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On the oscillation properties of certain power series

The object of this paper is to give some generalizations of theorems on the oscillation and non-oscillation of the trigonometric and hyperbolic functions.

DEFINITION 1. A continuous function $f(x)$ in $(-\infty, +\infty)$ will be called *oscillatory* if and only if for any given $r > 0$ there exists a point x_0 such that $|x_0| > r$ and the function $f(x)$ changes its sign at x_0.

Let us consider a power series of the form

$$(1)\qquad \sum_{i=0}^{\infty} \varepsilon_i \frac{x^i}{i!}, \quad \text{where } \varepsilon_i = 0, \ \pm 1.$$

Notice, that the functions e^x, e^{-x}, $\cosh x$ and $\sinh x$ are not oscillatory although they have developments of the form (1), and $\sin x$ and $\cos x$ are simultaneously oscillatory. We shall construct series of form (1) which have oscillatory sums.

Let

$$(2)\qquad s(x) = s_{a,k}(x) = \frac{x^a}{a!} - \frac{x^{a+k}}{(a+k)!} + \frac{x^{a+2k}}{(a+2k)!} - \frac{x^{a+3k}}{(a+3k)!} + \ldots,$$

where a is a non-negative integer and $k \geqslant 2$, $k > a$ is an integer. Obviously series (2) converges absolutely and almost uniformly and its sum is continuous for $x \in (-\infty, +\infty)$.

THEOREM 1. *The function* $s(x)$ *defined by* (2) *is an oscillatory function.*

Proof. One may easily prove that $s(x)$ satisfies the differential equation

$$(3)\qquad s^{(k)}(x) + s(x) = 0$$

with the initial condition

$$(4)\qquad s^{(j)}(0) = \delta_{j,a} \quad (j = 0, 1, \ldots, k-1).$$

Now we have to consider the two cases: (a) if k is even, (b) if k is odd. In the case (a) a general solution of (3) is of the form

$$s(x) = C_1 e^{\gamma_1 x} + \ldots + C_k e^{\gamma_k x},$$

where $\gamma_j = a_j + i\beta_j$ are k-th roots of -1, $j = 1, \ldots, k$, $\beta_j \neq 0$ and $-1 < a_1 < a_2 < \ldots < a_k < 1$. Condition (4) yields the following system of equations for the coefficients:

$$\sum_{j=1}^{k} C_j \gamma_j^m = \delta_{m,a} \quad (m = 0, 1, \ldots, k-1).$$

We see that $C_k = W^{-1} W_k$, where W and W_k are Vandermonde determinants. Hence we have

$$s(x) = \sum_{j=1}^{k} e^{a_j x}(d_j \cos\beta_j x + l_j \sin\beta_j x) = \sum_{j=1}^{k} e^{a_j x} M_j \sin(\beta_j x + N_j), \tag{5}$$

where N_j and M_j are constants, and M_j satisfy the condition $\sum_{j=1}^{k} M_j^2 > 0$. Let $M_p = \min M_j$, $M_p \neq 0$, and $M_q = \max M_j$, $M_q \neq 0$. Then

$$s(x) = e^{a_p x} M_p [\sin(\beta_p x + N_p) + o(e^x)] \quad \text{for } x \to -\infty$$

or

$$s(x) = e^{a_q x} M_q [\sin(\beta_q x + N_q) + o(e^{-x})] \quad \text{for } x \to +\infty.$$

It follows from these formulae that $s(x)$ is oscillatory in $(-\infty, +\infty)$.

In case (b) the general integral of (3) is of the form

$$s(x) = D_0 e^{-x} + D_1 e^{\varrho_1 x} + \ldots + D_{k-1} e^{\varrho_{k-1} x}, \tag{6}$$

where $\varrho_j = a_j + ib_j$ are k-th roots of -1, $j = 1, 2, \ldots, k-1$, and $b_j \neq 0$ and $-1 < a_1 < a_2 < \ldots < a_{k-1} < 1$ and D_j are constants coefficients which satisfy (in view of (4)) the equations

$$\sum_{j=0}^{k-1} D_j \varrho_j^i = \delta_{i,s} \quad (i = 0, 1, \ldots, k-1).$$

We see that $D_0 = W^{-1} W_0$, where W and W_0 are Vandermonde determinants $\neq 0$ (1). Hence we conclude by (6) that

$$\begin{aligned} s(x) &= D_0 e^{-x} + \sum_{j=1}^{k-1} e^{a_j x}(f_j \cos b_j x + g_j \sin b_j x) \\ &= D_0 e^{-x} + \sum_{j=1}^{k-1} e^{a_j x} P_j \sin(b_j x + Q_j), \end{aligned} \tag{7}$$

(1) See A. P. Miszyna and I. W. Proskuriakow, *Algebra wyższa*, Warszawa 1966, p. 24.

where P_j and Q_j are constants, and P_j satisfy the condition $\sum_{j=1}^{k-1} P_j^2 > 0$. Let $P_p = \min P_j$, $P_p \neq 0$ and $P_q = \max P_j$, $P_q \neq 0$. Then

$$s(x) = D_0 e^{-x} + e^{a_p x} P_p [\sin(b_p x + Q_p) + o(x)] \quad \text{for } x \to -\infty$$

or

$$s(x) = D_0 e^{-x} + e^{a_q x} P_q [\sin(b_q x + Q_q) + o(x)] \quad \text{for } x \to +\infty.$$

Remark 1. If b is a non-negative integer, $l \geqslant \max(2, b)$ and l is even, then the function

$$(8) \qquad S(x) = S_{b,l}(x) = \frac{x^b}{b!} + \frac{x^{b+l}}{(b+l)!} + \ldots = x^b \left[\frac{1}{b!} + \frac{x^l}{(b+l)!} + \ldots\right]$$

is not oscillatory.

From Theorem 1 we infer

THEOREM 2. *If b is non-negative integer $l \geqslant \max(2, b)$ and l is odd, then the function $S(x)$ is oscillatory.*

Proof. In this case we have $S(-x) = (-1)^b s_{b,l}(x)$.

THEOREM 3. *If a, b, p and k are non-negative integers, $2p \geqslant 2$, $2p > b$, $2k \geqslant 2$ and $2k > a$, then the function*

$$(9) \qquad M(x) = S_{b,2p}(x) + s_{a,2k}(x)$$

is not oscillatory.

Proof. $S(x)$ satisfies the differential equation

$$(10) \qquad S^{(2p)}(x) - S(x) = 0$$

with the initial data

$$(11) \qquad S^{(j)}(0) = \delta_{j,b} \quad (j = 0, 1, \ldots, 2p-1).$$

Let

$$\mu_k = \cos\left(\frac{\pi k}{p}\right) + i \sin\left(\frac{\pi k}{p}\right) = t_k + i h_k \quad (k = 0, 1, \ldots, 2p-1).$$

The general solution of equation (10) is of the form

$$S(x) = T_0 e^{\mu_0 x} + T_1 e^{\mu_1 x} + \ldots + T_{2p-1} e^{\mu_{2p-1} x}.$$

From (11) it follows that the constants T_i satisfy the system of linear equations

$$\sum_{j=1}^{2p-1} T_i \mu_j^k = \delta_{b,k} \quad (k = 0, 1, \ldots, 2p-1).$$

The Vandermonde determinants

$$(12) \qquad \begin{aligned} W_0 &= (-1)^{b+1} \det \|\mu_j^k\|, \quad & k, j = 0, 1, \ldots, 2p,\ k \neq b,\ j \neq 0, \\ W_p &= (-1)^{b+p+1} \det \|\mu_j^k\|, \quad & k, j = 0, 1, \ldots, 2p,\ k \neq b,\ j \neq b, \end{aligned}$$

and $W_0 \neq 0$, $W_p \neq 0$ (see footnote on p. 268). Hence W_0 and W_p are Cramer determinants. Thus

$$(13) \qquad S(x) = T_0 e^x + T_p e^{-x} + \sum_{\substack{j=1 \\ j\neq p}}^{2p-1} e^{t_j x}(m_j \cos h_j x + n_j \sin h_j x),$$

where $-1 < t_1 < \ldots < t_{p-1} < t_{p+1} < \ldots < t_{2p-1} < 1$, and m_j, n_j, T_0 and T_p are constants and $T_0 \neq 0$, $T_p \neq 0$.

Hence $S(x) = e^x[T_0 + O(e^{t_{2p-1}x})]$ for $x \to +\infty$ and $S(x) = e^{-x}[T_p + O(e^{-x})]$ for $x \to -\infty$ From (5) we have $s(x) = O(e^{a_1 x})$ for $x \to -\infty$ and $s(x) = O(e^{a_k x})$ for $x \to +\infty$. Hence

$$M(x) = S(x) + s(x) = e^x[T_0 + O(e^{t_{2p-1}x})] + O(e^{a_1 x}) \to (\operatorname{sgn} T_0)_\infty \quad \text{for } x \to \infty$$

and

$$M\ x) = S(x) + s(x) = e^{-x}[T_p + O(e^{t_1 x})] + O(e^{a_k x}) \to (\operatorname{sgn} T_p)_\infty \text{ for } x \to -\infty.$$

Let

$$(14) \qquad \sum_{j=0}^{2k} E_j \eta_j^l = \delta_{a,l},$$

where $l = 0, 1, \ldots, 2k$ and $\eta_j = \cos\left(\frac{\pi + 2\pi j}{2k+1}\right) + i \sin\left(\frac{\pi + 2\pi j}{2k+1}\right) = p_j + iq_j$,

$$(15) \qquad \begin{aligned} V &= \det \|\eta_j^l\|, \quad j, l = 0, 1, \ldots, 2k, \\ V_0 &= (-1)^{a+1} \det \|\eta_j^l\|, \quad j, l = 0, 1, \ldots, 2k,\ j \neq 0,\ l \neq b. \end{aligned}$$

THEOREM 4. *If a, b, p and k are non-negative integers and $2p \geqslant 2$, $2p \geqslant b$, $2k+1 \geqslant 2$, $2k+1 \geqslant a$, and $WW_0^{-1} \neq -VV_0^{-1}$, where W_0 and W are defined by* (12) *and V_0 and V by* (15), *then the function $M(x) = S_{b,2p}(x) + s_{a,2k+1}(x)$ is not oscillatory.*

Proof. The function $s(x)$ satisfies the differential equation

$$(16) \qquad s^{(2k+1)}(x) + s(x) = 0$$

and initial conditions

$$(17) \qquad s^{(l)}(0) = \delta_{a,l} \quad (l = 0, 1, \ldots, 2k).$$

The general integral of (16) is the function

$$(18) \qquad s(x) = E_0 e^{\eta_0 x} + E_1 e^{\eta_1 x} + \ldots + E_{2k} e^{\eta_{2k} x}.$$

From (17) it follows that the constants E_i satisfy the system of linear equations (14). From (13) and (18), we have

$$M(x) = T_0 e^x + (T_p + E_0) e^{-x} + \\ + \sum_{\substack{j=1 \\ j\neq p}}^{2p-1} e^{t_j x}(m_j \cos h_j x + n_j \sin h_j x) + \sum_{\substack{j=0 \\ j\neq k}}^{2k} e^{\gamma_1 x}(M_j \cos \sigma_j x + N_j \sin \sigma_j x)$$

where M_j, N_j, γ_j and σ_j are constants and $|\gamma_j| < 1$. Hence $M(x)$ is not oscillatory.

THEOREM 5. *If a, b, p and k are non-negative integers, $2p+1 \geqslant 2$, $2p+1 \geqslant b$, $2k+1 \geqslant 2$, $2k+1 \geqslant a$, then the functions $M(x) = S_{b,2p+1}(x) + s_{a,2k+1}(x)$ is not oscillatory.*

Proof. The function $S(x)$ satisfies the differential equation $S^{(2k+1)}(x) - S(x) = 0$ and initial data $S^{(j)}(0) = \delta_{a,j}$ $(j = 0, 1, \ldots, 2k)$. Then

$$S(x) = T_0 e^x + \sum_{j=1}^{2k} e^{c_j x}(g_j \cos d_j x + h_j \sin d_j x), \tag{19}$$

where T_0, g_j, d_j and h_j are constants and $|c_j| < 1$. From formulae (7) and (19) we get theorem 5.

THEOREM 6. *If a, b, p and k are non-negative integers and $2p+1 \geqslant 2$, $2p+1 \geqslant b$, $2k+1 \geqslant 2$, $2k+1 \geqslant a$, then the function $M(x) = S_{b,2p+1}(x) + s_{a,2k}(x)$ is not oscillatory.*

Proof. Thus

$$s(x) = G_0 e^{-x} + \sum_{j=1}^{2k} e^{r_j x}(A_j \cos b_j x + B_j \sin b_j x), \tag{20}$$

where A_j, B_j, b_j, G_0 and r_j are constants and $G_0 \neq 0$, and $|r_j| < 1$. From formulae (19) and (20) we get Theorem 6.

Let $g(x)$ be a continuous and increasing function in $(-\infty, +\infty)$ and $g(x) \to +\infty$ for $x \to +\infty$, and $g(x) \to -\infty$ for $x \to -\infty$. Theorems 1-6 immediately imply the following theorem:

THEOREM 7. *The function $s\big(g(x)\big)$, where $s(x)$ is defined by formula (3), is oscillatory.*

Proof. The function $s\big(g(x)\big)$ is continuous for $(-\infty, +\infty)$. Let $s(z)$ change the sign at the point z_1. Then we find a point x_1 such that $g(x_1) = z_1$. The function $s\big(g(x)\big)$ changes the sign at x_1. If z_2 is another zero of $s(z)$ of this kind, then $s\big(g(x)\big)$ changes the sign at the point x_2 for which $g(x_2) = z_2$. In virtue of monotony of $g(x)$ we obtain $x_1 \neq x_2$. Then $s\big(g(x)\big)$ oscillates at the points $x_1, x_2, \ldots$ for which $g(x_1) = z_1$, $g(x_2) = z_2, \ldots$

Under the assumptions of Remark 1, $S\big(g(x)\big)$ is not oscillatory. Of course, for $z = g(x)$ and $|z| = |g(x)| > A$ for A sufficiently large the function $S(z)$ does not change its sign.

Under the assumptions of Theorem 2 we conclude that $S\big(g(x)\big)$ is an oscillatory function.

Under the assumptions of Theorems 3-6 we conclude that $M\big(g(x)\big)$ is not an oscillatory function.