uniformly in $x\in\langle a, b\rangle$, the convergence* (2) *is uniform in this interval, too.*

These facts are an immediate consequence of the identity

$$\frac{d}{dz}\{\sigma_n^a(z;f)-\sigma_n^a(z;F_x)\}/_{z=x} = \frac{A_{n-1}^a}{A_n^a}f'(x).$$

Retaining in the present Section the symbol $F_x(t)$, we shall first give an analogue of (1.7) in [4], II, p. 60.

THEOREM 1. *Suppose that $|f(t)|$ possesses a majorant $f^*(t)$ Riemann-integrable over $\langle -\pi, \pi\rangle$ in the improper sense, discontinuous infinitely only at the points $x_k = 2\pi w_k$ with rational w_k. Moreover, f^* is non-decreasing [non-increasing] in some left [right] neighbourhoods of x_k. Then if $a>1$, relation* (2) *holds for all these x at which $|f'(x)|<\infty$. The convergence is uniform on $\langle a, b\rangle$, whenever f' is continuous at every point of this interval.*

Proof. In the case $|f'(x)|<\infty$,

$$F_x(t) = \varrho_x(t)\sin(t-x), \quad \text{where} \quad \varrho_x(t) = o(1) \quad \text{as } t\to x.$$

The o-relation is uniform in x on the closed intervals of continuity of f'. Under the assumption $1/n \leqslant t \leqslant \pi$, the estimates

$$\left|\frac{d}{dt} K_n^a(t)\right| \leqslant \begin{cases} C_2(a)/n^{a-1}t^{a+1} & \text{if } 1 < a \leqslant 2, \\ C_2(a)/nt^3 & \text{if } a > 2 \end{cases} \tag{3}$$

hold ([4], II, pp. 60-61). In the case $a > 0$, the inequality

$$\left|\frac{d}{dt} K_n^a(t)\right| \leqslant n^2, \quad \text{for each } t \text{ and } n, \tag{4}$$

is obvious.

Given any $\varepsilon > 0$, we choose a positive $\delta = \delta(\varepsilon) < \pi$ such that

$$|\varrho_x(t)| < \varepsilon \quad \text{when } |t-x| \leqslant \delta,$$

and we write

$$\frac{d}{dz}\sigma_n^a(z; F_x)/_{z=x} = -\frac{1}{\pi}\Big(\int_{x-\delta}^{x+\delta} + \int_{x-\pi}^{x-\delta} + \int_{x+\delta}^{x+\pi}\Big) F_x(t)\frac{\partial}{\partial t} K_n^a(t-x)\, d\varphi_n(t)$$

$$= -\frac{1}{\pi}(J_1+J_2+J_3).$$

Evidently,

$$|J_1| \leqslant \varepsilon \int_{x-\delta}^{x+\delta} |\sin(t-x)| \left|\frac{\partial}{\partial t} K_n^a(t-x)\right| d\varphi_n(t)$$

$$\leqslant \varepsilon\Big(\int_{x-\pi}^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^{x+\pi}\Big) |t-x| \left|\frac{\partial}{\partial t} K_n^a(t-x)\right| d\varphi_n(t);$$

whence, by (3) and (4),

$$|J_1| < C_3(a)\varepsilon \quad \text{for } n = 1, 2, \ldots$$

Further, taking $n > 1/\delta$, we have

$$|J_2| + |J_3| \leqslant \frac{C_2(a)}{n^\beta \delta^{\beta+2}} \int_{x-\pi}^{x+\pi} |F_x(t)|\, d\varphi_n(t)$$

$$\leqslant \frac{C_2(a)}{n^\beta \delta^{\beta+2}} \Big\{\int_{-\pi}^{\pi} |f(t)|\, d\varphi_n(t) + 2\pi\big(|f(x)| + |f'(x)|\big)\Big\},$$

where

$$\beta = \begin{cases} a-1 & \text{if } 1 < a \leqslant 2, \\ 1 & \text{if } a > 2. \end{cases}$$

Since

$$\int_{-\pi}^{\pi} |f(t)|\, d\varphi_n(t) \leqslant \int_{-\pi}^{\pi} f^*(t)\, d\varphi_n(t) \leqslant C_1 \quad (n = 1, 2, \ldots),$$

our thesis is established.

Theorem 1 may be extended to the derivatives of higher order (see [4], II, pp. 60-61).

Passing to $\alpha = 1$, we shall now prove the following

LEMMA 2. *Let $f(t)$ be Riemann-integrable over $\langle -\pi, \pi\rangle$, and let the derivative $f'(x)$ be finite. Then, for any positive $\delta < \pi$,*

$$\lim_{n\to\infty}\Big(\int\limits_{x-\pi}^{x-\delta} + \int\limits_{x+\delta}^{x+\pi}\Big) F_x(t)\frac{\partial}{\partial t}K_n^1(t-x)\,d\varphi_n(t) = 0.$$

The convergence is uniform in x on $\langle a, b\rangle$ if $f'(x)$ is bounded in this interval.

Proof. Suppose that $f'(t)$ is bounded in $\langle a, b\rangle$, where $a \leqslant b$, and set

$$L = \sup_{a\leqslant t\leqslant b}|f'(t)|, \qquad M = \sup_{a\leqslant s\leqslant b}\{\sup_{-\pi\leqslant t\leqslant \pi}|F_s(t)|\}.$$

As it is easy to check,

$$\frac{d}{dz}K_n^1(z) = \frac{\sin(n+1)z}{4\sin^2\frac{1}{2}z} - \frac{\sin^2\frac{1}{2}(n+1)z}{2(n+1)\sin^3\frac{1}{2}z}\cos\tfrac{1}{2}z = P_n(z) - R_n(z)$$

and

$$\Big|\Big(\int\limits_{x-\pi}^{x-\delta} + \int\limits_{x+\delta}^{x+\pi}\Big) F_x(t)R_n(t-x)\,d\varphi_n(t)\Big| \leqslant \frac{\pi M}{(n+1)\sin^3\frac{1}{2}\delta} \qquad (a\leqslant x\leqslant b).$$

Write

$$U_n(x) = \int\limits_{x+\delta}^{x+\pi} F_x(t)P_n(t-x)\,d\varphi_n(t), \qquad \tilde{F}_x(t) = \frac{F_x(t)}{4\sin^2\frac{1}{2}(t-x)}.$$

Then

$$U_n(x) = \int\limits_{x+\delta}^{x+\pi} \tilde{F}_x(t)\sin(n+1)(t-x)\,d\varphi_n(t) \qquad (a\leqslant x\leqslant b).$$

Given a positive $\lambda < \pi - \delta$, let us choose a partition

$$a+\delta = z_1 < z_2 < \ldots < z_k < z_{k+1} < \ldots < z_m < z_{m+1} = b+\pi \qquad (m\geqslant 2)$$

such that

$$\max_{1\leqslant k\leqslant m}(z_{k+1}-z_k) < \lambda \quad\text{ and }\quad \sum_{k=1}^{m}(z_{k+1}-z_k)\operatorname*{Osc}_{z_k\leqslant t\leqslant z_{k+1}} f(t) < \lambda.$$

If $z_\varrho \leqslant x+\delta \leqslant z_{\varrho+1}$, $z_\mu \leqslant x+\pi \leqslant z_{\mu+1}$, we have

$$\sum_{k=\varrho+1}^{\mu-1}(z_{k+1}-z_k)\operatorname*{Osc}_{z_k\leqslant t\leqslant z_{k+1}}\tilde{F}_x(t) < \left\{\frac{1+(b-a+\pi)L}{4\sin^2\frac{1}{2}\delta} + \frac{(b-a+\pi)M}{4\sin^4\frac{1}{2}\delta}\right\}\lambda.$$

Consequently,

$$|U_n(x)| \leqslant \Big(\int_{x+\delta}^{z_{\varrho+1}} + \int_{z_\mu}^{x+\pi} \Big) |\tilde{F}_x(t)|\, d\varphi_n(t) + \sum_{k=\varrho+1}^{\mu-1} |\tilde{F}_x(z_k)| \Big| \int_{z_k}^{z_{k+1}} \sin(n+1)(t-x)\, d\varphi_n(t) \Big| +$$

$$+ \sum_{k=\varrho+1}^{\mu-1} \int_{z_k}^{z_{k+1}} |\tilde{F}_x(t) - \tilde{F}_x(z_k)|\, d\varphi_n(t) \leqslant \frac{M}{2\sin^2 \frac{1}{2}\delta} \Big\{ \lambda + \frac{2\pi}{2n+1} \Big\} +$$

$$+ \frac{M}{4\sin^2 \frac{1}{2}\delta} \cdot \frac{2\pi(m-2)}{(2n+1)\cos\{\pi/(4n+2)\}} + \Big\{ \frac{1+(b-a+\pi)L}{4\sin^2 \frac{1}{2}\delta} +$$

$$+ \frac{(b-a+\pi)M}{4\sin^4 \frac{1}{2}\delta} \Big\} \lambda + \frac{M}{\sin^2 \frac{1}{2}\delta} \cdot \frac{\pi(m-2)}{2n+1}.$$

The integral

$$V_n(x) = \int_{x-\pi}^{x-\delta} \tilde{F}_x(t) \sin(n+1)(t-x)\, d\varphi_n(t)$$

can be estimated similarly. Thus

$$\lim_{n\to\infty} \Big(\int_{x-\pi}^{x-\delta} + \int_{x+\delta}^{x+\pi} \Big) F_x(t) P_n(t-x)\, d\varphi_n(t) = 0$$

uniformly on $\langle a, b \rangle$, and, the result follows.

Applying Lemmas 1, 2 and reasoning as in [1], pp. 197-198, 64-66, we obtain

THEOREM 2. *Suppose that the function $f(t)$, Riemann-integrable over $\langle -\pi, \pi \rangle$, is absolutely continuous in an interval $\langle A, B \rangle$. Then if $f'(x)$ is finite for a certain $x \in (A, B)$ and if*

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f'(t) - f'(x)|\, dt = 0, \tag{5}$$

we have

$$\lim_{n\to\infty} \frac{d}{dz} \sigma_n^1(z; f)/_{z=x} = f'(x). \tag{6}$$

Relation (6) *holds uniformly on $\langle a, b \rangle \subset (A, B)$, whenever $f'(x)$ is continuous at every $x \in \langle a, b \rangle$.*

In the case $0 < \alpha < 1$, $0 < t < 2\pi$,

$$K_n^\alpha(t) = \frac{\sin\{(n+\frac{1}{2}+\frac{1}{2}\alpha)t - \frac{1}{2}\pi\alpha\}}{A_n^\alpha (2\sin\frac{1}{2}t)^{\alpha+1}} + \frac{2A_n^{\alpha-1} + A_n^{\alpha-2}}{2A_n^\alpha (2\sin\frac{1}{2}t)^2} +$$

$$+ \operatorname{Im} \Big\{ \frac{e^{i(n+3/2)t}}{A_n^\alpha (2\sin\frac{1}{2}t)^3} \sum_{k=n+1}^{\infty} A_k^{\alpha-3} e^{-ikt} \Big\}.$$

Therefore, for these a and t,

$$\frac{d}{dt}K_n^a(t) = \Phi_n^a(t) - \Psi_n^a(t) + \Lambda_n^a(t),$$

where

$$\Phi_n^a(t) = \frac{(n+\frac{1}{2}+\frac{1}{2}a)\cos(\frac{1}{2}at-\frac{1}{2}\pi a)}{A_n^a(2\sin\frac{1}{2}t)^{a+1}}\cos(n+\tfrac{1}{2})t,$$

$$\Psi_n^a(t) = \frac{(n+\frac{1}{2}+\frac{1}{2}a)\sin(\frac{1}{2}at-\frac{1}{2}\pi a)}{A_n^a(2\sin\frac{1}{2}t)^{a+1}}\sin(n+\tfrac{1}{2})t$$

and

$$|\Lambda_n^a(t)| \leqslant C_4(a)\left\{\frac{1}{n^a(2\sin\frac{1}{2}t)^{a+2}} + \frac{1}{n(2\sin\frac{1}{2}t)^3} + \frac{1}{n^2(2\sin\frac{1}{2}t)^4}\right\}.$$

Put

$$M_n^\gamma(t) = n^{-\gamma}(2\sin\tfrac{1}{2}t)^{-\gamma-2} \qquad (\gamma > 0).$$

We shall now give three further lemmas needed in the proof of our next theorem.

LEMMA 3. *Let $f'(x)$ be finite (at a fixed x), and let ε and γ be two positive numbers. Then there is a positive $\eta = \eta(\varepsilon, \gamma) < \pi$ such that for any positive $\sigma < \eta$ and for all integers $n > 1/\sigma$,*

$$\int_{x+1/n}^{x+\sigma} |F_x(t)| M_n^\gamma(t-x)\, d\varphi_n(t) < \varepsilon. \tag{7}$$

The estimate is uniform in $x \in \langle a, b\rangle$ if f' is continuous at every point of this interval.

Proof of the second part. Given an arbitrary $\lambda > 0$, we choose a positive $\eta = \eta(\lambda) < \pi$ such that

$$|F_x(t)| < \lambda|t-x| \qquad \text{when } |t-x| < \eta,\ a \leqslant x \leqslant b.$$

Then if $\sigma < \eta$, $n > 1/\sigma$, the left-hand side of (7) does not exceed

$$\int_{x+1/n}^{x+\sigma} \frac{\lambda(t-x)}{n^\gamma\{2\sin\frac{1}{2}(t-x)\}^{\gamma+2}}\, d\varphi_n(t)$$

$$\leqslant \frac{\lambda}{n^\gamma}\left(\frac{\pi}{2}\right)^{\gamma+2} \int_{x+1/n}^{x+\sigma} \frac{1}{(t-x)^{\gamma+1}}\, d\varphi_n(t) < \lambda\left(\frac{\pi}{2}\right)^{\gamma+2}\left(\pi + \frac{1}{\gamma}\right).$$

Taking

$$\lambda = \left(\frac{2}{\pi}\right)^{\gamma+2}\frac{\varepsilon\gamma}{\pi\gamma+1},$$

we get (7) for all $x \in \langle a, b\rangle$.

LEMMA 4. *Let $f(t)$ be absolutely continuous in an interval $\langle A, B\rangle$, and let condition* (5) *be fulfilled at a certain $x \in (A, B)$. Then, given positive ε and $\alpha < 1$, there is a positive $\eta = \eta(\varepsilon, \alpha) < \pi$ such that for any positive $\sigma < \eta$ and for all $n > 1/\sigma$,*

$$\left|\int_{x+1/n}^{x+\sigma} F_x(t)\Phi_n^\alpha(t-x)\,d\varphi_n(t)\right| < \varepsilon. \tag{8}$$

Estimate (8) *holds uniformly in $x \in \langle a, b\rangle$ if the derivative f' is continuous at every x of the interval $\langle a, b\rangle$ interior to (A, B).*

Proof of the first part. It is enough to consider

$$\hat{\Phi}_n^\alpha(t) = \frac{n^{1-\alpha}\cos(\frac{1}{2}\alpha t - \frac{1}{2}\pi\alpha)}{(2\sin\frac{1}{2}t)^{\alpha+1}}\cos(n+\tfrac{1}{2})t$$

instead of $\Phi_n^\alpha(t)$. Write

$$J_n(x) = \int_{x+1/n}^{x+\sigma} F_x(t)\hat{\Phi}_n^\alpha(t-x)\,d\varphi_n(t) \qquad (1/n < \sigma)$$

and

$$\bar{F}_x(t) = F_x(t)\cos\{\tfrac{1}{2}\alpha(t-x) - \tfrac{1}{2}\pi\alpha\}.$$

Evidently, condition (5) implies

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} |F_x'(t)|\,dt = 0.$$

Observing that

$$\int |\bar{F}_x'(t)|\,dt \leqslant \int |F_x'(t)|\,dt + \tfrac{1}{2}\alpha\int |F_x(t)|\,dt$$

over the intervals with the end-points $x, x+h$, we obtain

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} |\bar{F}_x'(t)|\,dt = 0. \tag{9}$$

The partial integration leads to

$$J_n(x) = \left[\frac{n^{1-\alpha}}{\{2\sin\frac{1}{2}(z-x)\}^{\alpha+1}}\int_{x+1/n}^{z} \bar{F}_x(t)\cos(n+\tfrac{1}{2})(t-x)\,d\varphi_n(t)\right]_{z=x+1/n}^{z=x+\sigma} +$$
$$+(\alpha+1)\int_{x+1/n}^{x+\sigma}\frac{n^{1-\alpha}\cos\frac{1}{2}(z-x)}{\{2\sin\frac{1}{2}(z-x)\}^{\alpha+2}}\int_{x+1/n}^{z}\bar{F}_x(t)\cos(n+\tfrac{1}{2})(t-x)\,d\varphi_n(t)\,dz.$$

Further (see [1], p. 66),

$$\left|\int_{x+1/n}^{z} \bar{F}_x(t)\cos(n+\tfrac{1}{2})(t-x)\,d\varphi_n(t)\right| \leqslant \frac{2\pi}{2n+1}\int_{x+1/n}^{z} |\bar{F}'_x(t)|\,dt$$

for $z \in \langle x+1/n,\ x+\sigma\rangle$, and

$$\left|\int_{x+1/n}^{z} \bar{F}_x(t)\cos(n+\tfrac{1}{2})(t-x)\,d\varphi_n(t)\right| \leqslant \frac{2\pi}{2n+1}\left|\bar{F}_x\left(x+\frac{1}{n}\right)\right|$$

if $|z-x-1/n| < 2\pi/(2n+1)$. Moreover, for an arbitrary $\lambda > 0$, we can find a positive $\mu < \pi$ such that

$$|\bar{F}_x(x+1/n)| < \lambda/n \qquad \text{when } n > 1/\mu.$$

Hence, if $1/n < \sigma < \mu$,

$$\begin{aligned}
|J_n(x)| \leqslant{} & \frac{2\pi n^{1-\alpha}}{(2n+1)(2\sin\frac{1}{2}\sigma)^{\alpha+1}}\int_{x+1/n}^{x+\sigma} |\bar{F}'_x(t)|\,dt + \\
& + \frac{2\pi\lambda}{(2n+1)\,n^{\alpha}\left(2\sin\dfrac{1}{2n}\right)^{\alpha+1}} + \\
& + \frac{2\pi(\alpha+1)n}{(2n+1)\,n^{\alpha}}\int_{x+1/n}^{x+\sigma}\left\{\frac{1}{\left(2\sin\frac{1}{2}(z-x)\right)^{\alpha+2}}\int_{x+1/n}^{z}|\bar{F}'_x(t)|\,dt\right\}dz \\
\leqslant{} & \frac{\pi^{\alpha+2}}{2^{\alpha+1}\sigma}\int_{x}^{x+\sigma}|\bar{F}'_x(t)|\,dt + \frac{\pi^{\alpha+2}\lambda}{2^{\alpha+1}} + \\
& + \frac{\pi^{\alpha+3}(\alpha+1)}{2^{\alpha+2}n^{\alpha}}\int_{x+1/n}^{x+\sigma}\left\{\frac{1}{(z-x)^{\alpha+2}}\int_{x}^{z}|\bar{F}'_x(t)|\,dt\right\}dz.
\end{aligned}$$

By virtue of (9),

$$|J_n(x)| \leqslant \frac{\pi^{\alpha+2}\lambda}{2^{\alpha}} + \frac{\pi^{\alpha+3}(\alpha+1)\lambda}{2^{\alpha+2}n^{\alpha}}\int_{x+1/n}^{x+\sigma}\frac{1}{(z-x)^{\alpha+1}}\,dz < \frac{\pi^{\alpha+3}\lambda}{2^{\alpha}\alpha},$$

provided σ is small enough.

Taking

$$\lambda = \frac{2^{\alpha-1}\alpha\varepsilon}{\pi^{\alpha+3}\Gamma(\alpha+1)},$$

we get the required assertion.

Clearly, Φ_n^{α} in (8) can be replaced by Ψ_n^{α}.

LEMMA 5. *Let the functions $f(t)$, $\Omega(t)$ be absolutely continuous in any finite interval. Suppose that $f'(x)$ is bounded in $\langle a, b\rangle$, where $a \leqslant b$, and write*

$$G_x(t) = \frac{F_x(t)\,\Omega(t-x)}{\{2\sin\frac{1}{2}(t-x)\}^p} \qquad (p>0).$$

Then, for every positive $\delta < \pi$,

$$\int_{x+\delta}^{x+\pi} G_x(t)\cos(n+\tfrac{1}{2})t\,d\varphi_n(t) = O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty, \tag{10}$$

uniformly in $x\in\langle a, b\rangle$.

Proof. Denote by $L_n(x)$ the left-hand side of (10).

It is easily seen that

$$L_n(x) = -\int_{x+\delta-t_1^{(n)}}^{x+\pi-t_1^{(n)}} G_x(t+t_1^{(n)})\cos(n+\tfrac{1}{2})t\,d\varphi_n(t),$$

where $t_1^{(n)} = 2\pi/(2n+1)$. Therefore,

$$2L_n(x) = \int_{x+\delta}^{x+\pi-t_1^{(n)}} \{G_x(t)-G_x(t+t_1^{(n)})\}\cos(n+\tfrac{1}{2})t\,d\varphi_n(t) + O\left(\frac{1}{n}\right).$$

Integrating by parts, we obtain

$$2L_n(x) = \left[\{G_x(z)-G_x(z+t_1^{(n)})\}\int_{x+\delta}^{z}\cos(n+\tfrac{1}{2})t\,d\varphi_n(t)\right]_{z=x+\delta}^{x+\pi-t_1^{(n)}} -$$
$$-\int_{x+\delta}^{x+\pi-t_1^{(n)}}\{G_x'(z)-G_x'(z+t_1^{(n)})\}\int_{x+\delta}^{z}\cos(n+\tfrac{1}{2})t\,d\varphi_n(t)\,dz + O\left(\frac{1}{n}\right).$$

Further,

$$\left|\int_{x+\delta}^{z}\cos(n+\tfrac{1}{2})t\,d\varphi_n(t)\right| \leqslant \frac{2\pi}{2n+1} \qquad \text{for } z\geqslant x+\delta$$

and

$$\int_{x+\delta}^{x+\pi-t_1^{(n)}} |G_x'(z)-G_x'(z+t_1^{(n)})|\,dz \leqslant 2\int_{x+\delta}^{x+\pi}|G_x'(t)|\,dt.$$

Hence the Lemma.

THEOREM 3. *Considering the case $0<a<1$, suppose that the function $f(t)$ is absolutely continuous in $\langle -2\pi, 2\pi\rangle$. Then, under the assumption $|f'(x)|<\infty$, condition (5) implies (2). Relation (2) is uniform in $x\in\langle a, b\rangle$ if f' is continuous in this interval.*

Proof. Given an arbitrary $\varepsilon > 0$, we can find a positive $\delta < \pi$ such that

$$(11) \qquad \left| \int_{x-\delta}^{x+\delta} F_x(t) \frac{\partial}{\partial t} K_n^a(t-x)\, d\varphi_n(t) \right| < 2\varepsilon$$

whenever $n > 1/\delta$. If f' is continuous at every point of the interval $\langle a, b\rangle$, inequality (11) holds uniformly in $x \epsilon \langle a, b\rangle$. Indeed, the derivative of $K_n^a(t)$ is an odd function. Hence, by Lemmas 3, 4 and inequality (4),

$$\left| \left(\int_{x-\delta}^{x-1/n} + \int_{x+1/n}^{x+\delta} \right) F_x(t) \frac{\partial}{\partial t} K_n^a(t-x)\, d\varphi_n(t) \right| < \varepsilon$$

and

$$\left| \int_{x-1/n}^{x+1/n} F_x(t) \frac{\partial}{\partial t} K_n^a(t-x)\, d\varphi_n(t) \right| \leqslant \frac{\varepsilon n^2}{\pi} \int_{x-1/n}^{x+1/n} |t-x|\, d\varphi_n(t) < \varepsilon$$

for small δ and $n > 1/\delta$.

Applying Lemma 5, we obtain

$$\left| \left(\int_{x-\pi}^{x-\delta} + \int_{x+\delta}^{x+\pi} \right) F_x(t) \frac{\partial}{\partial t} K_n^a(t-x)\, d\varphi_n(t) \right| < \varepsilon$$

if n is large enough. The estimate holds uniformly in $\langle a, b\rangle$ when $f'(x)$ is continuous therein.

Now, the result follows at once from Lemma 1 (cf. the case $a > 1$).

2. Riesz summability. The Riesz means of $I_n(z; f)$, considered here, are of the form

$$S_n^r(z; f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} \left\{ 1 - \frac{k^2}{(n+1/2)^2} \right\}^r (a_k^{(n)} \cos kz + b_k^{(n)} \sin kz),$$

where r is a positive number. We set

$$g(t) = f(\pi t) \qquad \text{for } t \epsilon (-\infty, \infty),$$

$$T_n^r(u; g) = S_n^r(\pi u; f) \qquad \text{for } u \epsilon (-\infty, \infty).$$

The function $g(t)$ is of period 2, since $f(t)$ is 2π-periodic. Denote by $\omega(\delta; f)$, $\omega(\delta; f, \langle a, b\rangle)$ the moduli of continuity of f in the intervals $(-\infty, \infty)$ and $\langle a, b\rangle$, respectively.

Let

$$v_j^{(n)} = 2j/(2n+1) \qquad (j = 0, \pm 1, \pm 2, \ldots)$$

and

$$\psi_n(v) = v_j^{(n)} \quad \text{when} \quad v \in \langle v_{j-1}^{(n)}, v_j^{(n)}).$$

Suppose that

$$v_{a-1}^{(n)} < a \leqslant v_a^{(n)} < v_{a+1}^{(n)} < \ldots < v_\beta^{(n)} < b \leqslant v_{\beta+1}^{(n)}.$$

Then, we shall write (cf. section 1)

$$\int_a^b \varphi(v)\,d\psi_n(v) = \frac{2}{2n+1}\sum_{j=a}^{\beta}\varphi(v_j^{(n)})$$

for any function $\varphi(v)$ defined in $\langle a, b\rangle$. In particular,

$$T_n^r(u; g) = \int_{-1}^{1} g(v)Q_n^r(v-u)\,d\psi_n(v) = \int_{u-1}^{u+1} g(v)Q_n^r(v-u)\,d\psi_n(v),$$

where

$$Q_n^r(t) = \frac{1}{2} + \sum_{k=1}^{n}\left\{1-\frac{k^2}{(n+1/2)^2}\right\}^r \cos k\pi t.$$

Putting $B_n = (n+1/2)\pi$, we have

$$Q_n^r(t) = \frac{1}{2} + \sum_{k=1}^{n}\operatorname*{res}_{w=k\pi}\left\{1-\frac{w^2}{B_n^2}\right\}^r \frac{\cos(1-t)w}{\sin w}$$

$$= \frac{1}{2\pi i}\int_{B_n-\infty i}^{B_n+\infty i}\left(1-\frac{w^2}{B_n^2}\right)^r \frac{\cos(1-t)w}{\sin w}\,dw$$

when $0 < t < 2$ (cf. [3], §§ 18, 21, 22, 51). Hence, under the assumption $1/n \leqslant t \leqslant 1$,

$$|Q_n^r(t)| \leqslant \frac{C_5(r)}{n^r t^{r+1}} \quad \text{if } r > 0.$$

In case $r > 0$, the estimate

$$|Q_n^r(t)| \leqslant 2n,$$

for all t and $n = 1, 2, \ldots$, is obvious.

Applying these inequalities, we obtain

LEMMA 6. *For any real u and $n = 1, 2, \ldots$*

(i) $$\int_{u-1}^{u+1} |Q_n^r(v-u)|\,d\psi_n(v) \leqslant C_6(r) \quad \textit{if } r > 0,$$

(ii) $$n\int_{u-1}^{u+1} |v-u|\,|Q_n^r(v-u)|\,d\psi_n(v) \leqslant C_7(r) \quad \textit{if } r > 1.$$

For example, the proof of (i) runs as follows:

$$\int_{u-1}^{u+1} |Q_n^r(v-u)|\,d\psi_n(v)$$

$$\leqslant \int_{u-1}^{u-1/n} \frac{C_5(r)}{n^r(u-v)^{r+1}}\,d\psi(v) + \int_{u-1/n}^{u+1/n} 2n\,d\psi_n(v) + \int_{u+1/n}^{u+1} \frac{C_5(r)}{n^r(v-u)^{r+1}}\,d\psi_n(v)$$

$$\leqslant \frac{2C_5(r)}{n^r}\left\{\frac{2n^{r+1}}{2n+1} + \int_{u-1}^{u-1/n} \frac{dv}{(u-v)^{r+1}}\right\} + \frac{12n+4}{2n+1} \leqslant C_6(r).$$

Now, a result of Jackson's type will be given.

THEOREM 4. *Suppose that f is continuous in $(-\infty, \infty)$. Then, for* $n = 1, 2, \ldots,$

$$\max_{-\pi\leqslant x\leqslant\pi} |S_n^r(x;f)-f(x)| \leqslant C_8(r)\,\omega\left(\frac{1}{n};f\right) \quad \textit{if } r>1.$$

Proof. Since

$$\int_{u-1}^{u+1} Q_n^r(v-u)\,d\psi_n(v) = 1,$$

we have

$$T_n^r(u;g)-g(u) = \int_{u-1}^{u+1} \{g(v)-g(u)\}Q_n^r(v-u)\,d\psi_n(v).$$

Hence

$$|T_n^r(u;g)-g(u)| \leqslant \omega\left(\frac{1}{n};g\right)\left\{n\int_{u-1}^{u+1} |v-u||Q_n^r(v-u)|\,d\psi_n(v) + \right.$$

$$\left. + \int_{u-1}^{u+1} |Q_n^r(v-u)|\,d\psi_n(v)\right\},$$

and, by Lemma 6, the conclusion follows.

Consider an arbitrary function $g(t)$ of period 2, absolutely continuous in $\langle -2, 2\rangle$. Then, partial integration gives

$$\frac{d}{du}T_n^r(u;g) = \int_{u-1}^{u+1} g'(z)\,\Phi_n^r(u,z)\,dz,$$

where

$$\Phi_n^r(u,z) = \int_{u-1}^{z} \frac{\partial}{\partial v}Q_n^r(v-u)\,d\psi_n(v) = \int_{u-3}^{z} \frac{\partial}{\partial v}Q_n^r(v-u)\,d\psi_n(v) = \ldots$$

The identity

$$\frac{d}{dt} Q_n^r(t) = \frac{1}{2\pi i} \int_{B_n-\infty i}^{B_n+\infty i} \left(1-\frac{w^2}{B_n^2}\right)^r \frac{w\sin(1-t)w}{\sin w}\, dw$$

leads to

$$\left|\frac{d}{dt} Q_n^r(t)\right| \leqslant \frac{C_9(r)}{n^{r-1}t^{r+1}}$$

for $r \geqslant 1$ and $t \in \langle 1/n, 1\rangle$. In the general case

$$\left|\frac{d}{dt} Q_n^r(t)\right| \leqslant \pi n^2.$$

Consequently,

$$(\tilde{12}) \qquad |\Phi_n^r(u, z)| \leqslant \begin{cases} \dfrac{C_{10}(r)}{n^{r-1}(u-z)^r} & \text{when } u-1 \leqslant z \leqslant u-\dfrac{1}{n}, \\ C_{10}(r)\, n & \text{when } u-\dfrac{1}{n} \leqslant z \leqslant u+\dfrac{1}{n}, \\ \dfrac{C_{10}(r)}{n^{r-1}(z-u)^r} & \text{when } u+\dfrac{1}{n} \leqslant z \leqslant u+1. \end{cases}$$

Applying estimates $(\tilde{12})$, we obtain

LEMMA 7. *For any* $u \in (-\infty, \infty)$ *and* $n = 1, 2, \ldots$

(i) $$\int_{u-1}^{u+1} |\Phi_n^r(u, z)|\, dz \leqslant C_{11}(r) \quad \textit{if } r > 1,$$

(ii) $$n \int_{u-1}^{u+1} |u-z|\, |\Phi_n^r(u, z)|\, dz \leqslant C_{12}(r) \quad \textit{if } r > 2.$$

Next, the second auxiliary result will be proved:

LEMMA 8. *For any* u *and* $n = 1, 2, \ldots,$

$$\left|\int_{u-1}^{u+1} \Phi_n^r(u, z)\, dz - 1\right| \leqslant \frac{C_{13}(r)}{n} \quad \textit{if } r \geqslant 2.$$

Proof. Taking

$$g_u(t) = \sin\pi(t-u),$$

we observe that

$$\frac{d}{dv} T_n^r(v; g_u)\big|_{v=u} = \pi \int_{u-1}^{u+1} \cos\pi(z-u)\, \Phi_n^r(u, z)\, dz$$

$$= \pi \left\{1-\frac{1}{(n+1/2)^2}\right\}^r \cos\pi(v-u)\big|_{v=u},$$

and

$$\left|\int_{u-1}^{u+1} \Phi_n^r(u,z)\,dz - \int_{u-1}^{u+1} \cos\pi(z-u)\,\Phi_n^r(u,z)\,dz\right|$$

$$\leqslant \frac{\pi^2}{2}\left(\int_{u-1}^{u-1/n} + \int_{u-1/n}^{u+1/n} + \int_{u+1/n}^{u+1}\right)(u-z)^2|\Phi_n^r(u,z)|\,dz \leqslant \frac{C_{14}(r)}{n},$$

by (12). Hence the Lemma.

THEOREM 5. *Suppose that the derivative $f'(t)$ is continuous in $(-\infty,\infty)$. Then, for $n = 1, 2, \ldots$,*

$$\max_{-\pi\leqslant x\leqslant\pi}\left|\frac{d}{dx}S_n^r(x;f) - f'(x)\right| \leqslant C_{15}(r)\,\omega\left(\frac{1}{n};f'\right) \qquad \text{if } r > 2.$$

Proof. Clearly,

$$\frac{d}{dx}S_n^r(x;f) - f'(x) = \frac{1}{\pi}\left[\frac{d}{du}T_n^r(u;g) - g'(u)\right]_{u=x/\pi},$$

and

$$\frac{d}{du}T_n^r(u;g) - g'(u) = \int_{u-1}^{u+1}\{g'(z)-g'(u)\}\Phi_n^r(u,z)\,dz +$$
$$+ g'(u)\left\{\int_{u-1}^{u+1}\Phi_n^r(u,z)\,dz - 1\right\}.$$

In view of Lemmas 7-8,

$$\left|\frac{d}{du}T_n^r(u;g) - g'(u)\right| \leqslant C_{16}(r)\left\{\omega\left(\frac{1}{n};g'\right) + \frac{1}{n}|g'(u)|\right\}.$$

Let u_0 be a point in $\langle -1, 1\rangle$ such that $g'(u_0) = 0$. Then if $|u| \leqslant 1$,

$$\frac{1}{n}|g'(u)| \leqslant \frac{1}{n}\,\omega(|u-u_0|;g') \leqslant \frac{n|u-u_0|+1}{n}\,\omega\left(\frac{1}{n};g'\right) \leqslant 3\omega\left(\frac{1}{n};g'\right).$$

Since

$$\omega\left(\frac{1}{n};g'\right) \leqslant \pi\omega\left(\frac{\pi}{n};f'\right) \leqslant \pi(\pi+1)\,\omega\left(\frac{1}{n};f'\right),$$

our result is established.

By a slight modification of the last proof, we obtain

THEOREM 6. *Let $f(t)$ satisfy the Lipschitz condition*

$$\omega(\delta;f) = O(\delta) \qquad \text{as } \delta \to 0+,$$

and let $f'(t)$ be continuous in an open interval (a, b). Then, for each positive $\varepsilon \leqslant \min\{\pi, (b-a)/2\}$, $\eta < \varepsilon$ and $n = 1, 2, \ldots$,

$$\max_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\frac{d}{dx} S_n^r(x;f)-f'(x)\right|$$
$$\leqslant C_1(r, \varepsilon-\eta, f')\,\omega\left(\frac{1}{n}; f', \langle a+\eta, b-\eta\rangle\right) \quad \text{if } r > 2.$$

In the case $r > 1$, an analogue of Theorem 1 is true. This result can easily be extended to derivatives of higher order. For example, we have

THEOREM 7. *Suppose that $|f(t)|$ has the majorant $f^*(t)$ as in Theorem 1, and that $r > 2$. Then,*

$$\lim_{n\to\infty}\frac{d^2}{dz^2}S_n^r(z;f)\big|_{z=x} = f''(x) \tag{13}$$

for all these x at which $|f''(x)| < \infty$. The convergence (13) is uniform on $\langle a, b\rangle$ when the derivative $f''(x)$ is continuous at every point of this interval.

Proof. Set

$$F_x^2(t) = f(t)-f(x)-f'(x)\sin(t-x)-\tfrac{1}{2}f''(x)\sin^2(t-x),$$
$$\eta_x(t) = F_x^2(t)/\sin^2(t-x).$$

The assumption $|f''(x)| < \infty$ implies

$$\lim_{t\to x}\eta_x(t) = 0.$$

The last relation holds uniformly in $x \in \langle a, b\rangle$ if $f''(x)$ is continuous at every $x\in\langle a, b\rangle$. Since

$$\frac{d^2}{dz^2}[S_n^r(z;f)-S_n^r(z;F_x^2)]_{z=x} = \left\{1-\frac{4}{(n+1/2)^2}\right\}^r f''(x),$$

it is sufficient to prove that

$$\lim_{n\to\infty}\frac{d^2}{dz^2}S_n^r(z;F_x^2)\big|_{z=x} = 0.$$

Evidently,

$$\frac{d^2}{dz^2}S_n^r(z;F_x^2)\big|_{z=x} = \frac{1}{\pi^2}\int_{u-1}^{u+1}F_x^2(\pi v)\frac{\partial^2}{\partial v^2}Q_n^r(v-u)\,d\psi_n(v),$$

where $u = x/\pi$. By a simple calculation,

$$\left|\frac{d^2}{dt^2}Q_n^r(t)\right| \leqslant \begin{cases} C_{17}(r)/(n^{r-2}|t|^{r+1}) & \text{when } 1/n\leqslant |t|\leqslant 1,\\ \pi^2 n^3 & \text{always}.\end{cases}$$

Choose, for a given $\varepsilon > 0$, a positive $\delta = \delta(\varepsilon)$ such that

$$|F_x^2(\pi v)| < \varepsilon(\pi v - x)^2 \quad \text{if } |\pi v - x| \leqslant \delta,$$

and write

$$\frac{d^2}{dz^2} S_n^r(z; F_x^2)/_{z=x} = \frac{1}{\pi^2}\left(\int_{u-\delta}^{u+\delta} + \int_{u-1}^{u-\delta} + \int_{u+\delta}^{u+1}\right) F_x^2(\pi v)\frac{\partial^2}{\partial v^2} Q_n^r(v-u)\, d\psi_n(v).$$

Reasoning as in the proof of Theorem 1, we get the desired assertion.

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