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On a generalization of Jensen's formula

Let $f(z)$ be an analytic function in the disc $|z| < R$ such that $f(0) \neq 0$, and let $n(r)$ be the number of zeros of $f(z)$ in the disc $|z| \leqslant r < R$. The function $n(r)$ is connected with $f(z)$ by means of the Jensen's formula

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\varphi})|\,d\varphi = \ln|f(0)| + \int_0^r \frac{n(x)}{x}\,dx.$$

This formula may be treated as a formula defining the coefficient $a_0(r)$ of the expansion of the function $\ln|f(re^{i\varphi})|$ in the Fourier series. We shall prove that calculating the remaining coefficients

$$(2) \qquad a_n(r) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\varphi})|\,e^{-in\varphi}\,d\varphi \qquad (n = 1, 2, \ldots)$$

we obtain formulae which enable us to simplify the proofs of some classical theorems in the theory of analytic functions, in particular, Hadamard's theorem on factorization of an entire function of finite order. This proof of Hadamard's theorem does not require any estimations of the canonical product or of the real part of the function $f(z)$.

1. Calculation of the coefficients $a_n(r)$. If $z_1, z_2, \ldots$ are zeros of $f(z)$ arranged in the order of increasing moduli $r_1 \leqslant r_2 \leqslant \ldots$, then the function $\ln|f(re^{i\varphi})|$ possesses continuous derivatives with respect to r and φ in the disc $|z| < r_1$, and in every ring $r_{n-1} < r < r_n$ $(n = 2, 3, \ldots)$, $0 \leqslant \varphi < 2\pi$. Applying the identity

$$\ln|f(re^{i\varphi})| = 1/2\,[\log f(re^{i\varphi}) + \log\overline{f(re^{i\varphi})}]$$

we obtain

$$d\ln|f(re^{i\varphi})| = \mathrm{Re}\left[\frac{df(re^{i\varphi})}{f(re^{i\varphi})}\right] = \mathrm{Re}\left[\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}e^{i\varphi}\right]dr - \mathrm{Im}\left[r\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}e^{i\varphi}\right]d\varphi.$$

Hence

$$(3) \qquad \begin{aligned} \frac{\partial}{\partial r}[\ln|f(re^{i\varphi})|] &= \mathrm{Re}\left[\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}e^{i\varphi}\right], \\ \frac{\partial}{\partial \varphi}[\ln|f(re^{i\varphi})|] &= -r\,\mathrm{Im}\left[\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}e^{i\varphi}\right]. \end{aligned}$$

Integrating (2) by parts and applying (3) we get

$$a_n(r) = \frac{ri}{2\pi n} \int_0^{2\pi} \operatorname{Im}\left[\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} e^{i\varphi}\right] e^{-in\varphi} d\varphi .$$

Differentiating (2) under the sign of the integral we have

$$a_n'(r) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left[\frac{f'(re^{i\varphi})}{f(re^{i\varphi})} e^{i\varphi}\right] e^{-in\varphi} d\varphi .$$

Writing

$$\varphi_n(r) = \frac{1}{2\pi i} \int_{K(0,r)} [f'(z)/f(z)] z^{-n} dz,$$

the last formulae yield the linear differential equation

$$a_n'(r) + \frac{n}{r} a_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f'(re^{i\varphi})}{f(re^{i\varphi})} e^{i\varphi} e^{-in\varphi} d\varphi = r^{n-1} \varphi_n(r),$$

which gives after integration

$$(4) \qquad a_n(r) = a_n(r_0')(r_0'/r)^n + r^{-n} \int_{r_0'}^{r} t^{2n-1} \varphi_n(t)\, dt$$

$(r_m < r_0' \leqslant t \leqslant r < r_{m+1},\ m = 0, 1, \ldots;\ r_0 = 0)$.

Since $f(0) \neq 0$ we have in a neighbourhood of zero

$$(5) \qquad f'(z)/f(z) = c_0' + c_1' z + \ldots + c_{n-1}' z^{n-1} + \ldots$$

and so

$$\varphi_n(t) = \begin{cases} n(t) & \text{for } n = 0, \\ c_{n-1}' + \sum\limits_{|z_\nu| \leqslant t} z_\nu^{-n} & \text{for } n > 0. \end{cases}$$

Hence it follows that $\varphi_n(t)$ is a step function having steps at the points $t_\nu = r_\nu$. Thus the right-hand side of formula (4) is continuous at the points r_ν. Analogously to [3], p. 149, we prove that the left-hand side of this formula (as defined by (2)) is continuous for $0 \leqslant r < R$. Hence formula (4) remains valid for $0 < r < R$. Setting $n = 0$ and $r_0' = 0$ we obtain formula (1). If $n > 0$ and $r_0' \to 0$ we have

$$(6) \qquad a_n(r) = \frac{c_{n-1}'}{2n} r^n + r^{-n} \int_0^r t^{2n-1} \sum_{|z_\nu| \leqslant t} z_\nu^{-n} dt \qquad (n = 1, 2, \ldots).$$

If the function $f(z)$ possesses no zeros, the second term on the right-hand side of this equality is omitted. The integral in this term may be removed integrating by parts. Namely, we have

$$\int_0^r t^{2n-1} \sum_{|z_\nu|\leqslant t} z_\nu^{-n}\, dt = \frac{r^{2n}}{2n} \sum_{|z_\nu|\leqslant r} z_\nu^{-n} - \frac{1}{2n} \sum_{r_\nu\leqslant r} r_\nu^{2n} \sum_{\mu} z_{a_\mu^\nu}^{-n},$$

where $z_{a_\mu^\nu}$ are zeros of the function $f(z)$ lying on the circle $|z| = r_\nu$. Dividing both sides of (6) by r^n we thus obtain

(7) $$r^{-n} a_n(r) = (1/2n)\Big[c'_{n-1} + \sum_{|z_\nu|\leqslant r} z_\nu^{-n} - r^{-n} \sum_{|z_\nu|\leqslant r} (\bar{z}_\nu/r)^n\Big].$$

2. Applications.

2.1. From (6) we may immediately obtain formulae expressing the coefficients of the power series by means of the real part of the function. For this purpose let us write $f(z) = \exp(h(z))$, $h(z) = \sum_{n=0}^{\infty} c_n z^n$. Then $f'(z)/f(z) = h'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$, $c'_{n-1} = n c_n$, $\operatorname{Re} h(z) = U(r, \varphi) = \ln|f(re^{i\varphi})|$, $\ln|f(0)| = \operatorname{Re} c_0$, $n(t) \equiv 0$. From (1), (2) and (6) we thus obtain

$$\frac{1}{\pi}\int_0^{2\pi} U(r, \varphi) e^{-in\varphi} d\varphi = \begin{cases} 2\operatorname{Re} c_0 & \text{for } n = 0, \\ r^n c_n & \text{for } n > 0 \end{cases}$$

(compare [2], p. 321).

Further applications will be based upon an estimation of the coefficients $a_n(r)$. Let us write $M(r) = \max_{|z|\leqslant r} |f(z)|$, $\ln|f(re^{i\varphi})| = \Phi(r, \varphi)$, $E_1(r) = \{\varphi\colon \Phi(r, \varphi) \geqslant 0\}$, $E_2(r) = \{\varphi\colon \Phi(r, \varphi) < 0\}$. Then

$$|a_n(r)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |\Phi(r, \varphi)|\, d\varphi$$

$$= \frac{1}{2\pi}\int_{E_1} \Phi(r, \varphi)\, d\varphi + \frac{-1}{2\pi}\int_{E_2} \Phi(r, \varphi)\, d\varphi = A_1 + A_2.$$

It follows from formula (1) that $0 \leqslant A_1 \leqslant \ln M(r)$, $A_1 - A_2 \geqslant \ln|f(0)|$; hence

(8) $$A_2 \leqslant A_1 - \ln|f(0)|, \qquad |a_n(r)| \leqslant 2\ln M(r) + \big|\ln|f(0)|\big|.$$

2.2. First let us assume that $f(z)$ is an entire function of finite order ϱ, possessing no zeros. If $n > \varrho$, then $\lim_{r\to\infty} r^{-n} \ln M(r) = 0$, by the definition of the order. Dividing both sides of (6) by r^n and taking the limit as $r \to \infty$ and $n > \varrho$ we have $c'_{n-1} = 0$, by (8). In view of (5) this means that

$f'(z)/f(z)$ is a polynomial of degree $k \leqslant \varrho - 1$. Hence $f(z) = e^{Q(z)}$, where $Q(z)$ is a polynomial of degree $q \leqslant \varrho$. In particular, each polynomial possessing no zeros is constant ($\varrho = 0$). Thus we obtain the fundamental theorem of algebra.

2.3. Let us assume that $f(z) = H(z)P(z)$, where $H(z)$ is an entire function without zeros, and $P(z) = \prod_{n=1}^{\infty}(1 - z/z_n)\exp\big(\sum_{k=1}^{p} k^{-1}(z/z_n)^k\big)$ is the canonical product of $f(z)$. Then $c_k' = c_k'(f) = c_k'(H) + c_k'(P)$ and $c_k'(P) = -\sum_{\nu=1}^{\infty} z_\nu^{-k-1}$ for $k \geqslant p$. Hence we obtain from (7) for $n > p$

$$(9) \qquad r^{-n} a_n(r) = \frac{1}{2n}\Big[c_{n-1}'(H) - \sum_{\nu=1}^{\infty} z_\nu^{-n} + \sum_{|z_\nu| \leqslant r} z_\nu^{-n} - r^{-n} \sum_{|z_\nu| \leqslant r} (\bar{z}_\nu/r)^n\Big].$$

Moreover, $\big|\sum_{|z_\nu| \leqslant r} (\bar{z}_\nu/r)^n\big| \leqslant n(r)$ and $\lim_{r\to\infty} r^{-n} n(r) = 0$ for $n > \mu$, where $\mu \leqslant \varrho$ is the exponent of convergence of zeros (compare e.g. [1]). Taking in (9) the limit as $r \to \infty$ and $n > \varrho$ we obtain $c_{n-1}'(H) = 0$, by (8). This proves just as in 2.2 that $H(z) = e^{Q(z)}$, where $Q(z)$ is a polynomial of degree $q \leqslant \varrho$.

References

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