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On existence theorems for some class of non-linear Tchebycheff approximations

1. In the sequel the following notations will be used: X denotes an arbitrary compact topological space; $C[X]$ the linear space of real-valued continuous functions, defined on the space X, with the Tchebycheff's norm:

$$\|f\|_T = \sup_{x\in X} |f(x)|;$$

E_n denotes a given n-dimensional real Euclidean space; we say that a set of vectors $a = (a_1, a_2, \ldots, a_n) \in E_n$ is *bounded* if $\max_i |a_i| \leqslant M$ holds for every element of this set with some M. B denotes an arbitrary compact space; a mapping $B\times X\times E_1 \to E_1$ is denoted by $w(b, x, y)$; $g = (g_1, \ldots, g_n)$ denotes an n-tuple of real-valued continuous functions defined on X; $\big(a, g(x)\big) := \sum_{i=1}^{n} a_i g_i(x)$; $r_{ab}(x) := w\Big(b, x, \big(a, g(x)\big)\Big)$.

Let us assume:

1.1. For any given $b_0 \in B$ the exists a subset $X_{b_0} \subset X$, dense in X such that w is continous at (b_0, x, y) for $b\in B$, $x\in X_{b_0}$, $y\in E_1$, simultaneously.

1.2. For every $c > 0$ there exists $M_c > 0$ such that for every $b\in B$ and $x\in X_b$ the inequality $|y| > M_c$ implies $|w(b, x, y)| > c$.

2. With respect to 1.1 it is seen that the function

$$p_{ab}(x_0) := \begin{cases} r_{ab}(x_0) & \text{for } x_0\in X_b, \\ \overline{\lim\limits_{X_b\ni x\to x_0}}\; r_{ab}(x) & \text{for } x_0 \in X\setminus X_b \ (^1), \end{cases}$$

is well defined for every $a\in E_n$ and $b\in B$.

Let us now put $P := \{p_{ab}\colon a\in E_n, b\in B\}$. As usually the real number $d := \inf_{p_{ab}\in P} \|f - p_{ab}\|_T$ is called the *distance* of f from P: a function $p_{a^*b^*}\in P$

(1) We take $\underline{\lim}\, r_{ab}(x)$ if $|\underline{\lim}\, r_{ab}(x)| \geqslant |\overline{\lim}\, r_{ab}(x)|$ and $\overline{\lim}\, r_{ab}(x)$ otherwise.

such that $\|f-p_{a^*b^*}\|_T = d$ is called a *best approximation* to f in P; let us set $\hat{w}(b, x, y) = \min\big(|w(b, x, |y|)|, |w(b, x, -|y|)|\big)$.

2.1. It is seen that

$(+)$ $\hat{w}(b, x, y) \geqslant 0$ for any $(b, x, y) \in B\times X\times E_1$;

$(++)$ for any $b\in B$ and any $c > 0$ there exist a positive constant M_c independent of b, and a subset X_b dense in X such that the inequality $\hat{w}(b, x, y) > c$ holds for $|y| > M_c$ and $x\in X_b$;

$(+++)$ $\hat{w}(b, x, y) \leqslant |w(b, x, y)|$ for any $(b, x, y)\in B\times X\times E_1$.

2.2. *If $g_1, \ldots, g_n$ are linearly independent on X, then there exists a positive constant u such that $\max\limits_i |a_i| = 1$ implies $\|(a, g)\|_T \geqslant u > 0$.*

Proof. See e.g. [9], p. 24.

2.3. *If $f\in C[X]$ is bounded and the functions $g_1, \ldots, g_n$ are linearly independent on X, then the set $\{(a_k, b_k): a_n = (a_1^k, \ldots, a_n^k)\in E_n,\ b_k\in B,\ k = 1, 2, \ldots\}$ of parameters for which*

$$(+)\qquad \lim_k \|f-p_{a_kb_k}\|_T = d < \infty.$$

has a cluster point (a_0, b_0), $a_0\in E_n$, $b_0\in B$, and all components of the vector a_0 are finite.

Proof. From 2.2. it follows that for any positive constant $e < u$ there exists an open subset X_{ae} dependent on $a\in E_n$ and such that $|(a, g(x))| > u-e$ for any $x\in X_{ae}$ with $\max|a_i| = 1$. From 2.1 $(++)$ it follows that for $c = \|f\|_T+d+1$ there exists a constant $u_c > 0$ such that for all $|y| > u_c$ and all $x\in X_b \cap X_{ae} \neq \theta$, $b\in B$, the inequality

$$(++)\qquad \hat{w}(b, x, y) > c$$

holds.

If $a = (a_1, \ldots, a_n)$ with $\max|a_i| > u_c/(u-e) = M_c$, then

$$|(a, g(x))| = \max_j |a_j| \Big|\sum_{i=1}^{n} (a_i/\max_j |a_j|) g_i(x)\Big| > \max_j |a_j| (u-e) \geqslant u_c$$

for $x\in X_{a_1e}$, $a_1 = a/\max\limits_j |a_j|$.

Hence, in view of 2.1 $(+++)$ and 2.3 $(++)$, and the definition of p_{ab} we get

$$(+++)\qquad c < \hat{w}\big(b, x, (a, g(x))\big) \leqslant \big|w\big(b, x, (a, g(x))\big)\big| \leqslant \|p_{ab}\|_T.$$

On the other hand, for sufficiently large K, $k \geqslant K$, by our assumption 2.3 $(+)$ we get $\|f-p_{a_kb_k}\|_T \leqslant d+1$. Hence $\|p_{a_kb_k}\|_T \leqslant \|f\|_T+d+1 = c$. So, by 2.3 $(+++)$ it follows that $\max\limits_i |a_i^k| \leqslant M_c$. By the Bolzano-

Weierstrass Theorem the sequence a_k has a cluster point $a_0 = (a_1^0, \ldots, a_n^0)$ with $\max |a_i^0| \leqslant M_c$.

3. Simple examples show that in general there does not exist a best approximation by a function from P; see e.g. [2] and [5].

3.1. *If $f, g_1, \ldots, g_m$ are bounded functions in $C[X]$, X an arbitrary topological space, w a real-valued function satisfying* 1.1 *and* 1.2, *and there exist $a \in E_n$ and $b \in B$ such that $\|p_{ab} - f\|_T < \infty$, then there exists a best Tchebycheff approximation $p_{a^*b^*}$ to f on X.*

Proof. Let $g_1, \ldots, g_m$ be a maximal linearly independent subset of $g_1, \ldots, g_n$, Let (a_k, b_k) be the sequence of vectors $a_k \in E_m$, $b_k \in B$ with $\|p_{a_k b_k} - f\|_T \leqslant d + 1/k$. For this sequence, by 2.3, there exists the cluster point (a_0, b_0), $a_0 \in E_m$, $b_0 \in B$. We have

$$|p_{a_0 b_0}(x) - f(x)| \leqslant |p_{a_0 b_0}(x) - p_{a_k b_k}(x)| + |p_{a_k b_k}(x) - f(x)|$$
$$\leqslant |p_{a_0 b_0}(x)| - p_{a_k b_k}(x)| + d + 1/k.$$

As $(a, g(x))$ is continuous in $x \in X$, then $|p_{a_0 b_0}(x) - p_{a_k b_k}(x)| \to 0$ for $k \to \infty$ and $x \in X_{b_0}$, and thus

$$(+) \qquad |p_{a_0 b_0}(x) - f(x)| \leqslant d \quad \text{for } x \in X_{b_0}.$$

Hence, from the definition of $p_{ab}(x)$ and the continuity of f we may choose a sequence $x_j \to x \in X \cap X_{b_0}$, $x_j \in X_{b_0}$, such that $|p_{a_0 b_0}(x) - p_{a_0 b_0}(x_j)| \leqslant 1/j$, $|f(x) - f(x_j)| \leqslant 1/j$ holds. So $|p_{a_0 b_0}(x) - f(x)| \leqslant |p_{a_0 b_0}(x) - p_{a_0 b_0}(x_j)| + |p_{a_0 b_0}(x_j) - f(x_j)| + |f(x_j) - f(x)| \leqslant 2/j + d$ for $x \in X \setminus X_{b_0}$. Hence $|p_{a_0 b_0}(x) - f(x)| \leqslant d$ for $x \in X \setminus X_{b_0}$. So, from 3.1 (+) and the last inequality we obtain $\|p_{a_0 b_0} - f\|_T \leqslant d$. Since $a_0 \in E_m$ and $b_0 \in B$, there exists a best approximation $p_{a_0 b_0}$ to f in P.

4. Let us measure a distance between f and p_{ab} by means of the functional $\|s(p_{ab} - f)\|_T$, where $s \in C[X]$ is a given weight function see e.g. [1]. This problem of weighted Tchebycheff approximation may be reduced to the previous one by the following substitutions: $f_1 = s \cdot f$, $w_1 = s \cdot w$. The proof of 3.1 is valid if the coefficients are restricted to a closed set $R \subset E_m \times B$ containing at least one vector (a_0, b_0) such that $\|s(p_{a_0 b_0} - f)\|_T < \infty$.

4.1. *If $f, s, g_1, \ldots, g_n$ denote functions belonging to $C[X]$ such that $s, f, g_1, \ldots, g_n$ are bounded on X and $s \cdot w = w_1$ is a real-valued function satisfying* 1.1 *and* 1.2, *then for any closed subset $R \subset E_n \times B$ including a vector (a_0, b_0) with $\|s(p_{a_0 b_0} - f)\|_T < \infty$ there exists a best weighted Tchebycheff approximation $p_{a^*b^*}$ to f, and*

$$\|s(p_{a^*b^*} - f)\|_T = \inf_{(a,b) \in R} \|s(p_{ab} - f)\|_T.$$

From 3.1 and this theorem we may obtain the existence of the best Tchebycheff approximation by functions of the form

$$r_{ab}(x) = \frac{p_1\left(x, \sum_{i=1}^{n} a_i \cdot g_i(x)\right)}{p_2\left(x, \sum_{j=1}^{m} b_j \cdot h_j(x)\right)},$$

where g_i and h_j $(i = 1, \ldots, n,\ j = 1, \ldots, m)$ are continuous real-valued functions on X; h_j have the dense non-zero property in X for a given closed, bounded subset $B \subset E_m$ (for the definition of dense non-zero property see e.g. [2]) and h_j are bounded in X; $p_i(x, y)$, $i = 1, 2$, are real-valued continuous functions defined for $(x, y) \in X \times E_1$, $p_i(x, y) = 0$ if and only if $y = 0$; $p_i(x, y) \to \infty$ if $|y| \to \infty$ uniformly on X, $i = 1, 2$. Moreover, we assume that $p_2(x,y) < \infty$ for $|y| < \infty$ and $x \in X$.

The results presented in this note generalize some theorems on the existence of approximations which may be found in [1]-[10].

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