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Proximity mappings and induced homomorphisms

Let (X, δ) and (Y, σ) be proximity spaces, with proximity relations δ and σ for X and Y, respectively. By $P^*(X)$ we denote the algebra of bounded, real-valued δ-mappings on X, where the proximity relation on the real numbers is that induced by the usual metric. In part 1 of this paper we consider the duality between δ-mappings t of X into Y and the corresponding induced homomorphisms t' of $P^*(Y)$ into $P^*(X)$.

In [6] Smirnov showed that every proximity space (X, δ) is a dense subspace of a compact Hausdorff space δX, where $A\delta B$ in X if and only if the closures of A and B in δX meet. Other proofs of this result have been given by Leader [5] and Császár and Mrówka [1]. In part 2 we obtain δX as an upper semi-continuous decomposition of βX, the Stone-Cech compactification of X. From this results follow concerning the structure of maximal ideals in $P^*(X)$.

1. Induced homomorphisms of $P^*(X)$. We say that a proximity space (X, δ) is *P^*-embedded in a proximity space* (X_1, δ_1) if (X, δ) is a subspace of (X_1, δ_1) and if every function f in $P^*(X)$ has an extension f^* in $P^*(X_1)$.

If $A\bar{\delta}B$, where $\bar{\delta}$ denotes the negation of δ, there exists f in $P^*(X)$ such that $f[A] = 0$ and $f[B] = 1$. (See [2].)

THEOREM 1. *Let t be a δ-mapping of (X, δ) into (Y, σ) and let t' be the induced homomorphism $t'(f) = f\circ t$ from $P^*(Y)$ into $P^*(X)$.*

(A) *t' is an isomorphism if and only if $t[X]$ is dense in Y.*

(B) *t' is onto if and only if t is an equimorphism and $t[X]$ is P^*-embedded in Y.*

Proof. (A) The mapping t' is an isomorphism if and only if $t'(f) = \mathbf{0}$ implies $f = \mathbf{0}$. The latter condition holds if and only if $t[X]$ is dense in Y, since if a point y is remote from $t[X]$, then $\{y\}$ and $t[X]$ can be functionally separated by a member of $P^*(Y)$.

(B) For each $f \in P^*(X)$, suppose there exists $g \in P^*(Y)$ such that $t'(g) = f$. If $t(x_1) = t(x_2)$, then $(t'g)(x_1) = (t'g)(x_2)$ for all $g \in P^*(Y)$, which implies $f(x_1) = f(x_2)$, for all $f \epsilon P^*(X)$. Then $\{x_1\}\delta\{x_2\}$, so that $x_1 = x_2$. Thus t^{-1} is a well-defined mapping of $t[X]$ onto X.

Now let $C\sigma D$ in $t[X]$, and let $A = t^{-1}[C]$ and $B = t^{-1}[D]$. If $A\bar{\delta}B$, there exists $f \epsilon P^*(X)$ such that $f[A]$ is remote from $f[B]$. But $t'(g) = g \circ t = f$, for some $g \epsilon P^*(Y)$. Then $f[A] = g \circ t(t^{-1}[C]) = g[C]$ is remote from $f[B] = g \circ t(t^{-1}[D]) = g[D]$, which is a contradiction. Thus $A\delta B$, and t^{-1} is δ-mapping, so that t is an equimorphism.

Let $g_1 \epsilon P^*(t[X])$. Then $g_1 \circ t \epsilon P^*(X)$, and there exists $g \epsilon P^*(Y)$ such that $t'(g) = g \circ t = g_1 \circ t$. Evidently g is an extension of g_1 to Y, and $t[X]$ is P^*-embedded in Y.

Conversely, let $f \epsilon P^*(X)$. Since t^{-1} is a δ-mapping, $f \circ t^{-1} \epsilon P^*(t[X])$. Then there exists $g \epsilon P^*(Y)$ satisfying $g | t[X] = f \circ t^{-1}$. Thus $t'(g) = g \circ t = f \circ t^{-1} \circ t = f$, and t' is onto. This completes the proof.

It is easily verified that t is a δ-mapping of (X, δ) into (Y, σ) if and only if $t'[P^*(Y)] \subseteq P^*(X)$. From Theorem 1 it follows that if t is a homeomorphism of (X, δ) into (Y, σ), then t is an equimorphism and $t[X]$ is P^*-embedded in Y if and only if $t'[P^*(Y)] = P^*(X)$.

Because two spaces X and Y may be homeomorphic but not equimorphic under a mapping t, we may have $C(X)$ and $C(Y)$ isomorphic by t' without $t'[P^*(Y)] = P^*(X)$.

We next consider when a homomorphism of $P^*(Y)$ into $P^*(X)$ is induced by a δ-mapping of (X, δ) into (Y, σ).

THEOREM 2. *Let ψ be a homomorphism of $P^*(Y)$ into $P^*(X)$ with $\psi(\mathbf{1}) = \mathbf{1}$. If Y is compact, there exists a unique δ-mapping t of X into Y such that $t' = \psi$.*

Proof. For each $x \epsilon X$, let λ_x be the homomorphism of $P^*(Y)$ into the real numbers defined by $\lambda_x(g) = (\psi g)(x)$. Since $\lambda_x(\mathbf{1}) = 1$, λ_x is not the zero homomorphism, and since $P^*(Y)$ contains the constant functions, λ_x is onto. Now $P^*(Y) = C(Y)$ since Y is compact, and the kernel of λ_x is a fixed real maximal ideal of $C(Y)$. (See [3].) Thus there is a unique point $y \epsilon Y$ such that $(\psi g)(x) = g(y)$, for all $g \epsilon C(Y)$.

Define $t(x) = y$ for each $x \epsilon X$. Then t satisfies $t'(g) = \psi(g)$, for every $g \epsilon C(Y)$. Let $A\delta B$ in X. If $t[A]\bar{\sigma}t[B]$ in Y, there exists $g \epsilon C(Y)$ which completely separates $t[A]$ and $t[B]$. Then $t'(g) = \psi(g) \epsilon P^*(X)$, which is a contradiction. Hence t is a δ-mapping. The uniqueness of t follows immediately, and the proof is complete.

Theorem 2 fails if Y is not compact. Let δX be the Smirnov compactification of a non-compact proximity space (X, δ). Then there is an isomorphism ψ of $P^*(X)$ onto $P^*(\delta X)$, but by Theorem 1 ψ cannot be induced by an equimorphism of δX into X.

We now have the following corollary which concerns the extension of δ-mappings on X to δX:

COROLLARY 3. *Let (X, δ) be a dense subspace of (X_1, δ_1). Then X is P^*-embedded in X_1 if and only if every δ-mapping t from X into a compact proximity space (Y, σ) has an extension to a proximity mapping of X_1 into Y.*

Proof. For necessity, let $g \in C(Y)$. Then $g \circ t \in P^*(X)$, and $g \circ t$ has an extension $(g \circ t)_1$ in $P^*(X_1)$. The mapping $\psi(g) = (g \circ t)_1$ is a homomorphism of $C(Y)$ into $P^*(X_1)$, and $\psi(\mathbf{1}) = \mathbf{1}$. By Theorem 2 there exists a δ-mapping t_1 of X_1 into Y such that $t_1' = \psi$. For $x \in X$, then $g(t_1(x)) = (\psi g)(x) = (g \circ t)_1(x) = g(t(x))$, for all $g \in C(Y)$. Hence $t(x) = t_1(x)$, and t_1 is an extension of t.

Sufficiency is trivial. This completes the proof.

If Y is compact and ψ is a homomorphism of $P^*(Y)$ onto $P^*(X)$ with $\psi(\mathbf{1}) = \mathbf{1}$, then X is equimorphic to a P^*-embedded subspace of Y. Now Theorem 1 and Corollary 3 yield the following algebraic characterization of the Smirnov compactification of (X, δ).

THEOREM 4. *A compact Hausdorff space Y is the Smirnov compactification of (X, δ) if and only if there exists an isomorphism ψ from $C(Y)$ onto $P^*(X)$ with $\psi(\mathbf{1}) = \mathbf{1}$.*

2. The Smirnov compactification. We next obtain the Smirnov compactification [6] of (X, δ) as an upper semi-continuous decomposition of βX, the Stone-Čech compactification of X.

THEOREM 5. *A proximity space (X, δ) is equimorphic to a dense subspace (X_1, δ_1) of a compact Hausdorff space δX. Every δ-mapping of X_1 into a compact proximity space (Y, σ) has an extension mapping δX into Y. Moreover, δX is an upper semi-continuous decomposition of βX.*

Proof. If $f \in P^*(X)$, then $f[X]$ is bounded and $\mathrm{Cl} f[X]$ is compact. Let f^* denote the extension of f to βX. For points x and y of βX, define $x \varrho y$ if and only if x and y are not distinguished by f^*, for all $f \in P^*(X)$. Denote the equivalence class of x with respect to ϱ by $[x]$. Let δX be the quotient space of βX with respect to ϱ and the projection π of βX onto the decomposition. Since $P^*(X)$ distinguishes points of X, the restriction π_0 of π to X is injective. Let $X_1 = \pi_0[X]$.

Each f^*, where $f \in P^*(X)$, induces a continuous mapping f_1 of δX into R by $f_1([x]) = f^*(y)$, where $y \in [x]$. If $[x] \neq [y]$, there exists $f \in P^*(X)$ such that $f^*(x) \neq f^*(y)$, so that $f_1([x]) \neq f_1([y])$. Since f_1 is continuous, it follows that δX is a compact Hausdorff space, and X_1 is dense in δX.

Let δ_1 be the proximity relation on X_1 induced by the unique proximity relation on δX compatible with the topology on δX. If $A \bar{\delta} B$ in X, there exists $f \in P^*(X)$ such that $f[A]$ is remote from $f[B]$. Set $\pi_0[A] = A_1$

and $\pi_0[B] = B_1$. Since $f_1 \circ \pi = f^*$, the closures of A_1 and B_1 in δX do not meet, so that $A_1 \bar{\delta}_1 B_1$. Thus π_0^{-1} is a δ-mapping.

Now let $A \delta B$ in X, and suppose that $A_1 \bar{\delta}_1 B_1$. Then $\mathrm{Cl}_{\delta X} A_1 \cap \mathrm{Cl}_{\delta X} B_1 = \emptyset$. Since $\pi[\mathrm{Cl}_{\beta X} A] = \mathrm{Cl}_{\delta X} A_1$ and $\pi[\mathrm{Cl}_{\beta X} B] = \mathrm{Cl}_{\delta X} B_1$, $a \in \mathrm{Cl}_{\beta X} A$ implies $a \notin [b]$, for each $b \in \mathrm{Cl}_{\beta X} B$. Thus a and b are separated by some f^*, where $f \in P^*(X)$, so that in βX there are disjoint neighborhoods for a and b whose intersections with X are remote. Since $\mathrm{Cl}_{\beta X} B$ is compact, there are open sets O_a and V_a containing a and $\mathrm{Cl}_{\beta X} B$, respectively, where $(O_a \cap X) \bar{\delta} (V_a \cap X)$. Now $\mathrm{Cl}_{\beta X} A$ can be covered with a finite collection of sets $O_1, \ldots, O_n$ constructed in this manner, where $\mathrm{Cl}_{\beta X} B$ is contained in the intersection of the corresponding V_i, $i = 1, \ldots, n$. Then $A \subseteq \bigcup_{i=1}^{n} (O_i \cap X)$ and $B \subseteq \bigcap_{i=1}^{n} (V_i \cap X)$. But this requires that $(O_i \cap X) \delta (V_i \cap X)$, for some i, which is a contradiction. Thus $A_1 \delta_1 B_1$ in X_1, and π_0 is an equimorphism.

Let $g_1 \in P^*(X_1)$. Then $g = g_1 \circ \pi_0 \in P^*(X)$, and g has an extension g^* in $C(\beta X)$. Define $g_1^*([x]) = g^*(y)$, for $y \in [x]$. Then g_1^* is a continuous extension of g_1 to δX. By Corollary 3, δX has the desired extension property.

Finally, since π is a closed mapping, δX is an upper semi-continuous decomposition of βX, and the proof is complete.

Since $\pi^{-1}([x])$ is compact for each $[x] \in \delta X$, many of the properties of βX are inherited by δX. (See [4].)

The homomorphism π_0' of $C(\delta X)$ onto $P^*(X)$ is an isomorphism, since X_1 is dense and P^*-embedded in δX. The maximal ideals of $C(\delta X)$ are fixed and have the form $M_{[x]}^* = \{f_1 | f_1 \in C(\delta X) \text{ and } f_1([x]) = 0\}$. (See [3].) Let $f_1 \in M_{[x]}^*$. Then $\pi_0'(f_1) = f_1 \circ \pi_0 \in P^*(X)$, and $f_1 \circ \pi$ is the extension of $f_1 \circ \pi_0$ to βX. If $y \in [x]$, clearly $f_1 \circ \pi \in M^{*y}$, where $M^{*y} = \{f | f \in C^*(X) \text{ and } f^*(y) = 0\}$. Thus $\pi_0'[M_{[x]}^*] \subseteq M^{*y}$. It now follows that all maximal ideals of $P^*(X)$ have the form $M^{*y} \cap P^*(X)$, where $y \in \beta X$. Let $M^{*y} \cap P^*(X) = P^{*y}$.

Theorem 6. *If A and B are subsets of X, then $A \delta B$ if and only if there exist $a \in \mathrm{Cl}_{\beta X} A$ and $b \in \mathrm{Cl}_{\beta X} B$, where $P^{*a} = P^{*b}$.*

Proof. If $A \delta B$ in X, then there exists $[x] \in \mathrm{Cl}_{\delta X}(\pi A) \cap \mathrm{Cl}_{\delta X}(\pi B)$. Thus there exist $a \in \mathrm{Cl}_{\beta X} A$ and $b \in \mathrm{Cl}_{\beta X} B$, where $a, b \in [x]$. Then $\pi_0'[M_{[x]}^*] = P^{*a} = P^{*b}$.

Conversely, if $P^{*a} = P^{*b}$, then a and b are in $[x]$, for some $[x] \in \delta X$. Thus $[x] \in \pi[\mathrm{Cl}_{\beta X} A] \cap \pi[\mathrm{Cl}_{\beta X} B]$, so that $\mathrm{Cl}_{\delta X}(\pi A)$ meets $\mathrm{Cl}_{\delta X}(\pi B)$. Since π_0^{-1} is a δ-mapping, $A \delta B$, and the proof is complete.

It follows that βX and δX are homeomorphic under π if and only if $P^{*x} \neq P^{*y}$ whenever $x \neq y$ in βX. Thus the ideals P^{*x} are distinct for each $x \in \beta X$ if and only if $P^*(X) = C^*(X)$.

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