ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XIII (1969)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XIII (1969)

Aleksander Waszak (Poznań)

Linear functionals in Orlicz spaces of strongly $|A, \varphi|$-summable sequences

1.1. In this paper we shall use the following notation. Let T, T_0 and T_f be the spaces of all real sequences, real sequences convergent to 0, and "finite" real sequences (i.e. sequences whose all terms with the exception of a finite number are equal to 0), respectively. Sequences belonging to T will be denoted by $x = \{t_\nu\}$, $y = \{s_\nu\}$, etc. Moreover, e will denote the sequence $1, 1, \ldots$; e_ν — the sequence $0, 0, \ldots, 0, 1, 0, \ldots$ having 1 at the ν-th place; $|x| = \{|t_\nu|\}$; $x_k = \{t_\nu^k\}$; x^n and x_p^q denote the sequences $t_1, t_2, \ldots, t_n, 0, \ldots$ and $0, \ldots, 0, t_p, \ldots, t_{p+q-1}, 0, \ldots$, respectively.

$A = (a_{n\nu})$ will denote in this paper a matrix satisfying the following conditions: $a_{n\nu} \geqslant 0$ for $n, \nu = 1, 2, \ldots$; $(a_{n\nu})$ do not possess columns consisting of zeros only; $a_{n\nu} \to 0$ as $n \to \infty$ for $\nu = 1, 2, \ldots$; $a_{n1}+a_{n2}+\ldots \leqslant K$ for $n = 1, 2, \ldots$

By a *φ-function* we understand a continuous non-decreasing function $\varphi(u)$ defined for $u \geqslant 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. φ-functions will be denoted by Greek letters $\varphi, \psi, \ldots$ Let φ be a convex φ-function satisfying the conditions:

$$(0_1) \quad \frac{\varphi(u)}{u} \to 0 \text{ as } u \to 0_+, \qquad (\infty_1) \quad \frac{\varphi(u)}{u} \to \infty \text{ as } u \to \infty.$$

Let φ^* be the φ-function complementary to φ in the sense of Young, defined by the formula

$$\varphi^*(v) = \sup_{u \geqslant 0} \big(uv - \varphi(u)\big).$$

Functions inverse to the above ones will be denoted by $\varphi_{-1}, \varphi_{-1}^*, \ldots$, respectively ([2], pp. 16-26).

1.2. Let a matrix A be given and let φ be a convex φ-function. It is known [4] that the following functional may be defined in T_f:

$$(1) \qquad \varrho_\varphi^b(x) = \sum_{\nu=1}^{\infty} \varphi(|t_\nu|).$$

It is easily verified that this functional is a modular in T_f in the sense of [3]. Basing on this modular, a norm may be defined in T_f by means of the following formula:

$$(*) \qquad |||x|||_\varphi = \inf\{\varepsilon > 0:\ \varrho_\varphi^b(x/\varepsilon) \leqslant 1\}.$$

This norm is homogeneous and monotone, i.e. $|||\lambda x|||_\varphi = |\lambda|\, |||x|||_\varphi$ and if $x = \{t_\nu\}$, $y = \{s_\nu\}$ and $|t_\nu| \leqslant |s_\nu|$ for every ν, then $|||x|||_\varphi \leqslant |||y|||_\varphi$ (see [4], p. 134).

We can define the following norm for $x = \{t_\nu\} \in T_f$:

$$(**) \qquad |||x|||_\varphi^* = \sup_y \sum_{\nu=1}^{\infty} |t_\nu| s_\nu,$$

where the supremum is taken over all non-negative sequences $y = \{s_\nu\}$ satisfying the inequality $\sum\limits_{\nu=1}^{\infty} \varphi^*(s_\nu) \leqslant 1$ (see [7], p. 453).

By the Young inequality ([2], pp. 24, 89-91), for arbitrary two sequences x, y the following analogy of Hölder inequality holds:

$$(2) \qquad \sum_{\nu=p}^{p+q-1} |t_\nu s_\nu| \leqslant |||x_p^q|||_\varphi^*\, |||y_p^q|||_{\varphi^*}^*.$$

It is well known that norm $(**)$ is homogeneous and satisfies for all $x \in T_f$ the inequality

$$(3) \qquad \tfrac{1}{2}|||x|||_\varphi^* \leqslant |||x|||_\varphi \leqslant |||x|||_\varphi^*.$$

1.3.1. Applying norm $(*)$ we define the following class of sequences, strongly summable to zero. Let T_φ^b denote the class of sequences satisfying the condition $|||\bar{x}^n|||_\varphi \to 0$ as $n \to \infty$, where

$$\bar{x}^n = \begin{cases} \varphi_{-1}(a_{n\nu})t_\nu & \text{if} \quad n \geqslant \nu, \\ 0 & \text{if} \quad n < \nu. \end{cases}$$

Obviously, the class T_φ^b depends on the matrix A, however, this dependence is not pointed out in the notation because in this paper we shall deal with a fixed matrix A only.

We shall say that a sequence $\{t_\nu\}$ is *strongly* $|A, \varphi|$*-summable to* g, in writing $|A, \varphi|\text{-}\lim\limits_{\nu\to\infty} t_\nu = g$, if $\{t_\nu - g\} \in T_\varphi^b$.

A sequence $\{t_\nu\}$ is called φ*-summable to* g, in writing $\varphi\text{-}\lim\limits_{\nu\to\infty} t_\nu = g$, if $\lim\limits_{n\to\infty} \varphi_{-1}(1/n)\, |||x^n - g|||_\varphi^* = 0$ (see [7], p. 454). Obviously, if $a_{n\nu} = 1/n$ for $n \geqslant \nu$ and $a_{n\nu} = 0$ for $n < \nu$, we have $|A, \varphi|\text{-}\lim\limits_{\nu\to\infty} t_\nu = \varphi\text{-}\lim\limits_{\nu\to\infty} t_\nu$.

A sequence $\{t_\nu\}$ is called $|C^a, 1|$*-summable to* g, in writing $|C^a, 1|\text{-}\lim\limits_{\nu\to\infty} t_\nu = g$, if $\lim\limits_{n\to\infty} (1/n) \sum\limits_{\nu=1}^{n} |t_\nu - g|^a = 0$, $a \geqslant 1$. Obviously, if $\varphi(u) = a^{-1}u^a, a > 1$, and $a_{n\nu} = 1/n$ for $n \geqslant \nu$, $a_{n\nu} = 0$ for $n < \nu$, we have $|C^{\alpha}, 1|\text{-}\lim\limits_{\nu\to\infty} t_\nu = \varphi\text{-}\lim\limits_{\nu\to\infty} t_\nu = |A, \varphi|\text{-}\lim\limits_{\nu\to\infty} t_\nu$.

It is easily verified that T_φ^b is a linear space. Moreover, if the matrix A satisfies the assumptions of 1.1, then $T_0 \subset T_\varphi^b$.

1.3.2. We define in T_φ^b a norm

$$(***) \qquad ||x||_\varphi^b = \sup_n |||\bar{x}^n|||_\varphi .$$

Obviously, the space T_φ^b with norm (***) is a Banach space, and the coordinates t_n are continuous with respect to this norm.

1.3.3. *If* $x \in T_\varphi^b$, *then* $||x - x^k||_\varphi^b \to 0$ *as* $k \to \infty$.

Let $y = x - x^k$. Obviously, we have $|||\bar{y}^n|||_\varphi = 0$ for $n \leqslant k$. Since $\bar{y}^n \leqslant \bar{x}^n$, it follows from the monotony of norm (*) that $|||\bar{y}^n|||_\varphi \leqslant |||\bar{x}^n|||_\varphi$. Hence $|||\bar{y}^n|||_\varphi \to 0$ as $n \to \infty$.

Remark. The space T_φ^b is separable (see [4], p. 135).

1.3.4. *Let* $x \in T$. *In order that* $x \in T_\varphi^b$ *it is necessary and sufficient that* $||x^k - x^m||_\varphi^b \to 0$ *as* $k, m \to \infty$.

The necessity of 1.3.4 follows directly from 1.3.3, and the sufficiency is obvious (see [4], p. 135).

1.4. The norms of the sequences e^n, e_p^q, e_k may be easily calculated to be given by the following formulae:

$$|||e_k|||_\varphi = [\varphi_{-1}(1)]^{-1},$$

$$|||e^n|||_\varphi = [\varphi_{-1}(1/n)]^{-1},$$

$$|||e_p^q|||_\varphi = [\varphi_{-1}(1/q)]^{-1},$$

$$|||e^n|||_\varphi^* = n\varphi_{-1}^*(1/n)$$

(see [7], p. 453, and [4], p. 144),

$$|||\overline{e_k}^n|||_\varphi = \begin{cases} \varphi_{-1}(a_{nk})[\varphi_{-1}(1)]^{-1} & \text{for } k \leqslant n, \\ 0 & \text{for } k > n, \end{cases}$$

$$\frac{1}{n}\left[\varphi_{-1}\left(\frac{1}{n}\right)\right]^{-1} \sum_{\nu=1}^{n} \varphi_{-1}(a_{n\nu}) \leqslant |||\bar{e}^n|||_\varphi \leqslant [\varphi_{-1}(1)]^{-1} \sum_{\nu=1}^{n} \varphi_{-1}(a_{n\nu});$$

$$\frac{1}{r-p+1}\left[\varphi_{-1}\left(\frac{1}{r-p+1}\right)\right]^{-1} \sum_{\nu=p}^{r} \varphi_{-1}(a_{n\nu}) \leqslant |||\overline{e_p^{\varrho}}^n|||_\varphi \leqslant [\varphi_{-1}(1)]^{-1} \sum_{\nu=p}^{r} \varphi_{-1}(a_{n\nu}),$$

where $r = n$ if $p \leqslant n < p+q-1$ and $r = p+q-1$ if $p+q-1 \leqslant n$.

Moreover, writting $A_k = \sup_n a_{nk}$, we have

$$||e_k||_\varphi^b = \varphi_{-1}(A_k)[\varphi_{-1}(1)]^{-1},$$

$$\frac{1}{k}\left[\varphi_{-1}\left(\frac{1}{k}\right)\right]^{-1} \sup_n \sum_{\nu=1}^{k} \varphi_{-1}(a_{n\nu}) \leqslant ||e^k||_\varphi^b \leqslant [\varphi_{-1}(1)]^{-1} \sum_{\nu=1}^{k} \varphi_{-1}(A_\nu).$$

2.1. Let the matrix A possess the following property:

(+) For every $p, q > 0$ there exists n_0 such that for every $\nu \in \langle p, p+q-1\rangle$ and $n \geqslant n_0$ there exist constants $\mu' = \mu'(n, p, q)$ and $\mu'' = \mu''(n, p, q)$ satisfying the conditions $\mu', \mu'' \in \langle p, p+q-1\rangle$ and $a_{n+1,\mu'} \leqslant a_{n\nu} \leqslant a_{n-1,\mu''}$.

From this property we conclude that if $p = 2^k$, $q = 2^k$, $n = 2^{k+1}-1$, then for every $\nu \in \langle 2^k, 2^{k+1}-1\rangle$ there exist constants $\mu', \mu'' \in \langle 2^k, 2^{k+1}-1\rangle$ such that

$$\{\varphi_{-1}(a_{2^{k+2}-2,\mu''})\}^{-1} \leqslant \{\varphi_{-1}(a_{2^{k+2}-1,\nu})\}^{-1} \leqslant \{\varphi_{-1}(a_{2^{k+2},\mu'})\}^{-1}$$

for $a_{n\nu} \neq 0$. We write $a_{2^{k+2},\nu_k'} = \min a_{2^{k+2},\mu'}$ and $a_{2^{k+2}-2,\nu_k''} = \max a_{2^{k+2}-2,\mu''}$, where both the minimum and the maximum are taken over all $\mu', \mu'' \in \langle 2^k, 2^{k+1}-1\rangle$. In order to be brief we shall write

$$D(2^{k+2}-2, \nu_k'') = \{\varphi_{-1}(a_{2^{k+2}-2,\nu_k''})\}^{-1}, \qquad C(2^{k+2}, \nu_k') = \{\varphi_{-1}(a_{2^{k+2},\nu_k'})\}^{-1}.$$

In the sequel we shall suppose that the matrix possesses besides property (+), the following property:

$$\sup_{n>2} \sum_{k=0}^{n-1} \frac{C(2^{k+2}, \nu_k')}{D(2^n, \nu_k'')} < \infty. \tag{++}$$

It is easily seen that all matrices defining methods of strong (A, φ)-summability of sequences investigated in [9], pp. 243-246 possess the property (+). Examples of matrices possessing property (++) are given by matrices defining strong Cesàro summability of sequences of orders 1 and 2 and, by additional assumptions, matrices defining strong Riesz summability $(R, p, 1)$. The proofs of these properties require simple calculations only, and will not be performed here.

2.2. *Let*

$$P_l = \sum_{k=l}^{\infty} C(2^{k+2}, \nu_k') \, |||y_{2^k}^{2^k}|||_{\varphi*}.$$

For every sequence $x \in T_\varphi^b$ we have

$$\sum_{k=2^l}^{\infty} |t_k s_k| \leqslant 4 P_l Q_l(x) \tag{4}$$

where

$$Q_l(x) = \sup_{n \geqslant 2^l} |||\overline{(x - x^{2l-1})}^n|||_\varphi .$$

By (2) and (3) we obtain

$$\sum_{k=2^l}^{\infty} |t_k s_k| = \sum_{k=l}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1} |t_\nu s_\nu| \leqslant \sum_{k=l}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1} C(2^{k+2}, \nu_k') \varphi_{-1}(a_{2^{k+2}-1,\nu}) |t_\nu| |s_\nu|$$

$$\leqslant 4 \sum_{k=l}^{\infty} C(2^{k+2}, \nu_k') |||\overline{(x - x^{2^k-1})}^{2^{k+2}-1}|||_\varphi |||y_{2^k}^{2k}|||_{\varphi *} .$$

But

$$|||\overline{(x - x^{2^k-1})}^{2^{k+2}-1}|||_\varphi \leqslant |||\overline{(x - x^{2^l-1})}^{2^{k+2}-1}|||_\varphi \leqslant Q_l(x) .$$

Hence

$$\sum_{k=2^l}^{\infty} |t_k s_k| \leqslant 4 Q_l(x) \sum_{k=l}^{\infty} C(2^{k+2}, \nu_k') |||y_{2^k}^{2k}|||_{\varphi *} = 4 Q_l(x) P_l .$$

2.3. *Let $y = \{s_\nu\}$ be a sequence such that the series $S(x) = \sum_{\nu=1}^{\infty} t_\nu s_\nu$ is convergent for every sequence $x = \{t_\nu\} \in T_\varphi^b$. Then S is a linear functional over T_φ^b and the norm of S is $||S|| = \sup_x |S(x)|$, where $||x||_\varphi^b \leqslant 1$. It satisfies the inequality*

$$K \sum_{k=0}^{\infty} C(2^{k+2}, \nu_k') |||y_{2^k}^{2k}|||_{\varphi *} \leqslant ||S|| \leqslant 4 \sum_{k=0}^{\infty} C(2^{k+2}, \nu_k') |||y_{2^k}^{2k}|||_{\varphi *}, \tag{5}$$

where K is a positive constant.

Let $S_l(x) = \sum_{\nu=1}^{l} t_\nu s_\nu$. Obviously, S_l is a linear functional over T_φ^b, whence the functional S defined by the formula $S(x) = \lim_{l \to \infty} S_l(x)$ is linear over T_φ^b. The right-hand side of inequalities (5) is a special case of (4). The norm of S is estimated as follows. Arguing as in the case of functionals defined by an integral ([2], pp. 146, 150-152) we see that for every positive integer k there exists a sequence $x_k = \{t_\nu^k\}$ such that

$$|||(x_k)_{2^k}^{2k}|||_\varphi^* = 1 \quad \text{and} \quad \sum_{\nu=2^k}^{2^{k+1}-1} |t_\nu^k s_\nu| \geqslant \tfrac{1}{4} |||y_{2^k}^{2k}|||_{\varphi *}^* .$$

We define the following sequence $u = \{u_\nu\}$:

$$u_\nu = \begin{cases} C(2^{k+2}, \nu_k') |t_\nu^k| \operatorname{sign} s_\nu & \text{for } 2^k \leqslant \nu < 2^{k+1},\ k = 0, 1, \ldots, p-1, \\ 0 & \text{for remaining } k. \end{cases}$$

Obviously, $u \in T^b_\varphi$. Given n, we choose a number r such that $2^r \leqslant n < 2^{r+1}$. If $r \leqslant p-1$, then

$$|||\overline{u}^n|||_\varphi \leqslant |||\overline{u}^n|||^*_\varphi \leqslant \sup_y \Big(\sum_{k=0}^{r-1} \sum_{\nu=2^k}^{2^{k+1}-1} \varphi_{-1}(a_{n\nu}) u_\nu s_\nu + \sum_{\nu=2^r}^{n} \varphi_{-1}(a_{n\nu}) u_\nu s_\nu\Big)$$

$$\leqslant \sup_y \sum_{k=0}^{r-1} C(2^{k+2}, \nu_k') \sum_{\nu=2^k}^{2^{k+1}-1} \varphi_{-1}(a_{n\nu}) |t_\nu^k s_\nu| + C(2^{r+2}, \nu_r') \sup_y \sum_{\nu=2^r}^{n} \varphi_{-1}(a_{n\nu}) |t_\nu^r s_\nu|$$

$$\leqslant \sum_{k=0}^{r-1} \frac{C(2^{k+2}, \nu_k')}{D(2^r, \nu_k'')} |||(x_k)_{2^r}^{2^r}|||^*_\varphi + C(2^{r+2}, \nu_r') \varphi_{-1}(K) |||(x_r)_{2^r}^{2^r}|||^*_\varphi$$

$$= \sum_{k=0}^{r-1} \frac{C(2^{k+2}, \nu_k')}{D(2^r, \nu_k'')} + C(2^{r+2}, \nu_r') \varphi_{-1}(K) \leqslant L,$$

where the supremum is taken over all non-negative sequences y for which $\sum_{\nu=1}^{2^{k+1}-1} \varphi^*(s_\nu) \leqslant 1$, and the constant L depends on r; such a constant exists by condition (++).

If $r > p-1$, then

$$|||\overline{u}^n|||_\varphi \leqslant |||\overline{u}^n|||^*_\varphi \leqslant \sup_y \sum_{k=0}^{r} \sum_{\nu=2^k}^{2^{k+1}-1} \varphi_{-1}(a_{n\nu}) u_\nu s_\nu$$

$$= \sup_y \sum_{k=0}^{p-1} C(2^{k+2}, \nu_k') \sum_{\nu=2^k}^{2^{k+1}-1} \varphi_{-1}(a_{n\nu}) |t_\nu^k s_\nu|$$

$$\leqslant \sum_{k=0}^{p-1} \frac{C(2^{k+2}, \nu_k')}{D(2^p, \nu_k'')} |||(x_k)_{2^k}^{2^k}|||^*_\varphi = \sum_{k=0}^{p-1} \frac{C(2^{k+2}, \nu_k')}{D(2^p, \nu_k'')} \leqslant L.$$

Moreover,

$$S(u) = \sum_{k=0}^{\infty} \sum_{\nu=2^k}^{2^{k+1}-1} u_\nu s_\nu = \sum_{k=0}^{p-1} C(2^{k+2}, \nu_k') \sum_{\nu=2^k}^{2^{k+1}-1} |t_\nu^k| |s_\nu| \geqslant \tfrac{1}{4} \sum_{k=0}^{p-1} C(2^{k+2}, \nu_k') |||y_{2^k}^{2^k}|||^*_{\varphi^*}.$$

However, $L||S|| \geqslant |S(u)|$. Hence

$$||S|| \geqslant \frac{1}{4L} \sum_{k=0}^{p-1} C(2^{k+2}, \nu_k') |||y_{2^k}^{2^k}|||^*_{\varphi^*} \geqslant \frac{1}{4L} \sum_{k=0}^{p-1} C(2^{k+2}, \nu_k') |||y_{2^k}^{2^k}|||_{\varphi^*}.$$

Since p is arbitrary, we obtain the left-hand side of inequalities (5).

3. We may formulate the following theorem:

(a) *If* S *is a linear functional over* T_{φ}^{b}, *then there exists a sequence* $y = \{s_\nu\}$ *such that*

$$S(x) = \sum_{\nu=1}^{\infty} t_\nu s_\nu \quad \textit{for every } x = \{t_\nu\} \in T_{\varphi}^{b} \tag{i}$$

and

$$\sum_{k=0}^{\infty} C(2^{k+2}, \nu_k') \, |||y_{2^k}^{2^k}|||_{\varphi*} < \infty. \tag{ii}$$

(b) *If* $y = \{s_\nu\}$ *is a real sequence satisfying inequality* (ii), *then formula* (i) *defines a linear functional* S *over* T_{φ}^{b}, *and its norm satisfies inequalities* (5); *series* (i) *is then absolutly convergent for every* $x = \{t_\nu\} \in T_{\varphi}^{b}$.

3.1. *The condition* $|A, \varphi|$- $\lim_{\nu\to\infty} t_\nu = g$ *implies*

$$\lim_{m\to\infty} \sum_{\nu=1}^{\infty} t_\nu s_\nu^m = g \tag{6}$$

for every $|A, \varphi|$*-summable sequence* $\{t_\nu\}$ *if and only if*

$$\lim_{m\to\infty} \sum_{\nu=r}^{\infty} s_\nu^m = 1 \quad \textit{for } r = 1, 2, \ldots \tag{7}$$

and

$$\overline{\lim_{m\to\infty}} \sum_{k=0}^{\infty} C(2^{k+2}, \nu_k') \, |||(y_m)_{2^k}^{2^k}|||_{\varphi*} < \infty, \tag{8}$$

where $y_m = \{s_\nu^m\}$.

To prove the necessity of (7) we choose $\{t_\nu\}$ as follows: $t_\nu = 0$ for $\nu < r$, $t_\nu = 1$ for $\nu \geqslant r$. (6) implies (7).

Inequality (8) follows from 2.3 and from the Banach-Steinhaus theorem.

Let us write

$$\sum_{\nu=1}^{\infty} t_\nu s_\nu^m = \sum_{\nu=1}^{2^l-1} (t_\nu - g) s_\nu^m + \sum_{\nu=2^l}^{\infty} (t_\nu - g) s_\nu^m + g \sum_{\nu=1}^{\infty} s_\nu^m.$$

Applying 2.2, the sufficiency of conditions (7) and (8) is easily verified.

3.2. In particular, if $a_{n\nu} = 1/n$ for $n \geqslant \nu$ and $a_{n\nu} = 0$ for $n < \nu$, we obtain the following theorem (see [7], p. 457):

The condition $\varphi\text{-}\lim\limits_{\nu\to\infty} t_\nu = g$ *implies* $\lim\limits_{m\to\infty} \sum\limits_{\nu=1}^{\infty} t_\nu s_\nu^m = g$ *for every* φ*-summable sequence* $\{t_\nu\}$ *if and only if*

$$\lim_{m\to\infty} \sum_{\nu=r}^{\infty} s_\nu^m = 1 \quad \textit{for } r = 1, 2, \ldots$$

and

$$\overline{\lim_{m\to\infty}} \sum_{k=0}^{\infty} \{\varphi_{-1}(2^{-(k+2)})\}^{-1} |||(y_m)_{2^k}^{2^k}|||_{\varphi*} < \infty,$$

where $y_m = \{s_\nu^m\}$.

3.3. In case when $\varphi(u) = \alpha^{-1}u^\alpha$, $\alpha > 1$, if the matrix A possesses the same properties as in 3.2, we have $C(2^{k+2}, \nu_k') = (\alpha^{-1}2^{k+2})^{1/\alpha}$. Then we get a theorem corresponding to the Tandori theorem for integrals (see [8], pp. 226-228), which is equivalent to the theorem given in [1], p. 629.

The condition $|C^\alpha, 1|\text{-}\lim\limits_{\nu\to\infty} t_\nu = g$, $\alpha > 1$, *implies* $\lim\limits_{m\to\infty} \sum\limits_{\nu=1}^{\infty} t_\nu s_\nu^m = g$ *for every* $|C^\alpha, 1|$*-summable sequence* $\{t_\nu\}$ *if and only if condition* (7) *is satisfied and*

$$\overline{\lim_{m\to\infty}} \sum_{k=0}^{\infty} 2^{k/\alpha} \Big(\sum_{\nu=2^k}^{2^{k+1}-1} |s_\nu^m|^\beta \Big)^{1/\beta} < \infty,$$

where $1/\alpha + 1/\beta = 1$.

References

[1] D. Borwein, *Linear functionals connected with strong Cesàro summability*, Journal London Math. Soc. 40 (1965), pp. 628-634.

[2] M. A. Krasnoselskiĭ and B. Rutickiĭ, *Convex functions and Orlicz spaces* (in Russian), Moscow 1958.

[3] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), pp. 49-65.

[4] — *On modular spaces of strongly summable sequences*, ibidem 22 (1962), pp. 127-147.

[5] W. Orlicz, *On some spaces of strongly summable sequences*, ibidem 22 (1963), pp. 331-336.

[6] R. Taberski, *O zbieżności całek osobliwych w punktach Lebesgue'a-Orlicza pewnych funkcji*, Prace Mat. 5 (1961), pp. 33-42 (in Polish). (On the convergence of singular integrals at Lebesgue-Orlicz points of some functions).

[7] — *A theorem of Toeplitz type for the class of M-summable sequences*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 8 (1960), pp. 453-458.
[8] K. Tandori, *Über die Konvergenz singulärer Integrale*, Acta Scient. Math. Szeged 15 (1954), pp. 223-230.
[9] A. Waszak, *On spaces of strongly summable sequences with an Orlicz metric*, Prace Mat. 11 (1967), pp. 229-246.

I DEPARTMENT OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY
Poznań