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A space whose regions are the simple domains of another space*

1. Introduction. In Theorem 6, Chapter VII, p. 356 of [1], Professor R. L. Moore gives a sequence of axioms which characterize the plane. So many theorems about the plane are proved in Chapter III and IV of [1], using only his Axioms 0-5 (stated below), that a space satisfying these axioms might at first be thought to be very similar to the plane. Example 1 at the end of this paper shows, however, that there is a space satisfying these axioms and containing a point P and a region R that contains P, such that no domain containing P *whose boundary is a simple closed curve* is a subset of R.

In the case of Example 1, a simple modification of the regions in the space will yield the plane: If Σ denotes the space in Example I, and Σ' denotes the space whose points are the points of Σ, and whose regions are the complementary domains of simple closed curves (i.e., the simple domains of the space Σ), then Σ' is the plane. In this paper we find, for a space Σ satisfying Axioms 0-5, which of these axioms are also satisfied by the space Σ', defined in terms of Σ as in the last sentence. It is shown in Theorem 19 that Axioms 2-5 are satisfied by every such space. However, an example (Example 2) exists of a space Σ satisfying Axioms 0-5 such that Σ' does not satisfy Axiom 1_3 (stated below), but a condition is given in Theorem 26 which is both necessary and sufficient in order that Σ' satisfy Axiom 1_3.

The axioms and certain definitions of Moore [1] used in this paper are the following. The letter S denotes the set to which X belongs if and only if X is a point.

AXIOM 0. *If R is a region, R is a point set.*

AXIOM 1_3. *There exists a sequence $G_1, G_2, G_3, \ldots$ such that*

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1. *for each positive integer n, G_n is a collection such that each element of G_n is a region and G_n covers S;*

2. *for each positive integer n, G_{n+1} is a subcollection of G_n;*

3. *if R is a region and A is a point of R and B is a point of R, there exists a positive integer n such that if g is a region of G_n containing A, then $\bar{g}$ is a subset of R and it does not contain B, unless B is A.*

AXIOM 1. *To Axiom 1_3 add:*

4. *If $M_1, M_2, M_3, \ldots$ is an infinite sequence of closed point sets such that for each n, M_n contains M_{n+1} and M_n is a subset of the closure of some region of G_n, the point sets of the sequence $M_1, M_2, M_3, \ldots$ have a point in common.*

AXIOM $1_3'$. *In Axiom 1_3, substitute for part 3:*

3'. *If R is a region and A is a point of R and B is a point of R, there exists a positive integer n such that if X is a region of G_n containing A, and Y is a region of G_n intersecting X, then $\overline{Y}$ is a subset of R and it does not contain B unless B is A.*

AXIOM 2. *If P is a point of a region R, there exists a non-degenerate connected domain containing P and lying in R.*

AXIOM 3. *If P is a point, $S-P$ is connected and non-degenerate.*

AXIOM 4. *If J is a simple closed curve, $S-J$ is the sum of two mutually separated connected point sets, each having J as its boundary.*

AXIOM 5. *If A and B are two points, every region that contains A contains a compact continuum that separates A from B* (indeed, by Theorem 5, Chapter IV, p. 162 of [1], on the basis of this and the preceding axioms, every region that contains A contains a simple closed curve that separates A from B).

The statement that the space Σ satisfies Axioms 0-5 will mean that Σ satisfies Axioms 0, 1, 2, 3, 4, and 5 (but not Axiom $1_3'$ unless explicitly stated).

Suppose w is a point.

DEFINITION. The point set M is said to be *bounded with respect to w* if and only if w is neither a point nor a limit point of M.

DEFINITION. If J is a bounded simple closed curve, then the complementary domain of J that contains w will be called the *exterior* of J with respect to w, and the complementary domain of J that does not contain w will be called the *interior* of J with respect to w.

If in a theorem some point has been denoted by the letter w, then, unless otherwise noted, *bounded* will mean bounded with respect to w, *exterior* of the simple closed curve J will mean exterior with respect to w, etc. The notation $I(J)$ and $E(J)$ will denote the interior and exterior, respectively, of the simple closed curve J (with respect to w).

DEFINITION. A ***simple domain*** is a domain whose boundary is a simple closed curve. A ***simple disk*** is a simple domain plus its boundary.

For definitions of other terms used in this paper, see [1].

2. Theorems.

DEFINITION. If the space Σ satisfies Axioms 0-5, then Σ' denotes the space obtained by interpreting *point* to mean *point of Σ*, and *region* to mean *simple domain in Σ*.

Throughout the following treatment it will be assumed that the space Σ satisfies Axioms 0-5.

The definitions of all concepts and properties in the space Σ apply without modification to the space Σ', with the following two exceptions: 1. An arc in Σ' is a *perfectly compact* continuum with only two non-cut-points. 2. A simple closed curve in Σ' is a *perfectly compact* continuum which is separated by the omission of any two of its points. Defined in this manner, the arcs and simple closed curves of the space Σ' satisfy most of the relevant theorems which are results of Axiom 1, e.g., Theorems 95-103, 107, and 108 of Chapter I, [1]. If a is an arc with endpoints A and B, then $S(a)$ denotes $a-(A+B)$.

THEOREM 1. *If the point P is a limit point of the point set M in the space Σ, then P is a limit point of M in the space Σ'.*

Proof. Suppose D is a region in Σ' containing P. Then D is a simple domain in Σ. Therefore, there exists a region R in Σ containing P and lying in D. Now since R contains a point of M distinct from P, D must also. Thus P is a limit point of M in Σ'.

Theorems 2-7 are direct consequences of Theorem 1.

THEOREM 2. *Every domain in the space Σ' is a domain in the space Σ.*

THEOREM 3. *Every closed point set in Σ' is closed in Σ.*

THEOREM 4. *If M is a point set, then the closure of M in Σ is a subset of the closure of M in Σ', and the boundary of M in Σ is a subset of the boundary of M in Σ'.*

THEOREM 5. *Every two mutually separated point sets in Σ' are mutually separated in Σ.*

THEOREM 6. *Every connected point set in Σ is connected in Σ'.*

THEOREM 7. *Every compact point set in Σ is perfectly compact in Σ'.* (See Theorem 37 of Chapter I, [1].)

THEOREM 8. *If M is a point set connected either in Σ or in Σ', and the point P is a limit point of M in Σ', then P is a limit point of M in Σ. Moreover, M is closed in Σ' if and only if M is closed in Σ.*

Proof. It follows from Theorem 6 that if M is connected in Σ, then M is connected in Σ'. Therefore, we need suppose only that M is connected in Σ'. Suppose P is a limit point of M in Σ' and R is a region in Σ that contains P. Then by Theorem 5 of Chapter IV, [1], R contains a simple closed curve J that separates P from some point of M. Now, the two complementary domains of J are regions in Σ', and since M is connected in Σ' and intersects both of them, M must intersect J in a point distinct from P. Thus R contains a point of M distinct from P, and P is a limit point of M in Σ.

If M is closed in Σ, then M is also closed in Σ', since, by this theorem, every limit point of M in Σ' is also a limit point of M in Σ, and therefore must belong to M. The converse is true by Theorem 3.

THEOREM 9. *If M is a point set connected either in Σ or in Σ', then the closure of M in the space Σ is the closure of M in the space Σ'.*

This is a direct consequence of Theorems 1 and 8.

THEOREM 10. *If P is a point, there exists a sequence of regions in the space Σ' closing down on P.*

Proof. Let w denote a point distinct from P.

By Theorem 7 of Chapter I, [1], there is a sequence $R_1', R_2', R_3', \ldots$ of regions in Σ closing down on P. If n is a positive integer, there is a simple closed curve J lying in R and separating P from w, and there is a positive integer $i > n$ such that R_i lies in the interior of J with respect to w. Note that, by Theorem 3, Chapter III, p. 142 of [1], the interior of any simple closed curve lying in R_i is a subset of the interior of J. So there is a subsequence $R_1, R_2, R_3, \ldots$ of $R_1', R_2', R_3', \ldots$ and a sequence $D_1, D_2, D_3, \ldots$ of bounded simple domains in Σ such that for each positive integer n, D_n contains D_{n+1} and P, and the boundary of D_n lies in R_n.

Now suppose P' is a point distinct from P and D' is a region in Σ' containing P. By Theorem 12 of Chapter IV, [1], there exists a simple closed curve J in Σ that separates P from P', w, and the boundary of D'. It follows that $I(J)$ lies in D' since $I(J)$ contains the point P of D', but no point of the boundary of D'. There exists a positive integer i such that R_i lies in $I(J)$. Therefore, the boundary of D_i lies in $I(J)$, and by Theorem 3 of Chapter III, [1], D_i lies in $I(J)$. Hence D_i lies in D' and does not contain P'.

Therefore, the sequence $D_1, D_2, D_3, \ldots$ closes down on P in the space Σ'.

THEOREM 11. *If D is a connected domain in the space Σ', then the boundary of D in Σ' is the boundary of D in Σ.*

Proof. By Theorem 2, D is also a domain in Σ. Since the boundary of a domain is the set of all points of the closure of the domain not in the domain, and by Theorem 9 the closure of D in Σ' is the closure of D in Σ, it follows that the boundary of D in Σ' is the boundary of D in Σ.

THEOREM 12. *If D is a connected domain in the space Σ', then D is a connected domain in Σ.*

Proof. By Theorem 2, D is a domain in Σ. Suppose D is the sum of two point sets, mutually separated in Σ, but such that one of them, H, contains a limit point P in Σ' of the other, K. There exists a region R in Σ' (i.e., a simple domain in Σ), containing P and lying wholly in D. Since P is a limit point in Σ' of K, R contains a point of K. That is, R is a subset of D and therefore of $H+K$, which is connected in Σ, but which intersects both H and K. But this contradicts the assumption that H and K are mutually separated in Σ.

THEOREM 13. *If the point set M is closed in the space Σ', then D is a complementary domain of M in the space Σ if and only if D is a complementary domain of M in Σ'.*

Proof. By Theorem 3, M is closed in Σ. Every region in Σ' is a simple domain in Σ and is therefore connected in Σ', by Theorem 6. Hence Σ' is locally connected, and every component of $S-M$ in Σ' is a domain in Σ'. Also, the boundary in Σ' of a component of $S-M$ is a subset of M.

Suppose D is a complementary domain of M in Σ'. By Theorem 12, D is a connected domain in Σ, so D is a subset of a complementary domain D' of M in Σ. But since D' is connected in Σ, and contains no point of the boundary of D in Σ (which lies in M, by the above observation and Theorem 11), D' is also a subset of D. So D is D'.

Suppose D is a complementary domain of M in Σ. Then D is connected in Σ' by Theorem 6. So D is a subset of a complementary domain D' of M in Σ'. But with respect to Σ, D' is a connected domain that contains no point of the boundary of D, which lies in M. So D is D'.

THEOREM 14. *If the point set M is connected, either in Σ or in Σ', and connected im kleinen in Σ at the point P, then it is connected im kleinen in Σ' at P.*

Proof. If M is connected in Σ, then, by Theorem 6, M is connected in Σ'. Suppose M is connected in Σ' and connected *im kleinen* in Σ at the point P.

Suppose D is a region in Σ' that contains P. Let R denote a region in Σ that contains P and lies in D. Since M is connected *im kleinen* in Σ at P, there exists a connected region R' in Σ containing P such that $\overline{R'}\cdot M$ is a subset of a component K of $R\cdot M$. (Note: Since R' is connected in Σ, $\overline{R'}$ denotes the closure of R' both in Σ and in Σ'; cf. Theorem 9.) There exists a region D' in Σ' containing P and lying in D whose boundary lies in R'.

Now suppose Q is a point of $M \cdot D'$. If Q lies in $\overline{R'}$, then Q lies together with P in the connected subset K of $D \cdot M$. If Q does not lie in $\overline{R}'$, then it lies in a complementary domain D_Q of $\overline{R}'$, which lies in D. (Note: By Theorem 13, D_Q is a complementary domain of $\overline{R}'$ both in Σ and in Σ'.) By Theorem 11, the boundary b of D_Q in Σ' is also the boundary of D_Q in Σ. Now, $b \cdot M$ is a subset of $\overline{R}'$ and consequently of K. Therefore, $K+M \cdot D_Q$ is connected in Σ', for if $K+M \cdot D_Q$ is the sum of two point sets, mutually separated in Σ', where H_1 denotes the one that contains K, and H_2 denotes the other, then $M = H_2 + [H_1 + (M - M \cdot \overline{D}_Q)]$. The point sets H_2 and $H_1 + (M - M \cdot \overline{D}_Q)$ are mutually separated in Σ', since (1) we are assuming that H_1 and H_2 are, and (2) H_2 and $M - M \cdot \overline{D}_Q$ are, because H_2 lies in D_Q, and $M - M \cdot \overline{D}_Q$ lies in $S - \overline{D}_Q$, two mutually separated point sets in Σ'. This contradicts the fact that M is connected in Σ'. Therefore, Q and P lie together in the connected subset $K+M \cdot D_Q$ of D.

Thus every point of $M \cdot D'$ lies, together with P, in some subset of $M \cdot D$ which is connected in Σ', and M is connected *im kleinen* at P in the space Σ'.

THEOREM 15. *Every arc in the space Σ is an arc in the space Σ'.*

Proof. Suppose AB is an arc in Σ. AB is connected in Σ' by Theorem 6, perfectly compact in Σ' by Theorem 7, and closed in Σ' as a result of Theorem 8. Neither A nor B is a cutpoint of AB in Σ', since the point sets $AB - A$ and $AB - B$ are connected in Σ and therefore in Σ'.

Suppose P is a point of $S(AB) = AB - (A+B)$. No point of either of the sets $AP - P$ and $PB - P$ is a limit point of the other set in Σ. Therefore, by Theorem 8, since $AP - P$ and $PB - P$ are connected in Σ, no point of one of them can be a limit point of the other in Σ'. Consequently, they are mutually separated in Σ', and P is a cutpoint of AB in Σ'.

THEOREM 16. *Every simple closed curve in Σ is a simple closed curve in Σ'.*

Proof. Suppose J is a simple closed curve in Σ. By an argument similar to the one in Theorem 15, J is a perfectly compact continuum in Σ'.

Suppose P and Q are two points of J. Then J is the sum of two arcs in Σ, PWQ and PZQ, having only the points P and Q in common. It follows from Theorem 8 that $S(PWQ)$ and $S(PZQ)$ are mutually separated not only in Σ but also in Σ', and therefore the omission of the points P and Q separates J.

THEOREM 17. *Every simple closed curve J in Σ separates S (the set of all points of Σ) into only two mutually exclusive connected domains with respect to Σ', each having J as its boundary in Σ'.*

This theorem follows from Theorems 8, 13, and 11.

The following two definitions are taken from [2]:

DEFINITION. If J is a simple closed curve, the componant I of $S-J$ is *flat* if and only if it is true that if α and α' are two arcs lying, except for their end points, in I, and the end points of α separate those of α' on J, then α and α' intersect.

DEFINITION. A *primitive* simple closed curve is a simple closed curve J that separates space into two connected sets each of which is a flat componant of $S-J$.

THEOREM 18. *Every simple closed curve in Σ is a primitive simple closed curve in Σ'.*

Proof. Suppose J is a simple closed curve in Σ. By Theorem 17, J has only two complementary domains with respect to Σ'. To show that J is primitive, we need only show that each of these is flat, so suppose D is a complementary domain of J, and ABC and HEF are two arcs in Σ' lying, except for their endpoints, in D, whose endpoints are in the order $AHCFA$ on J.

Suppose ABC and HEF do not intersect. By Theorem 8, HEF is closed with respect to Σ. With the aid of Theorem 4 of Chapter I, [1], the arc ABC may be covered by a finite collection G of regions in Σ', each bounded by a simple closed curve in Σ, and such that (1) none of these regions or their boundaries intersects HEF and (2) one region contains A, another C, but except for these two, no region of G or its boundary intersects J. There exists an arc WYZ in Σ, lying in the sum of the boundaries of the elements of G, such that W lies on the segment $S(HAF)$ of J, and Z lies on the segment $S(HCF)$ of J, and such that WYZ lies except for its endpoints in D.

Now, let w denote a point not in $D+J$. Then the segment $S(HEF)$ in Σ' lies in $I(WYZHW)+I(WYZFW)$ since, not intersecting the boundary of any element of G, it cannot intersect WYZ. But $I(WYZHW)$ and $I(WYZFW)$ are regions in Σ', which implies that $S(HEF)$ is not connected in Σ'. We have thus come to a contradiction.

THEOREM 19. *The space Σ' satisfies Axioms* 2, 3, 4, *and* 5.

Proof. Σ' satisfies Axiom 2, since every region in Σ' is a simple domain in Σ and is therefore connected in Σ', by Theorem 6. Axiom 3 is satisfied, since if P is a point, $S-P$ is non-degenerate and connected in Σ, and is therefore connected in Σ', again by Theorem 6. Since every region in Σ' is a simple domain in Σ, Axiom 5 also holds in Σ'. This leaves Axiom 4.

Slye has shown [2] that any space satisfying the following conditions (slightly reworded) satisfies Axiom 4:

1. There exists a region, and each region is a non-degenerate *arcwise* connected point set.

2. If P and Q are two points, R is a region containing P, and R' is a region containing P, then there is a region containing P but not Q, and lying in both R and R'.

3. If the point P is distinct from the point Q, and from the point Q', and R is a region containing P, then there exists a primitive simple closed curve in R that separates P and $Q+Q'$.

Now, a region in Σ' is a simple domain in Σ, so every two points of it are the endpoints of an arc in Σ, which is also an arc in Σ' by Theorem 15. Hence, Σ' satisfies Condition 1. That Σ' satisfies Condition 2 is a direct consequence of Theorem 10. In the case of Condition 3, suppose that P is distinct from the point Q and from the point Q', and that R is a region in Σ' containing P. Then, again as a result of Theorem 10, there is a region R_1 in Σ' such that $\overline{R_1}$ lies in R and contains neither Q nor Q'. So the boundary, J, of R_1 is a simple closed curve in Σ, lying in R, that separates P from $Q+Q'$, and by Theorem 18, J is a primitive simple closed curve in Σ'.

THEOREM 20. *The space Σ' is separable if and only if the space Σ is separable.*

Proof. If Σ is separable, there exists a countable point set K such that every point of S is a limit point of K in Σ. But then by Theorem 1, every point of S is a limit point of K in Σ', and therefore Σ' is also separable.

Suppose Σ' is separable but Σ is not. Then there exists a countable point set K such that every point of S is a limit point of K in Σ'. By Theorem 164 of Chapter I, [1], there exists in Σ an uncountable collection H of mutually exclusive regions. Let w denote a point, and for each region h with respect to Σ in H, let D_h denote the interior with respect to w of some bounded simple closed curve J_h that lies in h. Let G denote the collection of all domains D_h for elements h of H. If h_1 and h_2 are two elements of H, then h_1 and h_2 are mutually exclusive, and thus J_{h_1} and J_{h_2} are also mutually exclusive; therefore, D_{h_1} and D_{h_2} are distinct. Hence G is also uncountable.

Now, each element g of G contains a point P_g of K, and therefore, since K is countable, some point P of K lies in uncountably many elements of G. That is, there exists an uncountable subcollection H' of H such that if h belongs to H', then J_h separates w from P.

There exists an arc in Σ from P to w. The arc Pw must intersect every one of the regions in Σ of the collection H', and it therefore intersects uncountably many mutually exclusive regions in Σ. But this is a contradiction, since Pw is an arc in Σ and does not contain uncountably many mutually exclusive segments.

Therefore, Σ must also be separable.

In Theorems 21 to 23, only the space Σ is considered, and all concepts and properties are to be considered as defined with respect to that space.

THEOREM 21. *Suppose w is a point and $I_1, I_2, I_3, \ldots$ is a sequence of bounded simple domains such that for each positive integer n, the boundary of I_n lies in a region R_n of G_n that does not contain w, where $G_1, G_2, G_3, \ldots$ is a sequence of collections of regions satisfying Axiom 1_3. Suppose, moreover, that $I_1, I_2, I_3, \ldots$ have a point P in common. Then* (1) *no point other than P is common to all the regions of the sequence $R_1, R_2, R_3, \ldots$, and* (2) *if the domains $I_1, I_2, I_3, \ldots$ have a point Q distinct from P in common, then $R_1, R_2, R_3, \ldots$ have no point in common.*

Proof. Suppose some point X distinct from P is common to all the regions $R_1, R_2, R_3, \ldots$ There exists an arc from P to w which does not contain X. Since for each n, the boundary of I_n lies in R_n and separates P from w, R_n must intersect the arc Pw. But X does not lie on Pw, and therefore there exists a region R containing X but no point of Pw. Furthermore, there exists a positive integer n such that if g is a region of G_n that contains X, then $\bar{g}$ lies in R. Thus R_n lies in R, and cannot intersect the arc Pw. But this is a contradiction, since it was shown above that R_n does intersect Pw.

Therefore, no point other than P can be common to the regions $R_1, R_2, R_3, \ldots$

Now if the domains $I_1, I_2, I_3, \ldots$ have two points, P and Q, in common, then, as was just shown, no point other than P can be common to $R_1, R_2, R_3, \ldots$ But P cannot be common to these regions either, since no point other than Q can be common to all of them. Thus, the regions $R_1, R_2, R_3, \ldots$ can have no point whatsoever in common.

THEOREM 22. *Suppose the space Σ satisfies Axiom $1_3'$ (in addition, of course, to Axioms 0-5); w, w', and A are three points; and $J_1, J_2, J_3, \ldots$ is a sequence of simple closed curves such that for each n,* (1), *A lies in the interior, I_n, of J_n with respect to both w and w', and* (2), *J_n lies in some region of G_n, where $G_1, G_2, G_3, \ldots$ is a sequence of collections of regions satisfying Axiom $1_3'$. Then the point A is the only point common to I_1+J_1, I_2+J_2, $I_3+J_3, \ldots$*

Proof. Suppose some point B distinct from A also lies in I_1+J_1, I_2+J_2, $I_3+J_3, \ldots$ By Axiom 3 and Theorem 1, Chapter II, p. 84 of [1], there exist arcs from B to A and from B to w', neither of which contains w. The sum, M, of the arcs BA and Bw' is a compact continuum intersecting each of the simple closed curves $J_1, J_2, J_3, \ldots$ Therefore, M contains a point Z (distinct from w) and a sequence $P_1, P_2, P_3, \ldots$ such that (1) for each positive integer n, P_n is a point of $M \cdot J_n$, and (2) some infinite subsequence of $P_1, P_2, P_3, \ldots$ converges to Z (see Theorem 11, Chapter I, p. 5 of [1]). Let C denote one of the points A and B which is distinct from Z. There exists a simple closed curve J that separates Z from both C and w. There exists a region R containing Z whose closure lies in the interior of J, and, since the sequence $G_1, G_2, G_3, \ldots$ satisfies Axiom $1_3'$, there is a positive integer N such that if X and Y are intersecting regions of G_N, one of which contains Z, then $\bar{X}+\bar{Y}$ is a subset of R. Let R' denote a region of G_N that contains Z. Then, since some subsequence of $P_1, P_2, P_3, \ldots$ converges to Z, there exists a positive integer n greater than N such that P_n lies in R'. Since J_n lies in a region of G_n and intersects R', J_n must lie in R and therefore in $I(J)$. But the point C lies without J and therefore without J_n, contrary to the fact that C is one of the points A and B, each of which is common to $I_1+J_1, I_2+J_2, I_3+J_3, \ldots$

Therefore, A is the only point common to $I_1+J_1, I_2+J_2, I_3+J_3, \ldots$

THEOREM 23. *Suppose the space Σ satisfies Axiom $1_3'$, w and w' are points, H and K are mutually exclusive compact continua containing neither w nor w', and $G_1, G_2, G_3, \ldots$ is a sequence of collections of regions that satisfies Axiom $1_3'$. Then there exists a positive integer n such that if $\bar{D}$ is a simple disk containing neither w nor w' but intersecting H, and the boundary of D lies in some region of G_n, then D does not intersect K.*

Proof. Case 1. *The theorem is true if both H and K are non-degenerate.* For, suppose it is false. Then for each positive integer n there exists a simple disk $\overline{D_n}$ whose boundary lies in a region R_n of G_n, and such that $\overline{D_n}$ intersects both H and K, but contains neither w nor w'.

There exists a positive integer n_1 such that if n is an integer greater than n_1, the boundary, b_n, of D_n intersects both H and K; for otherwise there exists an increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers such that for each i, b_{n_i} does not intersect L, where L is one of the point sets H and K. Hence L is non-degenerate. For each i, let J_i denote b_{n_i}. Then, for each i, $I(J_i) = D_{n_i}$, and since L is connected and intersects $I(J_i)$ but not J_i, L must lie wholly in $I(J_i)$. So $J_1, J_2, J_3, \ldots$ is a sequence of simple closed curves such that for each n, J_n lies in a region of G_n, but $I(J_1), I(J_2), I(J_3), \ldots$ contains the non-degenerate point set L, contrary to the result of Theorem 22.

There exists a sequence $P_1, P_2, P_3, \ldots$ and a point Z of K such that (1) for each n greater than n_1, P_n is a point of b_n lying in K, and (2) some subsequence of $P_1, P_2, P_3, \ldots$ converges to Z (see Theorem 11, Chapter I, p. 5 of [1]). There exists a simple closed curve C separating Z from both H and w (by Theorem 12, Chapter IV, p. 169 of [1]) and a region R containing Z and lying in the interior of C with respect to w. Since the sequence $G_1, G_2, G_3, \ldots$ satisfies Axiom $1_3'$, there exists a positive integer n_2 such that if X and Y are intersecting regions of G_{n_2}, one of which contains Z, then $\bar{X}+\bar{Y}$ lies in R. Let R' denote a region of G_{n_2} that contains Z, and n' denote a positive integer greater than both n_1 and n_2 such that $P_{n'}$ lies in R'. Then $R_{n'}$ intersects R' and so $\bar{R}_{n'}$ lies in R, and since $b_{n'}$ lies in $R_{n'}$, $b_{n'}$ also lies in R, and therefore in $I(C)$. But then, by Theorem 3, Chapter III, p. 142 of [1], $\overline{D_{n'}}$ lies in $I(C)$ and cannot intersect H, contrary to the original assumption. This completes the proof of Case 1.

Case 2. *The theorem is true if K is degenerate, but H is not.* By Theorem 4, Chapter IV, p. 162 of [1], there is a simple closed curve J separating H from K. Since the continua H and J are both non-degenerate, we may apply the result of Case 1 to them. Hence, there exists a positive integer n such that if $\bar{D}$ is a simple disk intersecting H but containing neither w nor w', and the boundary of D lies in a region of G_n, then $\bar{D}$ does not intersect J. But since $\bar{D}$ is connected, $\bar{D}$ cannot intersect K without intersecting J, so Case 2 is proven.

Case 3. *The theorem is true if H is degenerate but K is not.* The proof is similar to that for Case 2.

Case 4. *The theorem is true if both H and K are degenerate.* Again, there is a simple closed curve separating H from K. Since H is degenerate, but J is not, we may apply the result of Case 3 to them. Hence, there exists a positive integer n such that if $\bar{D}$ is a simple disk intersecting H but containing neither w nor w', and the boundary of D lies in a region of G_n, then $\bar{D}$ does not intersect J, and consequently cannot intersect K. So Case 4 is proven, and this completes the proof of Theorem 23.

THEOREM 24. *If the space Σ satisfies Axiom $1_3'$, then so does the space Σ'.*

Proof. Let w and w' denote points; $C_1, C_2, C_3, \ldots$ a sequence of simple domains in Σ, such that for each n, C_n contains $\overline{C_{n+1}}$, and such that w is the only point common to all the elements of the sequence; and $C_1', C_2', C_3', \ldots$ a sequence defined similarly with respect to w' such that C_1 and C_1' are mutually exclusive. Let $G_1, G_2, G_3, \ldots$ denote a sequence of collections of regions in Σ satisfying Axiom $1_3'$. For each n, let Q_n denote the collection to which D belongs if and only if (1) D is one of the domains $C_n, C_n', C_{n+1}, C_{n+1}', \ldots$, or (2) D is a simple domain in Σ such that its boundary, J, lies in some region of G_n, and $D+J$ contains neither w nor w'. Then $Q_1, Q_2, Q_3, \ldots$ is a sequence of collections of regions in Σ'.

Parts 1 and 2 of Axiom $1_3'$ are clearly satisfied by the sequence Q_1, $Q_2, Q_3, \ldots$ Suppose D is a region in Σ' (i.e., a simple domain in Σ) and A and B are points of D. If A is either w or w' (say w), then there exists a positive integer n such that C_n lies in D and does not contain B. By

Theorem 23, there exists a positive integer n' greater than $n+1$ such that if $\bar{D}$ is a simple disk in Σ, containing neither w nor w', but whose boundary lies in some region of $G_{n'}$ (i.e., if D is a domain of $Q_{n'}$ other than $C_{n'}, C'_{n'}, C_{n'+1}, C'_{n'+1}, \ldots$), then $\bar{D}$ does not intersect the boundaries of both C_n and C_{n+1}. It follows that no region of Q'_n intersects the boundaries of both C_n and C_{n+1}, since $C_{n'}, C_{n'+1}, C_{n'+2}, \ldots$ all lie in C_{n+1}, and $C'_{n'}, C'_{n'+1}, C'_{n'+2}, \ldots$ all fail to intersect C_n. Now, if X and Y are intersecting domains of $Q_{n'}$ such that X contains A (i.e., w), then (1) X must be one of the domains $C_{n'}, C_{n'+1}, C_{n'+2}, \ldots$, since no other domains in $Q_{n'}$ contain w, and hence $\bar{X}$ lies entirely in C_{n+1}, and (2) since $\bar{Y}$ cannot intersect both X and the boundary of C_n without also intersecting the boundary of C_{n+1}, $\bar{Y}$ lies entirely in C_n. Thus, $\bar{X}+\bar{Y}$ lies in C_n, and therefore in D, and does not contain B.

Suppose A is neither w nor w'. There exist simple closed curves J and J' in Σ such that J lies in D and separates A from B, w, and w', and such that J' separates A from J. By Theorem 23, there exists a positive integer n such that (1) no simple disk in Σ which contains neither w nor w', but whose boundary lies in a region of G_n, intersects two of the sets A, J, and J', and (2) $C_n+C'_n$ does not intersect J. Now, suppose X and Y are two intersecting domains in Q_n such that X contains A. X is not one of the domains $C_1, C'_1, C_2, C'_2, \ldots$, since Q_n does not contain any of the domains $C_1, C'_1, C_2, C'_2, \ldots, C_{n-1}, C'_{n-1}$, and none of the domains $C_n, C'_n, C_{n+1}, C'_{n+1}, \ldots$ intersects J (which X would have to do if it contained w or w' as well as A). Thus, by the definition of the sequence $Q_1, Q_2, Q_3, \ldots$, $\bar{X}$ must be a simple disk, containing neither w nor w', whose boundary lies in some region of G_n. $\bar{X}$, then, cannot intersect J', and therefore lies entirely within J'. Since Y intersects X and therefore $I(J)$, Y cannot be one of the domains $C_1, C'_1, C_2, C'_2, \ldots$, by an argument similar to that given above for X. Therefore, $\bar{Y}$ is also a simple disk, containing neither w nor w', whose boundary lies in a region of G_n, and $\bar{Y}$ cannot intersect both J and J'. Since Y intersects $I(J')$ and cannot intersect J without also intersecting J', $\bar{Y}$ must lie wholly within J. Hence $\bar{X}+\bar{Y}$ lies in $I(J)$, and therefore in D, and does not contain B.

Thus the sequence $Q_1, Q_2, Q_3, \ldots$ satisfies Axiom $1'_3$.

DEFINITION. Let w and X denote two definite points. The sequence $Q_1, Q_2, Q_3, \ldots$ of collections of regions in Σ' is said to be *derived from the sequence* $G_1, G_2, G_3, \ldots$ of collections of regions in Σ if and only if, for each positive integer n, Q_n is the collection of simple domains in Σ to which D belongs if and only if either (1) D does not contain X and is the *exterior* with respect to w of some simple closed curve that lies, together with w, in some region of G_n, or (2) D is the *interior* with respect to w of some simple closed curve that lies in some region of G_n that does not contain w.

If Σ' satisfies Axiom 1, there is a sequence $Q_1, Q_2, Q_3, \ldots$ satisfying the requirements of that Axiom with respect to Σ'. In Theorem 25 we show that it is possible to select the sequence $Q_1, Q_2, Q_3, \ldots$ so that it is derived from a certain sequence $G_1, G_2, G_3, \ldots$ satisfying Axiom 1 with respect to Σ.

THEOREM 25. *Let w and X denote two definite points. If the space Σ' satisfies Axiom 1, then there is a sequence $G_1, G_2, G_3, \ldots$ of collections of regions in Σ satisfying Axiom 1 with respect to Σ, such that the sequence $Q_1, Q_2, Q_3, \ldots$ of collections of regions in Σ' derived from $G_1, G_2, G_3, \ldots$ satisfies Axiom 1 with respect to Σ'.*

Proof. Since Σ satisfies Axiom 1, there exists a sequence $G_1', G_2', G_3', \ldots$ of collections of regions in Σ satisfying the requirements of Axiom 1. By hypothesis, there exists a sequence $Q_1', Q_2', Q_3', \ldots$ of collections of simple domains in Σ satisfying the requirements of that axiom with respect to the space Σ'. Let $R_1, R_2, R_3, \ldots$ denote a sequence of regions in Σ closing down on w, such that for each n, R_n lies in some simple domain of Q_n'. For each n, let G_n denote the collection of regions in Σ to which R belongs if and only if (1) R is a member of the sequence R_n, $R_{n+1}, \ldots$, or (2) for some bounded simple domain q' of Q_n', R is a region of G_n' lying in q'. The sequence $G_1, G_2, G_3, \ldots$ satisfies Axiom 1 with respect to Σ.

Let $Q_1, Q_2, Q_3, \ldots$ denote the sequence of collections of regions in Σ' derived from the sequence $G_1, G_2, G_3, \ldots$ of regions in Σ. The sequence $Q_1, Q_2, Q_3, \ldots$ clearly satisfies the first two parts of Axiom 1. Suppose D is a simple domain, and A and B are points of D. If A is w, there exists a simple domain D' in Σ containing A and lying in D but not containing B. There exists a positive integer n such that R_n lies in D'. Suppose q is a domain of Q_n that contains w. Then the boundary of q must lie in R_n, since (1), R_n, $R_{n+1}, R_{n+2}, \ldots$ are the only regions of G_n that contain w, and (2) if the boundary of q does not lie in some region of G_n that contains w, then q itself cannot contain w, by the definition of the sequence $Q_1, Q_2, Q_3, \ldots$ Since R_n lies in D', the boundary of q lies in D'; thus $\bar{q}$ lies in D' and therefore in D, and does not contain B.

If A is distinct from w, there exists a simple domain D' in Σ lying in D, and containing A, but neither B nor w. Since the sequence Q_1', $Q_2', Q_3', \ldots$ satisfies Axiom 1, there exists a positive integer n such that if q' is a domain of Q_n', then $\overline{q'}$ lies in D' if q' contains A, and $\overline{q'}$ lies in $S-\overline{D'}$ but does not contain X if q' contains w. Now, suppose q is a domain of Q_n that contains A.

If q contains w, then by the definition of the sequence $Q_1, Q_2, Q_3, \ldots$, (1), q is the exterior with respect to w of a simple closed curve J in Σ lying in a region R of G_n which also contains w, and (2). q does not contain X.

But every region of G_n that contains w (i.e. $R_n, R_{n+1}, R_{n+2}, \ldots$) is a subset of R_n, and R_n itself lies in some simple domain q' of Q'_n. Therefore, since R contains w, q' contains w, and consequently $\overline{q'}$ lies in $S-\overline{D'}$ and does not contain X; i.e., if J' denotes the boundary of q', then both D' and X lie in the same complementary domain of J', which must be $I(J')$, since q' is $E(J')$. Now, the simple closed curve J, which is the boundary of q, lies in R_n, and hence in q', and therefore cannot separate any point of D' from X. Thus, since q does not contain X, q does not contain any point of D' either; in particular, q does not contain A. But since this contradicts the assumption made above, it must be false that q contains w.

Thus q must be the interior with respect to w of some simple closed curve J in Σ lying in a bounded region R of G_n, and R must lie in some domain q' of Q'_n whose boundary is a simple closed curve J' in Σ, such that $q'+J'$ does not contain w either. Thus q' is $I(J')$, and since J lies in q', $I(J)$ lies in q'; but $I(J)$ is q. Since q' is in Q'_n and contains A (q' contains q, and q contains A), $\overline{q'}$ lies in D'. Thus $\bar{q}$ lies in D', and therefore in D, and does not contain B.

To show that the sequence $Q_1, Q_2, Q_3, \ldots$ satisfies the fourth part of Axiom 1, the following argument suffices. Suppose $M_1, M_2, M_3, \ldots$ is a sequence of point sets, each closed in Σ', such that for each positive integer n, M_n contains M_{n+1}, and M_n lies in the closure of some domain of Q_n. Then, for each n, M_n also lies in the closure of some domain of Q'_n, and, since the sequence $Q'_1, Q'_2, Q'_3, \ldots$ satisfies Axiom 1, the point sets $M_1, M_2, M_3, \ldots$ have a point in common.

THEOREM 26. *The space Σ' satisfies Axiom 1_3 if and only if there exist a point w and a sequence $G_1, G_2, G_3, \ldots$ of collections of regions in the space Σ satisfying the conditions of Axiom 1 such that if A is a point distinct from w, and J is a simple closed curve that separates A from w, then there exists a positive integer n such that if R is a bounded region of G_n, and R contains a simple closed curve that separates A from w, then R lies entirely within J.*

Proof. Suppose there exist a point w and a sequence $G_1, G_2, G_3, \ldots$ satisfying all of these conditions. Let $I_1, I_2, I_3, \ldots$ denote a sequence of simple domains in Σ having only the point w in common, such that each contains the closure of the next. For each positive integer n, let Q_n denote the collection to which D belongs if and only if either (1) D is one of the domains $I_n, I_{n+1}, I_{n+2}, \ldots$, or (2) D is the interior with respect to w of some simple closed curve in Σ that lies in a bounded region of G_n.

The sequence $Q_1, Q_2, Q_3, \ldots$ clearly satisfies parts 1 and 2 of Axiom 1_3. Suppose A and B are points of the simple domain I. If A is w, then there exists a positive integer n such that $\overline{I_n}$ lies in I and does not contain B.

Since every member of Q_n that contains w belongs to the sequence $I_n, I_{n+1}, I_{n+2}, \ldots$, the closure of any domain of Q_n that contains A lies in $I-B$.

If A is distinct from w, there exists a simple closed curve J in Σ separating A from both B and w, and such that $I(J)$ also lies in I. Now, by hypothesis, there exists a positive integer n such that (1) if R is a bounded region of G_n which contains a simple closed curve that separates A from w, then R lies entirely within J, and (2) I_n does not contain A. Suppose q is a domain of Q_n that contains A. Then since q is not one of the domains $I_n, I_{n+1}, I_{n+2}, \ldots$, there exists a simple closed curve J' lying in a bounded region R of G_n such that q is $I(J')$. But R must lie wholly within J, and therefore J' lies within J. Thus $\bar{q}$ lies wholly within J, and therefore in I, and does not contain B.

The space Σ', then, satisfies Axiom 1_3.

Now, suppose the space Σ' satisfies Axiom 1_3. Let w denote a point, and $Q_1, Q_2, Q_3, \ldots$ a sequence of collections of simple domains in Σ satisfying the conditions of Axiom 1_3, and $G_1', G_2', G_3', \ldots$ a sequence of collections of regions in Σ satisfying Axiom 1. For each n, let G_n denote the collection to which R belongs if and only if R is a region of G_n' that lies in some domain q of Q_n, such that $\bar{q}$ does not contain w unless R does.

The sequence $G_1, G_2, G_3, \ldots$ satisfies Axiom 1, since the sequence $G_1', G_2', G_3', \ldots$ satisfies Axiom 1. Now, suppose A is a point distinct from w, and J is a simple closed curve that separates A from w. Since the sequence $Q_1, Q_2, Q_3, \ldots$ satisfies Axiom 1_3, there exists a positive integer n such that every domain of Q_n that contains A lies along with its boundary in $I(J)$. Suppose R is a bounded region of G_n containing a simple closed curve J' that separates A from w. Then R lies in a domain q of Q_n whose closure is bounded. For some simple closed curve C in Σ, q is $I(C)$. Thus J' lies in $I(C)$, and therefore $I(J')$, which contains A, is a subset of $I(C)$ (i.e., of q). Since q contains A and is a domain of Q_n, q lies along with its boundary in $I(J)$; and therefore, since R lies in q, R must also lie entirely within J.

3. Examples.

EXAMPLE 1. Consider the plane. For each positive integer n and each positive integer m, let P_m denote the point $(2^{1-2m}, 0)$, and $D(n, m)$ the circular disk centered at P_m, with radius $2^{-2m}\cdot n/(n+1)$. For each positive integer n, (1) let Q_n denote the sum of the circular disks $D(n, m)$, for all positive integers, m, (2) let O_n denote a circular domain centered at the origin with radius $2^{2(1-n)}$, and (3) let R_n denote the point set $O_n - O_n \cdot Q_n$.

Let Σ denote the space obtained by interpreting the word *point* to mean a point of the plane, and the word *region* to mean a subset R

of the plane such that either (1) for some circular domain D such that $\bar{D}$ does not contain the origin, R is D, or (2) for some positive integer n, R is R_n.

The space Σ satisfies Axioms 0-5, and the space Σ' derived from Σ in the manner indicated in this paper, is the plane. Let O denote the origin, M denote the set consisting of all the points $P_1, P_2, P_3, \ldots$, and I denote the interval having $(0,0)$ and $(1,0)$ as the coordinates of its endpoints.

The following examples show that the converses of a number of the earlier theorems in this paper are not true: (1) O is a limit point of the point set M in the space Σ', but not in the space Σ (cf. Theorem 1), (2) the point set R_1 is a region in Σ, but not in Σ', since the point O of R_1 is a limit point of $S-R_1$ in the space Σ' (cf. Theorem 2); (3) the point set M is closed in Σ (indeed, it has no limit point in Σ), but M is not closed in Σ' since O is a limit point of M with respect to Σ' which is not a point of M (cf. Theorem 3); (4) the point sets O and M are mutually separated in Σ, but not in Σ', (cf. Theorem 5); and (5) I is an arc in Σ', but not in Σ (cf. Theorem 15).

Moreover, if the condition that the point set M be connected is omitted from the hypothesis of Theorem 14, then the point set $M+O$ in this example is connected *im kleinen* in the space Σ, but not in the space Σ', contrary to the conclusion of Theorem 14. Also, the interval I in this example is a point set which is connected in both Σ and Σ', and connected *im kleinen* at the point O in the space Σ', but not in the space Σ.

EXAMPLE 2. The following is an example of a space Σ that satisfies Axioms 0-5, but such that the space Σ', derived from it in the manner described earlier, does not satisfy Axiom 1_3.

Consider the plane. Let I denote the unit interval, and M denote the Cantor (ternary) set on I. The set M is often obtained as the common part of the elements of a certain sequence $M_0, M_1, M_2, \ldots$ such that for each n, the components of M_n consist of 2^n intervals of length 3^{-n}. Let E denote the collection of all endpoints of the intervals of $M_0, M_1, M_2, \ldots$, and K denote $M-E$. If P is a point of M, and n is a positive integer, let $I_n(P)$ denote the component of M_n that does not contain P, but that lies together with P in a component of M_{n-1}.

Let T denote a reversible transformation of K into some collection of mutually exclusive circular disks of radius 1, none of which intersects the plane. For each point P of K, let L_P and R_P denote the endpoints of a diameter of $T(P)$; X_P, a third point of the boundary of $T(P)$; and C_P, the center of $T(P)$.

Suppose P is a point of K, $0 \leqslant \theta < 2\pi$, and n is a positive integer. Let $D_1(P, \theta, n)$ denote the domain in the plane (see Fig. 1) bounded by the radii PP_1 and PP_3 and the arc $P_1P_2P_3$ of a circle C centered at P, such that all the following requirements are satisfied: (1) $1/n$ is the least positive measure of both of the angles P_1PP_2 and P_2PP_3, (2) the length of the radius PP_1 of C is $1/n$, (3) if Q is a point on the x-axis to the right of P, then θ is the least positive measure of the angle QPP_2, measured

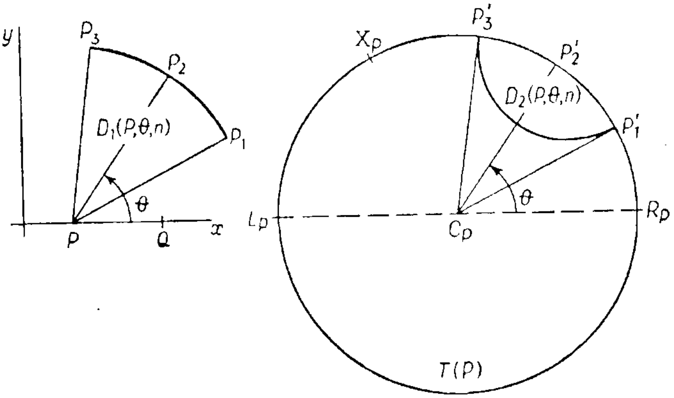

Fig. 1

counterclockwise from the arc PQ to the arc PP_2. Let P'_1, P'_2, and P'_3 denote three points of the boundary of $T(P)$ such that (1) the least positive measure of the angle $R_PC_PP'_2$, measured in the $R_PX_PL_P$ direction from the arc C_PR_P to the arc $C_PP'_2$, is θ, and (2) $1/n$ is the least positive measure of each of the angles $P'_1C_PP'_2$ and $P'_2C_PP'_3$. Let $P'_1ZP'_3$ denote the arc of the circle J such that (1) J is tangent to the intervals $C_PP'_1$ and $C_PP'_3$ at the points P'_1 and P'_3, respectively, and (2) $P'_1ZP'_3$ lies in $T(P)$. Finally, let $D_2(P, \theta, n)$ denote the component of $T(P)-P'_1ZP'_3$ that contains P'_2.

R is said to be a type I region of order n if and only if, for some point P of K and some number θ, $0 \leqslant \theta < 2\pi$, (1) R is $D_1(P, \theta, n)+$ $+D_2(P, \theta, n)$, and (2) R contains no point of I. Note that if P is a point of K, then for every point X of the boundary of $T(P)$, other than L_P and R_P, there is a positive integer N such that if $n > N$, some type I region of order n contains X; but the points L_P and R_P belong to no type I region whatsoever.

Now suppose P is a point of M, and $D_1, D_2, D_3, \ldots$ is a sequence of domains in the plane such that (1) if P is a point of E, then for each n, D_n is a circular domain centered at P, with radius $1/n$, and (2) if P is a point of K, then either (a) for each n, D_n is $D_1(P, O, n)$, or (b) for each n, D_n is $D_1(P, \pi, n)$. Let $I'_1, I'_2, I'_3, \ldots$ denote the subsequence of $I_1(P), I_2(P), I_3(P), \ldots$ consisting of all the terms of this sequence that intersect D_1. Let $Z(1), Z(2), Z(3), \ldots$ denote a sequence of mutually exclusive bounded simple domains in the plane such that for each positive integer m, (1) $Z(m)$ contains I'_m, (2) if I'_m does not intersect $\bar{D}_i$ for some positive integer i, then neither does $\overline{Z(m)}$, (3) if I'_m lies in D_i for some positive integer i, then so does $\overline{Z(m)}$, and (4) if I'_m intersects $\bar{D}_i$ for some positive integer i, $\overline{Z(m)}$ neither contains P nor separates D_i. For each positive integer m, let $Z(1, m), Z(2, m), Z(3, m), \ldots$ denote a sequence of simple domains such that for each n, (1) $Z(n, m)$ contains I'_m, and (2) $\overline{Z(n, m)}$ lies in $Z(n+1, m)$ and in $Z(m)$. Let Q_n denote the sum of the simple disks $\overline{Z(n, m)}$ for all positive integers m. Finally, let $D'_1, D'_2, D'_3, \ldots$ denote a sequence of point sets such that for each n, D'_n is $D_n - D_n \cdot Q_n$.

For each point P, let $D_n(P)$ denote the circular domain of radius $1/n$ centered at P. If P is a point of M, then by the process used in the last paragraph to obtain from the sequence $D_1, D_2, D_3, \ldots$ a sequence $D'_1, D'_2, D'_3, \ldots$ we obtain from the sequences $D_1(P), D_2(P), D_3(P), \ldots$ and $D_1(P, 0, 1), D_1(P, 0, 2), D_1(P, 0, 3), \ldots$ and $D_1(P, \pi, 1), D_1(P, \pi, 2), D_1(P, \pi, 3), \ldots$, the sequences $D'_1(P), D'_2(P), D'_3(P), \ldots$ and $D'_1(P, 0, 1), D'_1(P, 0, 2), D'_1(P, 0, 3), \ldots$ and $D'_1(P, \pi, 1), D'_1(P, \pi, 2), D'_1(P, \pi, 3), \ldots$ and R is said to be a type II region of order n if and only if for some point P of M, (1) if P belongs to E, R is $D'_n(P)$, and (2) if P belongs to K, then either (a) R is $D'_1(P, 0, n)+D_2(P, 0, n)$, or (b) R is $D'_1(P, \pi, n)+D_2(P, \pi, n)$. Note that if n is a positive integer and P is a point of E, or is $R_{P'}$ or $L_{P'}$ for some point P' of K, then some type II region of order n contains P; but if P_1 and P_2 are two points of M, and R is a type II region, then (1) R cannot contain both P_1 and P_2, if they are points of E, and (2) R cannot intersect both $T(P_1)$ and $T(P_2)$, if they are points of K.

R is a type III region of order n if and only if R is a circular domain of radius $1/n$ such that either (1) $\bar{R}$ lies in the plane, but does not intersect M, or (2) for some point P of K, $\bar{R}$ lies in $T(P)$, but does not intersect the boundary of $T(P)$. Note that if P is a point which is not on the boundary of $T(P')$ for any point P' of K, and which is not a point of E, then there is a positive integer N such that if $n > N$, some type III region of order n contains P.

Definition of the space Σ. P is a point of the space Σ if and only if P is either (1) a point in the plane, but not in K, or (2) a point of $T(P')$ for some point P' of K. For each positive integer n, let G_n denote the collection of all regions of type I, type II, or type III, of order n or more. R is a region of Σ if and only if R is an element of G_1.

The sequence $G_1, G_2, G_3, \ldots$ satisfies all the requirements of Axiom 1. Indeed, the space Σ satisfies all the Axioms 0-5. However, the space Σ', obtained from Σ in the usual manner, fails to satisfy Axiom 1_3. This may be seen by considering the point set J obtained in the following manner.

Let $ALBNA$ denote the circle in the plane that has I as a diameter, where the point L of this circle lies above I. Let U denote the collection to which x belongs if and only if either (1) x is the semicircle ALB, (2) for some positive integer n, x is the closure of some segment in H_n, or (3) for some point P of K, x is $L_P C_P R_P$. Then U is an uncountable collection of mutually exclusive arcs in the space Σ, but J, the sum of all the arcs in U, is a simple closed curve with respect to the space Σ'. Since no space that satisfies Axiom 1_3 can contain a simple closed curve which contains uncountably many mutually exclusive intervals, it follows that Σ' cannot satisfy Axiom 1_3.

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