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On the strong (L) summability of Fourier series

1.1. Let $f(t)$ be a function integrable in the sense of Lebesgue over $(-\pi, \pi)$ and periodic with period 2π. The Fourier series associated with $f(t)$ is

$$\tag{1.1.1} \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t).$$

We denote by S_ν the ν-th partial sum of (1.1.1) and write

$$\tag{1.1.2} \Phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2S\}$$

and

$$\tag{1.1.3} \Phi(t) = \int_0^t \varphi(u)\,du.$$

1.2. A series $a_0 + a_1 + \ldots + a_n + \ldots$ is said to be *strongly summable* $(C, 1)$ or *summable* $[C, 1]$ *to the sum* S, if

$$\tag{1.2.1} \sum_{\nu=0}^{n} |S_\nu - S| = o(n), \quad \text{as } n \to \infty,$$

S_ν being the sum of the first $(\nu+1)$ terms of the series $\sum_{n=0}^{\infty} a_n$ (Fekete [1]).

If, instead of condition (1.2.1), we have

$$\tag{1.2.2} \sum_{\nu=0}^{n} |S_\nu - S|^q = o(n), \quad \text{as } n \to \infty,$$

then the series is said to be *strongly summable* $(C, 1)$ with index q $(q > 0)$, or simply summable Hq to the sum S (Kuttner [5]).

The series is further said to be strongly summable by Riesz logarithmic mean of order one or summable $[R, \log n, 1]$ to the sum S, if

$$\tag{1.2.3} \sum_{\nu=1}^{n} \frac{|S_\nu - S|}{\nu} = o(\log n), \quad \text{as } n \to \infty.$$

It is strongly summable $(R, \log n, 1)$ with index q $(q > 0)$ to the sum S, if

$$\sum_{\nu=1}^{n} \frac{|S_\nu - S|^q}{\nu} = o(\log n), \quad \text{as } n \to \infty \tag{1.2.4}$$

(Hardy and Littlewood [4]).

Recently, Rai [6] has defined an analogue for strong summability of (L) summability method as follows:

DEFINITION. A series $\sum_{n=0}^{\infty} a_n$ with the sequence of partial sum $\{S_n\}$ is said to be *strongly (L) summable to the sum S*, if

$$\sum_{\nu=1}^{\infty} \frac{|S_\nu - S| x^\nu}{\nu} = o\big(|\log(1-x)|\big), \quad \text{as } x \to \bar{1} \tag{1.2.5}$$

for x in the open interval $(0, 1)$.

The object of this paper is to apply strong (L) summability to Fourier series. In fact we prove:

THEOREM. *If*

$$\int_0^t \varphi(u)\,du = o\left(t \log \frac{1}{t}\right) \tag{1.2.6}$$

and

$$\int_t^\pi \left|\frac{\Phi(u)}{u}\right| \log^+ \left|\frac{\Phi(u)}{u}\right| du = o\left(\log \frac{1}{t}\right) \ (^1), \tag{1.2.7}$$

then

$$\sum_{\nu=1}^{\infty} \frac{|S_\nu - S| x^\nu}{\nu} = o\big(|\log(1-x)|\big).$$

2.1. In order to prove this theorem, we require the following lemmas:

LEMMA 1 [3]. *If*

$$f_R(\theta) = \sum_{-R}^{R} C_n e^{ni\theta} \quad \text{and} \quad f_R^+(\theta) = \sum_{-R}^{R} C_n^+ e^{ni\theta},$$

where C_n^+ are the numbers $|C_n|$ rearranged in an order such that $C_0^+ \geqslant C_{-1}^+ \geqslant C_1^+ \geqslant C_{-2}^+ \geqslant C_2^+ \geqslant C_{-3}^+ \geqslant \ldots$ (²); then

$$\int_{-\pi}^{\pi} e^{b|f_R(\theta)|}\,d\theta \leqslant 2 \int_{-\pi}^{\pi} e^{b|f_R^+(\theta)|}\,d\theta$$

for every positive b.

(¹) $\log^+ x$ denotes the function which is equal to $\log x$ for $x > 1$ and to zero elsewhere.

(²) This notation is due to Gabriel [2].

LEMMA 2. $\sum_1^\infty \frac{x^\nu}{\nu} \cos\nu\theta = O(1)$, *for* $0 < x < 1$ *and* $0 < \theta \leqslant \pi$.

Proof. We have

$$\left|\sum_1^\infty \frac{x^\nu}{\nu} \cos\nu\theta\right| = \left|-\frac{1}{2}\log(1-2x\cos\theta+x^2)\right|$$
$$\leqslant |\log(1+x)|$$
$$< |\log 2|,$$

uniformly for $0 < x < 1$ and $0 < \theta \leqslant \pi$.

LEMMA 3. $\sum_1^\infty \frac{x^\nu}{\nu} \sin\nu\theta = O(1)$, *for* $0 < x < 1$ *and* $1-x \leqslant \theta \leqslant \pi$.

Proof. We have

$$\left|\sum_1^\infty \frac{x^\nu}{\nu} \sin\nu\theta\right| = \left|\tan^{-1}\frac{x\sin\theta}{1-x\cos\theta}\right| = O(1)$$

for $1-x \leqslant \theta \leqslant \pi$ and $0 < x < 1$.

3.1. Proof of the theorem. It is well known that

$$S_\nu - S = \frac{2}{\pi}\int_0^\pi \frac{\sin(\nu+1/2)u}{2\sin(u/2)}\varphi(u)\,du$$
$$= \frac{2}{\pi}\int_0^\pi \frac{\sin\nu u}{u}\varphi(u)\,du + o(1)$$

by Riemann-Lebesgue theorem.

Therefore

$$\text{(3.1.1)}\quad \sum_{\nu=1}^\infty \frac{|S_\nu - S|x^\nu}{\nu} = \frac{2}{\pi}\sum_{\nu=1}^\infty \frac{x^\nu}{\nu}\left|\int_0^\pi \frac{\varphi(u)}{u}\sin\nu u\,du\right| + o\big(|\log(1-x)|\big)$$
$$= \frac{2}{\pi}\sum_{\nu=1}^\infty \frac{x^\nu}{\nu}\left|\left(\int_0^{1-x} + \int_{1-x}^\pi\right)\frac{\varphi(u)}{u}\sin\nu u\,du\right| + o\big(|\log(1-x)|\big)$$
$$\leqslant \frac{2}{\pi}\sum_{\nu=1}^\infty \frac{x^\nu}{\nu}|T_1| + \frac{2}{\pi}\sum_{\nu=1}^\infty \frac{x^\nu}{\nu}|T_2| + o\big(|\log(1-x)|\big).$$

Consider $\sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} |T_1|$. We have from (1.2.6)

$$(3.1.2) \qquad \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} \left| \int_0^{1-x} \frac{\varphi(u)}{u} \sin \nu u \, du \right|$$

$$= \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} \left| \left(\Phi(1-x) \cdot \frac{\sin \nu(1-x)}{(1-x)} \right) - \int_0^{1-x} \left(\frac{\nu \cos \nu u}{u} - \frac{\sin \nu u}{u^2} \right) \Phi(u) \, du \right|$$

$$= o\left(\sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} (1-x) |\log(1-x)| \cdot \nu \right) + \sum_{\nu=1}^{\infty} \frac{x^\nu}{\nu} \left| \int_0^{1-x} o\left\{ \left(\frac{\nu}{u} + \frac{\nu u}{u^2} \right) \left(u \log \frac{1}{u} \right) \right\} du \right|$$

$$= o\big(|\log(1-x)|\big) + o\left(\sum_{\nu=1}^{\infty} x^\nu \left| \int_0^{1-x} \log \frac{1}{u} \, du \right| \right)$$

$$= o\big(|\log(1-x)|\big) + o\left(\sum_{\nu=1}^{\infty} x^\nu \left| \left\{ \left(u \log \frac{1}{u} \right)_0^{1-x} + \int_0^{1-x} du \right\} \right| \right)$$

$$= o\big(|\log(1-x)|\big) + o\left(\sum_{\nu=1}^{\infty} x^\nu (1-x) |\log(1-x)| \right) + o\left(\sum_{\nu=1}^{\infty} x^\nu (1-x) \right)$$

$$= o\big(|\log(1-x)|\big).$$

And,

$$\sum_{\nu=1}^{\infty} \frac{x^\nu |T_2|}{\nu} = \int_{1-x}^{\pi} \frac{\varphi(u)}{u} \sum_{\nu=1}^{\infty} \frac{\varepsilon_\nu \sin \nu u}{\nu} x^\nu \, du,$$

where $\varepsilon_\nu = \pm 1$ so as to make $\varepsilon_\nu T_2 \geqslant 0$ when $\nu = 1, 2, 3, \ldots$

We have from Young's inequality [3]

$$(3.1.3) \qquad \omega v \leqslant l \omega \log^+ \omega + l e^{(v-l)/l} \quad \text{for } \omega > 0, l > 0$$

and for all real v.

Setting

$$\omega = \left| \frac{\varphi(u)}{u} \right|, \quad v = |\Omega(u)| = \left| \sum_{\nu=1}^{\infty} \frac{\varepsilon_\nu \sin \nu u}{\nu} x^\nu \right|$$

and $l = 2$ in (3.1.3), we have in virtue of condition (1.2.7),

$$(3.1.4) \quad \left| \left(\sum_{\nu=1}^{\infty} \frac{x^\nu |T_2|}{\nu} \right) \right| = \left| \int_{1-x}^{\pi} \frac{\varphi(u)}{u} \sum_{\nu=1}^{\infty} \frac{\varepsilon_\nu \sin \nu u}{\nu} x^\nu \, du \right| \leqslant \int_{1-x}^{\pi} \left| \frac{\varphi(u)}{u} \right| |\Omega(u)| \, du$$

$$\leqslant 2 \int_{1-x}^{\pi} \left|\frac{\varphi(u)}{u}\right| \log^+ \left|\frac{\varphi(u)}{u}\right| du + A \int_{1-x}^{\pi} e^{|\Omega(u)|/2} du$$

$$= o\big(|\log(1-x)|\big) + A \int_{1-x}^{\pi} e^{|\Omega(u)|/2} du$$

$$= o\big(|\log(1-x)|\big) + K,$$

where A is some constant.

Now, it remains to estimate K.

Following Szász ([7], p. 708), if we put

$$\frac{\varepsilon_\nu x^\nu}{\nu} = P_\nu,$$

we have $\Omega_n(u) = I[r(u)]$ and $|\Omega_n(u)| \leqslant |r(u)|$, where

$$r(u) = \sum_{\nu=1}^{n} P_\nu e^{\nu i u}$$

is a polynomial of degree n whose coefficients have (in someorder) the absolute values $x, x^2/2, x^3/3, \ldots, x^n/n$. Hence, in the notation of Lemma 1

$$r^+(u) = xe^{-iu} + \frac{x^2}{2} e^{iu} + \frac{x^3}{3} e^{-2iu} + \ldots,$$

the last term having modulus x^n/n and an argument depending on the parity of n. The real and imaginary parts of $r^+(u)$ are bounded for all n and u $(1-x \leqslant u \leqslant \pi)$ by Lemmas 2 and 3 respectively; hence

(3.1.5) $$|r^+(u)| \leqslant A \quad \text{(constant)}.$$

We now obtain from Lemma 1 and (3.1.5)

$$\int_{1-x}^{\pi} e^{|\Omega_n(u)|/2} du \leqslant A \int_{1-x}^{\pi} e^{|r(u)|/2} du$$

$$\leqslant A \int_{1-x}^{\pi} e^{|r^+(u)|/2} du$$

$$\leqslant A \int_{1-x}^{\pi} du$$

$$\leqslant A \quad \text{(constant)},$$

for all n. Hence it follows that

(3.1.6) $$K = A \int_{1-x}^{\pi} e^{|\Omega(u)|/2} du \leqslant A.$$

Collecting (3.1.4) and (3.1.6), we have

$$\sum_{\nu=1}^{\infty} \frac{x^{\nu} |T_2|}{\nu} = o\big(|\log(1-x)|\big). \tag{3.1.7}$$

Finally, the collection of (3.1.1), (3.1.2) and (3.1.7) proves the theorem.

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