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On the osculating plane of a curve in an n-dimensional Euclidean space

Van der Waag [7] distinguishes the eight types of an osculating plane of a curve in the 3-dimensional Euclidean space and gives the conditions for their existence and the dependence between the osculating planes of different types. Radziszewski [5] gives in the terms of the metric of the space the necessary and sufficient conditions for the existence of the osculating planes of the different types in the sense of van der Waag. In [6] gives that conditions in the sense of Bouliguand [1]. But in the general theory of curve a necessary and sufficient condition for the existence of the osculating plane of a curve is usually given in the terms of the continuous second derivatives of the functions describing a curve. However, it is a rather strong condition, because an osculating plane exists also under weaker that.

In the present paper we formulate the sufficient conditions for the existence of the osculating plane without resolving to the second continuous derivatives of functions describing a curve.

1. Preliminaries. We consider a fixed rectangle cartesian coordinate system in an n-dimensional Euclidean space E^n and continuous curve L represented parametrically by the equations $x^i = x^i(t)$ $(i = 1, \ldots, n)$ with the orientation corresponding to increasing parameter t. By a regular triplet $(P_1|P_3|P_2)$ of L we mean three not collinear distinct points $P_1(x_1^1, \ldots, x_1^n)$, $P_2(x_2^1, \ldots, x_2^n)$, $P_3(x_3^1, \ldots, x_3^n)$ (where $x_r^i = x^i(t_r)$) of an arc of L such that P_3 lies between P_1 and P_2. The unspecified summation $\sum$ will mean the summation over the set $\{1, 2, \ldots, n\}$; the sum $\sum_j$ will mean the sum over all cyclic permutations of elements of the set $\{n-1, n, j\}$, e.g. $\sum_j V^{i_1 i_2}$ will mean the sum $V^{n-1\,n} + V^{nj} + V^{j\,n-1}$.

Let $(P_1|P_3|P_2)$ be an arbitrary regular triplet of L. The plane $H^2(t_1, t_2, t_3)$ containing points P_1, P_2, P_3 is uniquely oriented by the two vectors

(1.1) $$\boldsymbol{P_3P_s} = \sum (x_s^i - x_3^i)\boldsymbol{k}_i \quad (s = 1, 2),$$

where $\boldsymbol{k}_i$ is an unit vector in direction of i-th ax of coordinate system. Moreover, $H^2(t_1, t_2, t_3)$ can be described by the system of $n-2$ equations

$$(x^j - x_3^j)V^{n-1\,n} + (x^{n-1} - x_3^{n-1})V^{nj} + (x^n - x_3^n)V^{j\,n-1} = 0$$
$$(j = 1, 2, \ldots, n-2),$$

where $V^{pr} = V_1^p V_2^r - V_2^p V_1^r$ and $V_i^r = (x_i^r - x_3^r)/(t_i - t_3)$ and generally $V_{mq}^r = (x_m^r - x_q^r)/(t_m - t_q)$ $(m, q = 1, 2, 3;\ m \neq q;\ i = 1, 2;\ p, r = 1, 2, \ldots, n)$ (cf. [3]). Hence, $H^2(t_1, t_2, t_3)$ is the common part of $n-2$ $(n-1)$-dimensional hyperplanes H_j^{n-1}. Note that H_j^{n-1} $(j = 1, 2, \ldots, n-2)$ is oriented by the vector $\boldsymbol{H}_j$ with components:

$$(1.2) \qquad (\underbrace{0, 0, \ldots, 0}_{(j-1)}, V^{n-1\,n}, \underbrace{0, 0, \ldots, 0}_{(n-j-2)}, V^{nj}, V^{j\,n-1});$$

(1.2) determine the direction of $\boldsymbol{H}_j$. A square of lenght of $\boldsymbol{H}_j$ is

$$|\boldsymbol{H}_j|^2 = (V^{n-1\,n})^2 + (V^{nj})^2 + (V^{j\,n-1})^2 \overset{\text{df}}{=} \sum\nolimits_j (V^{i_1 i_2})^2$$

(cf. remark about the summation) and in consequence of this fact the unit vector in direction determined by (1.2) have the following not vanishing components:

$$(1.3) \qquad \left(\frac{V^{n-1\,n}}{\sqrt{\sum_j (V^{i_1 i_2})^2}}, \frac{V^{nj}}{\sqrt{\sum_j (V^{i_1 i_2})^2}}, \frac{V^{j\,n-1}}{\sqrt{\sum_j (V^{i_1 i_2})^2}}\right)$$

for respective j. The system of $n-2$ vectors $\boldsymbol{H}_j$ $(j = 1, 2, \ldots, n-2)$ gives the orientation of $H^2(t_1, t_2, t_3)$ as well as vectors (1.1).

Now consider the other case. By a regular pair $(P_1|P_2)$ of L we mean two distinct points P_1 and P_2 lying on an arc of L and such that P_1 precedes P_2 on L. Let $(P_1|P_2)$ be a regular pair of L. The plane $H^2(t_1, v_2)$ containing the points P_1, P_2 and the straight line v_2 tangent to L at P_2 is described by the system of $n-2$ equations

$$(x^j - x_2^j)^1V^{n-1\,n} + (x^{n-1} - x_2^{n-1})^1V^{nj} + (x^n - x_2^n)^1V^{j\,n-1} = 0$$
$$(j = 1, 2, \ldots, n-2)$$

(where ${}^1V^{pr} = V_1^p \dot{x}_2^r - V_1^r \dot{x}_2^p$ and $\dot{x}_2^p = \dot{x}^p(t_2)$; $p, r = 1, 2, \ldots, n$) if P_1 does not lie on v_2. Then the system of $n-2$ vectors ${}^1\boldsymbol{H}_j$ $(j = 1, 2, \ldots, n-2)$ with the not vanishing components:

$$(1.4) \qquad \left(\frac{{}^1V^{n-1\,n}}{\sqrt{\sum_j ({}^1V^{i_1 i_2})^2}}, \frac{{}^1V^{nj}}{\sqrt{\sum_j ({}^1V^{i_1 i_2})^2}}, \frac{{}^1V^{j\,n-1}}{\sqrt{\sum_j ({}^1V^{i_1 i_2})^2}}\right)$$

for respective j, gives the orientation of $H^2(t_1, v_2)$.

2. Definitions of the osculating planes. Let $\{P_1^\alpha | P_3^\alpha | P_2^\alpha\}$ $(\alpha = 1, 2, \ldots$ always) be a sequence of regular triplets of L. Since each element of $\{P_1^\alpha | P_3^\alpha | P_2^\alpha\}$ determine the plane $H^2(t_1, t_2, t_3)^\alpha$ for respective α, hence we have a sequence $\{H^2(t_1, t_2, t_3)^\alpha\}$ of planes corresponding to a sequence $\{P_1^\alpha | P_3^\alpha | P_2^\alpha\}$.

It is known that if points $P_1^\alpha, P_3^\alpha, P_2^\alpha$ of the triplet $(P_1^\alpha | P_3^\alpha | P_2^\alpha)$ converge along the curve to a point P_0 (indepedently one from another) and the limit H_r^2 of planes $H^2(t_1, t_2, t_3)^\alpha$ exists, then H_r^2 is the osculating plane of L at P_0. However, one may expect that the osculating plane H_r^2 depends on the position of P_0 with the respect to triplets $(P_1^\alpha | P_3^\alpha | P_2^\alpha)$ and just for this reason four definitions describing 4 types of convergence of $(P_1^\alpha | P_3^\alpha | P_2^\alpha)$ to P_0 are introduced. We shall prove the existence of an osculating plane in each of the cases described by these definitions, without referring to second-order continuous derivatives of the functions describing L.

DEFINITION 1. If $(P_1^\alpha | P_3^\alpha | P_2^\alpha)$ converges along the curve to a point $P_0 \in L$, then by the *osculating plane* ${}^0H_r^2$ *of* L *at* P_0 we mean the limit of the planes $H^2(t_1, t_2, t_3)^\alpha$, if it exists. Thus

$$ {}^0H_r^2 = \lim_{(P_1^\alpha | P_3^\alpha | P_2^\alpha) \to P_0} H^2(t_1, t_2, t_3)^\alpha \qquad (r = 1, 2, 3, 4) $$

and we have:

a) ${}^0H_1^2$ if P_0 does not belong to any arc $P_1^\alpha P_2^\alpha$;

b) ${}^0H_2^2$ if P_0 belongs to each arc $P_1^\alpha P_2^\alpha$;

c) ${}^0H_3^2$ if $P_3^\alpha = P_0$ in the regular triplet $(P_1^\alpha | P_3^\alpha | P_2^\alpha)$;

d) ${}^0H_4^2$ if $P_1^\alpha = P_0$ in the regular triplet.

Thus, ${}^0H_r^2$ $(r = 1, 2, 3, 4)$ exists if and only if the corresponding finite limits of (1.3) exist, the convergence being understood in the respective sense. These limits define the components of vectors giving the orientation of ${}^0H_1^2, {}^0H_2^2, {}^0H_3^2, {}^0H_4^2$, respectively.

Let $\{P_1^\alpha | P_2^\alpha\}$ be a sequence of the regular pairs of L and $\{H^2(t_1, v_2)^\alpha\}$ be a sequence of planes corresponding to $\{P_1^\alpha | P_2^\alpha\}$.

DEFINITION 2. If $(P_1^\alpha | P_2^\alpha)$ converges along the curve to a point $P_0 \in L$, then by the *osculating plane* ${}^1H_r^2$ *of* L *at* P_0 we mean the limit of the planes $H^2(t_1, v_2)^\alpha$, if it exists. Thus

$$ {}^1H_r^2 = \lim_{(P_1^\alpha | P_2^\alpha) \to P_0} H^2(t_1, v_2)^\alpha \qquad (r = 1, 2, 3, 4) $$

and we have:

a) ${}^1H_1^2$ if P_0 does not belong to any arc $P_1^\alpha P_2^\alpha$;

b) ${}^1H_2^2$ if P_0 belongs to each arc $P_1^\alpha P_2^\alpha$;

c) ${}^1H_3^2$ if $P_1^\alpha = P_0$ in the regular pair $(P_1^\alpha | P_2^\alpha)$;

d) ${}^1H_4^2$ if $P_2^\alpha = P_0$ in the regular pair.

It follows that ${}^1H^2_r$ $(r = 1, 2, 3, 4)$ exists if and only if the corresponding limits of (1.4) exist, the convergence being understood in the respective sense. These limits define the components of vectors giving the orientation of ${}^1H^2_1$, ${}^1H^2_2$, ${}^1H^2_3$, ${}^1H^2_4$, respectively.

It is clear that the existence of ${}^hH^2_1$ or ${}^hH^2_2$ $(h = 0, 1)$ implies the existence of both ${}^hH^2_3$ and ${}^hH^2_4$. We should note that these definitions depend on the fixed coordinate system; it may happen that ${}^hH^2_r$ $(h = 0, 1;$ $r = 1, 2, 3, 4)$ exists in one coordinate system and does not exist in another one.

The expressions (1.3) and (1.4) there are undeterminate in the limit always, because there we have fractions containing determinants $V^{i_1i_2}$ or ${}^1V^{i_1i_2}$ with the equal rows, as in the numerator, as in the denominator. To avoid that we must transform (1.3) [and (1.4)]. Substract the second row from the first one in each determinant $V^{i_1i_2}$ [resp. ${}^1V^{i_1i_2}$], appearing as in the numerator, as in the denominator, and each obtained difference divide by binomial $t_2 - t_1$. Introducing the designations

$$M^k_{12;3} = \frac{V^k_1 - V^k_2}{t_1 - t_2} \quad \text{and} \quad {}^1M^k_{12;2} = \frac{V^k_1 - \dot{x}^k_2}{t_1 - t_2} \quad \text{for a fixed } k$$

we can write (1.3) and (1.4) then in the form

$$\left(\frac{M^n_{12;3}V^{n-1}_2 - M^{n-1}_{12;3}V^n_2}{\sqrt{\sum_j (M^{i_1}_{12;3}V^{i_2}_2 - M^{i_2}_{12;3}V^{i_1}_2)^2}}, \frac{M^j_{12;3}V^n_2 - M^n_{12;3}V^j_2}{\sqrt{\sum_j (M^{i_1}_{12;3}V^{i_2}_2 - M^{i_2}_{12;3}V^{i_1}_2)^2}}, \frac{M^{n-1}_{12;3}V^j_2 - M^j_{12;3}V^{n-1}_2}{\sqrt{\sum_j (M^{i_1}_{12;3}V^{i_2}_2 - M^{i_2}_{12;3}V^{i_1}_2)^2}}\right), \tag{2.1}$$

$$\left(\frac{{}^1M^n_{12;2}\dot{x}^{n-1}_2 - {}^1M^{n-1}_{12;2}\dot{x}^n_2}{\sqrt{\sum_j ({}^1M^{i_1}_{12;2}\dot{x}^{i_2}_2 - {}^1M^{i_2}_{12;2}\dot{x}^{i_1}_2)^2}}, \frac{{}^1M^j_{12;2}\dot{x}^n_2 - {}^1M^n_{12;2}\dot{x}^j_2}{\sqrt{\sum_j ({}^1M^{i_1}_{12;2}\dot{x}^{i_2}_2 - {}^1M^{i_2}_{12;2}\dot{x}^{i_1}_2)^2}}, \frac{{}^1M^{n-1}_{12;2}\dot{x}^j_2 - {}^1M^j_{12;2}\dot{x}^{n-1}_2}{\sqrt{\sum_j ({}^1M^{i_1}_{12;2}\dot{x}^{i_2}_2 - {}^1M^{i_2}_{12;2}\dot{x}^{i_1}_2)^2}}\right), \tag{2.2}$$

respectively. It is easy to verify that (2.1) and (2.2) there are not always undetermined in the limit.

3. The sufficient conditions for the existence of the osculating plane. All the osculating planes ${}^hH^2_r$ $(h = 0, 1;\ r = 1, 2, 3, 4)$ of L at P_0 exist if the second derivatives $\ddot{x}^k(t)$ $(k = 1, 2, \ldots, n)$ are continuous at t_0. The last condition is very strong and, in fact, all those planes exist under weaker conditions.

Before we formulate the theorem about the existence of the osculating plane, we shall give some lemmas presented in [2] and [4].

LEMMA 1 (cf. [2], L. 2 and [4], L. 3). *The finite limit*

$$\lim_{\substack{t_1,t_2,t_3\to t_0\\(t_1-t_3)(t_2-t_3)<0}} M^k_{12;3} = \frac{1}{2}B^k \quad (k=1,2,\ldots,n) \tag{3.1}$$

exists if and only if there exists the finite limit

$$\lim_{\substack{t_1,t_2\to t_0\\t_1\neq t_2}} \frac{\dot{x}^k_1-\dot{x}^k_2}{t_1-t_2} = B^k. \tag{3.2}$$

LEMMA 2 (cf. [4], L. 3). *The finite limit* $\lim\limits_{\substack{t_1,t_2\to t_0\\t_1\neq t_2}} {}^1M^k_{12;2} = \frac{1}{2}B^k$ $(k=1,2,\ldots,n)$ *exists if and only if there exists the finite limit* (3.2).

LEMMA 3 (cf. [2], L. 2). *The finite limit* (3.1) *exists if and only if there exist the finite limits*

$$\lim_{\substack{t_p,t_3\to t_0\\t_p\neq t_3}} \frac{V^k_{p3}-\dot{x}^k_3}{t_p-t_3} = \frac{1}{2}B^k \quad (p=1,2;\ k=1,2,\ldots,n)$$

and is $(t_1-t_3)(t_2-t_3)<0$.

Seting t_0 instead of t_3 in Lemma 3 and applying Lemma 2 we get

LEMMA 4. *The finite limit* $\lim\limits_{\substack{t_1,t_2\to t_0\\t_1\neq t_2}} M^k_{12;0} = \frac{1}{2}B^k$ $(k=1,2,\ldots,n)$ *exists if and only if the finite derivative* $\ddot{x}^k(t_0)=B^k$ *exists at* t_0 (*is here* $M^k_{12;0}=(V^k_{10}-V^k_{20})/(t_1-t_2)$).

LEMMA 5. (cf. [4], L. 2). *If the derivative* $\dot{x}^k(t)$ *at* t_0 *exists* $(k=1,2,\ldots$ $\ldots,n)$ *and is finite, then the finite limit* $\lim\limits_{t_2\to t_0}[(\dot{x}^k_2-V^k_{02})/(t_2-t_0)]=\frac{1}{2}B^k$ *exists if and only if the finite second derivative* $\ddot{x}^k(t_0)=B^k$ *exists at* t_0.

DEFINITION. If the function $x^k(t)$ is defined in a closed interval $[y, y+c]$ $(c>0)$, then by the *right-side upper derivative of the function* $x^k(t)$ *at* y we mean

$$\overset{+}{x}{}^k(y) = \limsup_{h\to 0\ h>0} \frac{x^k(y+h)-x^k(y)}{h}.$$

LEMMA 6 (cf. [4], L. 4 and [2], L. 6). *Let the function* $x^k(t)$ *be continuous in a certain neighbourhood* (t_0-c, t_0+c) *of* t_0 $(c>0)$. *Then the finite limit* $\lim\limits_{\substack{t_2,t_3\to t_0\\(t_2-t_3)(t_0-t_3)<0}}[(V^k_{23}-V^k_{03})/(t_2-t_0)]=\frac{1}{2}B^k$ *exists if and only if there exists the finite limit*

$$\lim_{t\to t_0} \frac{\overset{+}{x}{}^k(t)-\dot{x}^k_0}{t-t_0} = \frac{1}{2}B^k \quad (k=1,2,\ldots,n). \tag{3.3}$$

THEOREM. *Let the curve L be described by the functions $x^k(t)$ $(k = 1, 2, \ldots, n)$, continuous in a certain neighbourhood of t_0, and let the derivatives $\dot{x}^k(t)$ exist at t_0 and satisfy $|\dot{x}^k(t_0)| < \infty$ for each superscript k. Then the osculating plane*

(A) ${}^1H_4^2$ *exists if the limits*

$$\lim_{t_1 \to t_0} \frac{V_{10}^k - \dot{x}_0^k}{t_1 - t_0} = \frac{1}{2} B^k \tag{3.4}$$

exist, are finite and $\sum |B^k| \neq 0$;

(B) ${}^0H_4^2$ *exists if the limits*

$$\lim_{\substack{t_2, t_3 \to t_0 \\ t_2 \neq t_3}} V_{23}^k = \dot{x}_0^k \tag{3.5}$$

and (3.3) *exist, are finite and* $\sum |B^k| \neq 0$;

(C) ${}^0H_3^2$ *and* ${}^1H_3^2$ *exist if the derivatives $\dot{x}^k(t)$ are continuous at t_0 and the second derivatives $\ddot{x}^k(t_0) = B^k$ exist, are finite and does not vanish simultaneously;*

(D) ${}^hH_1^2$ *and* ${}^hH_2^2$ *exist* $(h = 0, 1)$ *if the derivatives $\dot{x}^k(t)$ are continuous at t_0 and limits* (3.2) *exist, are finite and* $\sum |B^k| \neq 0$.

The osculating plane ${}^hH_r^2$ $(h = 0, 1;\ r = 1, 2, 3, 4)$ is described by the system of $n-2$ equations

$$(x^j - x_0^j)(B^n \dot{x}_0^{n-1} - B^{n-1}\dot{x}_0^n) + (x^{n-1} - x_0^{n-1})(B^j \dot{x}_0^n - B^n \dot{x}_0^j) + \\ + (x^n - x_0^n)(B^{n-1}\dot{x}_0^j - B^j \dot{x}_0^{n-1}) = 0 \tag{3.6}$$

$(j = 1, 2, \ldots, n-2)$ *and have the orientation given by $n-2$ vectors with the not vanishing components*:

(3.7)

$$\left(\frac{B^n \dot{x}_0^{n-1} - B^{n-1}\dot{x}_0^n}{\sqrt{\sum_j (B^{i_1}\dot{x}_0^{i_2} - B^{i_2}\dot{x}_0^{i_1})^2}}, \frac{B^j \dot{x}_0^n - B^n \dot{x}_0^j}{\sqrt{\sum_j (B^{i_1}\dot{x}_0^{i_2} - B^{i_2}\dot{x}_0^{i_1})^2}}, \frac{B^{n-1}\dot{x}_0^j - B^j \dot{x}_0^{n-1}}{\sqrt{\sum_j (B^{i_1}\dot{x}_0^{i_2} - B^{i_2}\dot{x}_0^{i_1})^2}} \right).$$

Proof. Since $\sum |B^k| \neq 0$, we assume that $B^n \neq 0$ without loss of generality of Theorem.

For (A) [resp. (B)] write t_0 instead of t_2 [resp. t_1] in expressions (2.2) [resp. (2.1)] and, applying (3.4) [resp. Lemma 6 and (3.5)], converge to the limit with (2.2) [resp. (2.1)].

For (D). Taking into account the continuity of $\dot{x}^k(t)$ we get

$$\lim_{t_2 \to t_0} \dot{x}_2^k = \dot{x}_0^k \tag{3.8}$$

and, using the mean-value theorem,

$$\lim_{\substack{t_p, t_3 \to t_0 \\ t_p \neq t_3}} V_{p3}^k = \dot{x}_0^k \quad (p = 1, 2). \tag{3.9}$$

Note, that by continuity of $\dot{x}^k(t)$ the convergence in the case $(t_1-t_3)(t_2-t_3) < 0$ is not essentially different from the one in the case $(t_1-t_3)(t_2-t_3) > 0$. Hence, for ${}^0H_2^2$ and ${}^1H_2^2$ we can as reason as for ${}^0H_1^2$ and ${}^1H_1^2$.

Now, converge to the limit with expressions (2.1) (for ${}^0H_1^2$) [resp. (2.2) for ${}^1H_1^2$] and apply Lemma 1 and (3.9) for $p = 2$ [resp. Lemma 2 and (3.8)].

For (C) write t_0 instead of t_3 (for ${}^0H_3^2$) [resp. t_1 for ${}^1H_3^2$] in expressions (2.1) [resp. (2.2)] and, applying Lemma 4 and (3.9) [resp. Lemma 5 and (3.8)], converge to the limit with (2.1) [resp. (2.2)].

Note, that in the case of ${}^0H_3^2$ we must to write 0 as the second subscript for symbols V, i.e. V_{10}^k, V_{20}^k and $M_{12;0}^k$, instead of the subscript 3, omited in the notation (cf. the remark after (1.2) in [3]). In case of ${}^1H_3^2$ we have V_{02}^k and ${}^1M_{02;2}^k = (\dot{x}_2^k - V_{02}^k)/(t_2-t_0)$ in the respective place.

Then, for (A), (B), (C) and (D), as the values in the limit we get (3.7), which are finite and fully determined and, by assumption that $B^n \neq 0$, simultaneously not vanishing. Hence, by the respective definition, exists ${}^hH_r^2$ and is described by the system of (3.6). QED.

Before we finish let us make two remarks. A quantity B^k appearing in each part of Theorem is the same quantity. It can be easily deduced from uncited lemmas of [4]. Moreover, the quantity B^k plays a role of the second-order derivative of $x^k(t)$ in formulated conditions. Hence, if functions $x^k(t)$ are of the class C^2, then we can write $\ddot{x}_0^k$ instead of B^k in formulas (3.6) and (3.7). Taking into account this fact and seting $n = 3$ and x, y, z instead of x^1, x^2, x^3, respectively, we get

$$(3.10)\quad (X-x_0)(\dot{y}_0\ddot{z}_0-\dot{z}_0\ddot{y}_0)+(Y-y_0)(\dot{z}_0\ddot{x}_0-\dot{x}_0\ddot{z}_0)+(Z-z_0)(\dot{x}_0\ddot{y}_0-\dot{y}_0\ddot{x}_0)=0$$

instead of system (3.6). (3.10) is the well-known equation of the osculating plane in E^3. Instead of (3.7) then we get the components of the binormal vector of L at t_0.

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