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Some remarks on the Dini-Lipschitz test

1. Let $\varrho(x)$ be a positive, Lebesgue-integrable function in $\langle a, b\rangle$, and let $L_\varrho^p\langle a, b\rangle$ be the class of all functions Lebesgue-integrable with p-th power with respect to the weight-function $\varrho(x)$ over $\langle a, b\rangle$. We shall consider a systems of polynomials $\{p_n(x)\}$, $n = 0, 1, 2, \ldots$, orthonormal with the weight-function $\varrho(x)$ in $\langle a, b\rangle$, i.e.

$$\int_a^b p_n(x)p_k(x)\varrho(x)\,dx = \begin{cases} 1 & \text{if } n \neq k, \\ 0 & \text{if } n = k, \end{cases}$$

where $p_n(x)$ is a polynomial of degree n having positive coefficient by x^n.

Alexits gives in [1], p. 31, the following test for convergence of the Fourier expansion of a given function with respect to the system of polynomials $\{p_n(x)\}$:

If a function $f \in L_\varrho^2\langle a, b\rangle$ satisfies at a point $\xi \in (a, b)$ the Dini-Lipschitz condition

$$|f(\xi+h)-f(\xi)| \leqslant \frac{K}{|\log|h||^a} \qquad (K = \text{const}, a > 1)$$

for sufficiently small $|h|$ and if the weight-function $\varrho(x)$ and the polynomials $p^n(x)$ are uniformly bounded in a neighbourhood of the point ξ, then the Fourier series

$$\sum_{n=0}^{\infty} c_n p_n(x), \tag{1}$$

where

$$c_n = \int_a^b f(t)p_n(t)\varrho(t)\,dt,$$

is convergent at $x = \xi$ to the value $f(\xi)$.

If the above assumptions are satisfied in an interval $\langle a_1, b_1\rangle \subset (a, b)$ uniformly, then the convergence of series (1) is uniform in each subinterval $\langle a_2, b_2\rangle \subset (a_1, b_1)$.

This theorem is an analogue of the well-known Dini-Lipschitz test for convergence of trigonometric Fourier series.

The subject of this paper is to extend the above theorem to the case when $f(x)$ is Lebesgue-integrable over subintervals and $f(x)$ together with $\varrho(x)$, $p_n(x)$ is subject to further restrictions specified below.

In the sequel we shall denote by $S_n(x)$ the n-th partial sum of the Fourier series (1) of a function $f(x)$ with respect to the system $\{p_n(x)\}$. Then

$$S_n(x) = \int_a^b f(t) K_n(t, x) \varrho(t) dt,$$

where the function

$$K_n(t, x) = \sum_{i=0}^{n} p_i(t) p_i(x)$$

is called the *kernel* of the above integral.

The following Christoffel-Darboux formula holds in case of the system of polynomials under consideration:

$$K_n(t, x) = \frac{a_n}{a_{n+1}} \frac{p_n(x)p_{n+1}(t) - p_n(t)p_{n+1}(x)}{t-x}, \tag{2}$$

where a_n and a_{n+1} are the positive coefficients by the highest power in polynomials $p_n(x)$ and $p_{n+1}(x)$, respectively. Moreover, it is known ([1], p. 33) that $a_n/a_{n+1} \leqslant \max(|a|, |b|)$.

We may limit ourselves without loss of generality to the interval $\langle -1, 1\rangle$ in place of $\langle a, b\rangle$, because the general case of $\langle a, b\rangle$ may be reduced to the case $\langle -1, 1\rangle$ substituting $t = -1 + 2(x-a)/(b-a)$.

2. Now, we formulate a new test of Dini-Lipschitz type.

THEOREM. *Let* $\{p_n(x)\}$ *be a system of polynomials orthonormal in* $\langle -1, 1\rangle$ *with a weight-function* $\varrho(x)$, *mentioned above, satisfying the following conditions: there exist constants* $c_1 > 0$, $c_2 \geqslant 0$, $c_3 > 0$, $c_4 \geqslant 0$, $c_2 + c_4 < 1$ *such that*

$$0 < \varrho(x) \leqslant \frac{c_1}{(1-x^2)^{c_2}}, \quad |p_n(x)| \leqslant \frac{c_3}{(1-x^2)^{c_4}}, \tag{3}$$

where $x \in (-1, 1)$, $n = 0, 1, 2, \ldots$ *Let a function* $f(x)$ *be Lebesgue-integrable over every interval* $\langle a, b\rangle \subset (-1, 1)$ *and*

$$\int_{-1}^{1} \frac{|f(t)|\varrho(t)}{(1-t^2)^{c_4}} dt < \infty.$$

Suppose further that, for a fixed $\xi \in (-1, 1)$,

$$f(\xi \pm h) - f(\xi) = O(|\log_1 h \cdot \log_2 h \ldots \log_{m-1} h|^{-1} |\log_m h|^{-a}) \tag{4}$$

as $h \to 0+$, *where* $a > 1$, $m \geqslant 1$ *and*

$$\log_m h = \underbrace{\log\log\ldots\log}_{m \text{ times}} h.$$

Then the Fourier series (1) *of* f *is convergent at* $x = \xi$ *to the value* $f(\xi)$.

The above assumptions are satisfed, e.g. by the normalized Jacobi polynomials $\{p_n^{(\alpha,\beta)}(x)\}$ with the weight-function $\varrho(x) = (1-x)^\alpha(1+x)^\beta$, where $\alpha > -1$, $\beta > -1$ (see [2], p. 89, footnote (²)).

The proof of the Theorem will be based on the following two auxiliary results.

LEMMA 1. *Suppose that condition* (3) *holds and that* $\langle a, b\rangle \subset (-1, 1)$. *Then there is a positive constant* C *such that for each* $n > k \geqslant 0$,

$$\left|\int_\alpha^\beta p_n(x)p_k(x)\varrho(x)\,dx\right| \leqslant \frac{C}{n-k}$$

uniformly in α, β, *provided* $a \leqslant \alpha < \beta \leqslant b$.

This Lemma is proved in [2], pp. 91-92.

LEMMA 2. *Let* $\{\Phi_n(t)\}$ *be a sequence of measurable functions uniformly bounded in a finite interval* $\langle a, b\rangle$. *Suppose that for every* $c \in \langle a, b\rangle$,

$$\lim_{n\to\infty}\int_a^c \Phi_n(t)\,dt = 0.$$

Then given any f *Lebesgue-integrable over* $\langle a, b\rangle$, *we have*

$$\lim_{n\to\infty}\int_a^b f(t)\Phi_n(t)\,dt = 0.$$

(see [3], pp. 240-42).

3. In order to prove our Theorem we first observe that, by (2), (3) and (4),

$$\lim_{\eta\to 0+}\int_{\xi-\eta}^{\xi+\eta} \{f(t)-f(\xi)\}K_n(t, \xi)\varrho(t)\,dt = 0$$

uniformly in n. Hence, for any positive ε there is a positive δ $(-1+\delta < \xi < 1-\delta)$ such that

$$\left|\int_{\xi-\delta}^{\xi+\delta} \{f(t)-f(\xi)\}K_n(t, \xi)\varrho(t)\,dt\right| < \varepsilon \qquad (n = 0, 1, 2, \ldots). \tag{5}$$

Write

$$S_n(\xi)-f(\xi) = \left(\int_{-1}^{\xi-\delta} + \int_{\xi-\delta}^{\xi+\delta} + \int_{\xi+\delta}^{1}\right)\{f(t)-f(\xi)\}K_n(t, \xi)\varrho(t)\,dt$$
$$= A_n + B_n + C_n.$$

The function $\{f(t)-f(\xi)\}/(1-t^2)^{c_4}$ is Lebesgue-integrable with the weight-function $\varrho(t)$ over $\langle -1, 1\rangle$; whence

$$\lim_{\eta\to 0+}\int_{-1}^{-1+\eta}\frac{|f(t)-f(\xi)|}{(1-t^2)^{c_4}}\varrho(t)\,dt = 0. \tag{6}$$

In view of (2),

$$\Big|\int_{-1}^{-1+\eta}\{f(t)-f(\xi)\}K_n(t,\xi)\varrho(t)\,dt\Big|$$
$$\leqslant \frac{a_n}{a_{n+1}}\int_{-1}^{-1+\eta}\frac{|f(t)-f(\xi)|}{|t-\xi|}|p_{n+1}(t)|\,|p_n(\xi)|\,\varrho(t)\,dt +$$
$$+\frac{a_n}{a_{n+1}}\int_{-1}^{-1+\eta}\frac{|f(t)-f(\xi)|}{|t-\xi|}|p_n(t)|\,|p_{n+1}(\xi)|\,\varrho(t)\,dt.$$

Thus, by (3) and (6), there is a positive δ_1 $(-1+\delta_1 < \xi-\delta)$ such that

$$\Big|\int_{-1}^{-1+\delta_1}\{f(t)-f(\xi)\}K_n(t,\xi)\varrho(t)\,dt\Big| < \varepsilon \quad (n = 0, 1, 2, \ldots). \tag{7}$$

It is easily seen that the functions

$$\Phi_n(t) = K_n(t,\xi)\varrho(t) \quad (n = 0, 1, 2, \ldots)$$

satisfy the assumptions of Lemma 2 in $\langle a, b\rangle = \langle -1+\delta_1, \xi-\delta\rangle$. Indeed, formula (2) and estimates (3) lead to

$$|\Phi_n(t)| \leqslant \frac{a_n}{a_{n+1}}\cdot\frac{|p_n(\xi)|\,|p_{n+1}(t)|+|p_n(t)|\,|p_{n+1}(\xi)|}{|t-\xi|}\varrho(t) \leqslant M$$

for every $t\in\langle a, b\rangle$ $(n = 0, 1, 2, \ldots)$, with a certain constant M. Taking $c\in\langle a, b\rangle$ and applying (2), we write

$$\int_a^c \Phi_n(t)\,dt = \frac{a_n}{a_{n+1}}p_{n+1}(\xi)\int_a^c\frac{p_n(t)}{\xi-t}\varrho(t)\,dt - \frac{a_n}{a_{n+1}}p_n(\xi)\int_a^c\frac{p_{n+1}(t)}{\xi-t}\varrho(t)\,dt.$$

By the second mean-value theorem,

$$\int_a^c\frac{p_n(t)}{\xi-t}\varrho(t)\,dt = \frac{1}{\xi-c}\int_{u_n}^c p_n(t)\varrho(t)\,dt \quad (a \leqslant u_n \leqslant c).$$

Then, Lemma 1 gives

$$\int_a^c\frac{p_n(t)}{\xi-t}\varrho(t)\,dt = O\left(\frac{1}{n}\right) \quad \text{as } n\to\infty,$$

and, consequently,

$$\lim_{n\to\infty}\int_a^c \Phi_n(t)\,dt = 0.$$

The function $f(t)-f(\xi)$ is Lebesgue-integrable over $\langle -1+\delta_1, \xi-\delta\rangle$. Hence, by Lemma 2,

$$\lim_{n\to\infty}\int_{-1+\delta_1}^{\xi-\delta}\{f(t)-f(\xi)\}K_n(t, \xi)\varrho(t)\,dt = 0.$$

Applying (7), we conclude that

$$|A_n| < 2\varepsilon$$

for sufficiently large n. Analogously, $|C_n| < 2\varepsilon$. Estimate (5) shows that $|B_n| < \varepsilon$ for all n. Thus

$$|S_n(\xi)-f(\xi)| < 5\varepsilon,$$

provided n is large enough, and the Theorem is proved completely.

It can easily be observed that if condition (4) of the Theorem holds uniformly with respect to ξ in the whole interval $\langle a_1, b_1\rangle \subset (-1, 1)$, then the convergence of the series (1) at $x = \xi$ to $f(\xi)$ is uniform in every closed subinterval of the interval (a_1, b_1).

References

[1] G. Alexits, *Konvergenzprobleme der Orthogonalreihen*, Berlin 1960.

[2] G. Freud, *Über die Konvergenz orthogonaler Polynomreihen*, Acta Math. Acad. Sci. Hungaricae 3 (1952), pp. 89-98.

[3] И. П. Натансон, *Теория функций вещественной переменной*, Москва--Ленинград 1950.