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Topologies of uniform convergence

1. Introduction. The space of all functions from the real line into itself is connected in the topology of pointwise convergence but is not connected in the uniform convergence topology. This suggests an investigation of topologies between pointwise and uniform convergence. The following is a study of the extensive collection of such topologies generated by covers of the domain space.

2. Cover topologies. Let S be a non-empty set, E a Hausdorff linear topological space, and F the set of all functions from S to E. Let $\mathscr{A}$ be a subset of the power set $\mathscr{P}(S)$ and for $A \in \mathscr{A}$ and U a neighborhood of $\mathbf{0}$ in E, let $N(A, U) = \{f \in F: f(A) \subset U\}$. Sets of the form $f + \bigcap_{i=1}^{k} N(A_i, U_i)$ yield a base for a neighborhood system of f. The resulting topology on F is called the *topology of uniform convergence* on members of $\mathscr{A}$ (see [2]). If $\mathscr{A}$ is the collection of all singletons we obtain the topology of pointwise convergence $\mathscr{I}_*$ and if $\mathscr{A}$ is $\{S\}$ we obtain the topology of uniform convergence $\mathscr{I}^*$. Intermediate topologies arise if we require that $S = \bigcup \mathscr{A}$; only such $\mathscr{A}$ will be considered. Covers of special interest will be those closed under finite union and the taking of subsets. Covers with this property will be called *standard covers*. Spaces $(F, \mathscr{I}_{\mathscr{A}})$ obtained from covers are topological groups and hence are homogeneous spaces. If every element of a cover $\mathscr{U}_1$ is a subset of the union of finitely many elements of a cover $\mathscr{U}_2$ we will write $\mathscr{U}_1 < f\mathscr{U}_2$.

The following theorem is basic in comparing topologies generated by covers:

THEOREM 1. *If $\mathscr{U}_1$ and $\mathscr{U}_2$ are covers of S, then $\mathscr{I}_{\mathscr{U}_1} \subset \mathscr{I}_{\mathscr{U}_2}$ if and only if $\mathscr{U}_1 < f\mathscr{U}_2$.*

Proof. If $\mathscr{U}$ is the collection of all finite unions from $\mathscr{U}_2$, then $\mathscr{I}_{\mathscr{U}_2} = \mathscr{I}_{\mathscr{U}}$ (see [2]) and if $A_2 \subset A_1$, then $N(A_1, U) \subset N(A_2, U)$. So if $\mathscr{U}_1 < f\mathscr{U}_2$, then $\mathscr{I}_{\mathscr{U}_1} \subset \mathscr{I}_{\mathscr{U}} = \mathscr{I}_{\mathscr{U}_2}$. Suppose $\mathscr{U}_1 \not< f\mathscr{U}_2$, that is, there is an $A \in \mathscr{U}_1$ which is not contained in any finite union of elements in $\mathscr{U}_2$. The collection $\mathscr{B}$ of all finite subsets of $\mathscr{U}_2$ becomes a directed set when ordered by inclusion. Consider the net $\{f_a : a \in \mathscr{B}\}$ defined as follows:

$$f_a(x) = \begin{cases} 0 & \text{if } x \in \bigcup a, \\ a \neq 0 & \text{if } x \notin \bigcup a. \end{cases}$$

Consider $N(B, U)$ with $B \in \mathscr{U}_2$, U a neighborhood of $0 \in E$. If $a > \beta = \{B\}$, then $f_a \in N(B, U)$; hence $f_a \to \theta$ in $\mathscr{I}_{\mathscr{U}_2}$ (θ the zero function in F). In $\mathscr{I}_{\mathscr{U}_1}$, look at $N(A, V)$ where V is a neighborhood of $0 \in E$ with $a \notin V$. Clearly, for all $a \in \mathscr{B}$, $f_a \notin N(A, V)$ since $a \in f_a(A)$ for all $a \in \mathscr{B}$. Hence f_a does not converge to θ in $\mathscr{I}_{\mathscr{U}_1}$ and $\mathscr{I}_{\mathscr{U}_1} \not\subset \mathscr{I}_{\mathscr{U}_2}$.

COROLLARY 1. *If $\mathscr{U}$ and $\mathscr{V}$ are standard covers, $\mathscr{I}_{\mathscr{U}} \subset \mathscr{I}_{\mathscr{V}}$ if and only if $\mathscr{U} \subset \mathscr{V}$.*

Note. Different covers may generate the same topology. However, Corollary 1 shows that in an equivalence class of covers generating a given cover topology, there is a unique standard cover. (Every equivalence class must contain a standard cover, for if $\mathscr{U}$ is in a given equivalence class then so is the standard cover whose elements are subsets of finite unions of $\mathscr{U}$.) Furthermore, a standard cover contains each cover in its equivalence class. The cover of S consisting of all finite sets is the standard cover generating pointwise convergence, while the power set of S is the standard cover generating uniform convergence on S.

THEOREM 2. *If $\mathscr{I}$ is any topology with $\mathscr{I}_* \subset \mathscr{I} \subset \mathscr{I}^*$ there is a finest cover topology coarser than $\mathscr{I}$.*

Proof. Let $\mathscr{A} = \{\mathscr{U} : \mathscr{I}_{\mathscr{U}} \subset \mathscr{I}\}$. $\mathscr{A}$ is not empty since $\mathscr{I}_* \subset \mathscr{I}$. If $\mathscr{V} = \bigcup \mathscr{A}$, then $\mathscr{V}$ is a cover and $\mathscr{I}_{\mathscr{V}} \subset \mathscr{I}$. For if $N = \bigcap_{i=1}^{n} N(A_i, U_i)$, $A_i \in \mathscr{U}_i \in \mathscr{A}$, is a neighborhood of 0 in $\mathscr{I}_{\mathscr{V}}$, we know there are neighborhoods N_i of 0 in $\mathscr{I}$ such that $N_i \subset N(A_i, U_i)$ since $\mathscr{I}_{\mathscr{U}_i} \subset \mathscr{I}$. Then $\bigcap_{i=1}^{n} N_i \subset N$ so that $\mathscr{I}_{\mathscr{V}} \subset \mathscr{I}$. This is the finest cover topology contained in $\mathscr{I}$.

THEOREM 3. *If $\mathscr{I}_{\mathscr{V}} \subset \mathscr{I}_{\mathscr{U}}$ properly, there exists a cover $\mathscr{W}$ such that $\mathscr{I}_{\mathscr{V}} \subset \mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{U}}$ with the last inclusion proper and no cover topology lies between $\mathscr{I}_{\mathscr{W}}$ and $\mathscr{I}_{\mathscr{U}}$.*

LEMMA 1. *If $\mathscr{A}$ is a cover of an infinite set A having no finite subcover, then there exists a cover containing $\mathscr{A}$ maximal with respect to having no finite subcover.*

Proof of Lemma 1. Let $\{\mathscr{A}_a\}$ be a chain of covers of A each containing $\mathscr{A}$ such that no $\mathscr{A}_a$ has a finite subcover. Then $\bigcup \{\mathscr{A}_a\}$ is a cover. Suppose $A \subset \bigcup_{i=1}^{k} A_i$ where $A_i \in \mathscr{A}_{a_i}$. Since $\{\mathscr{A}_a\}$ is a chain, some $\mathscr{A}_{a_j}$

contains each of the other $\mathscr{A}_{a_i}$ so $A_i \in \mathscr{A}_{a_j}$, $i = 1, \ldots, k$, and $\mathscr{A}_{a_j}$ has a finite subcover. The result follows from this contradiction and the maximal principle.

Proof of Theorem 3. For fixed $A \in \mathscr{U}$ and any $B \in \mathscr{U}$, (i) $B \subset A$, or (ii) $B \cap A = \emptyset$ or (iii) $B \cap A \neq \emptyset$ and $B \cap (S - A) \neq \emptyset$. Consider the cover $\mathscr{U}'$ obtained by taking all $B \in \mathscr{U}$ of types (i) and (ii) along with sets of the form $B_1 = B \cap A$ and $B_2 = B \cap (S - A)$, $B \in \mathscr{U}$ and of type (iii). By Theorem 1, $\mathscr{I}_{\mathscr{U}} = \mathscr{I}_{\mathscr{U}'}$. Since $\mathscr{I}_{\mathscr{V}} \subset \mathscr{I}_{\mathscr{U}}$ properly, there is an $A \in \mathscr{U}$ not a subset of a finite union of elements of $\mathscr{V}$. Relative to this set A, form $\mathscr{U}'$ and $\mathscr{V}'$ as above. So $\mathscr{V}' = \mathscr{V}_1 \cup \mathscr{V}_2$ where $B \in \mathscr{V}_1$ implies $B \subset A$ and $B \in \mathscr{V}_2$ implies $B \cap A = \emptyset$. Similarly write $\mathscr{U}' = \mathscr{U}_1 \cup \mathscr{U}_2$. By Lemma 1, there is a maximal family $\mathscr{W}_1$ of subsets of A covering A, such that $\mathscr{V}_1 \subset \mathscr{W}_1$ and $\mathscr{W}_1$ has no finite subcover. Let $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$ where $\mathscr{W}_2 = \mathscr{U}_2$. Using Theorem 1, $\mathscr{I}_{\mathscr{V}'} \subset \mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{U}'}$ where the last inclusion is proper since $A \in \mathscr{U}'$. But suppose $\mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{K}} \subset \mathscr{I}_{\mathscr{U}'}$ for some cover $\mathscr{K}$, first inclusion proper. Then $\mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{K}'} \subset \mathscr{I}_{\mathscr{U}'}$ (as above $\mathscr{K}' = \mathscr{K}_1 \cup \mathscr{K}_2$). Since $\mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{K}'}$ properly, there is a $B \in \mathscr{K}'$, with B not a subset of a finite union from $\mathscr{W}$. Now $B \notin \mathscr{K}_2$ since $\mathscr{U}_2 = \mathscr{W}_2$, and B is in a finite union of elements of $\mathscr{U}$. Thus $B \in \mathscr{K}_1$, $B \notin \mathscr{W}_1$. Because of the maximal property of $\mathscr{W}_1$, $A = B \cup B_1 \cup \ldots \cup B_n$, $B_i \in \mathscr{W}_1$. But each B_i is a subset of a finite union from $\mathscr{K}_1$, thus A is a finite union of elements of $\mathscr{K}_1$. Also for $C \in \mathscr{U}_2$, $C \in \mathscr{W}_2$ so that C is a subset of a finite union from $\mathscr{K}_2$, and $\mathscr{U}' < f\mathscr{K}'$. Thus $\mathscr{I}_{\mathscr{U}'} = \mathscr{I}_{\mathscr{K}'}$ and the theorem is established.

THEOREM 4. *If $\mathscr{I}_* \subset \mathscr{I}_{\mathscr{U}}$ properly, there is a cover topology $\mathscr{I}_{\mathscr{W}}$ such that $\mathscr{I}_* \subset \mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{U}}$ with all inclusions proper.*

Proof. If $\mathscr{I}_* \subset \mathscr{I}_{\mathscr{U}}$ properly, then there exists $A \in \mathscr{U}$ with A infinite. Let $\mathscr{W}$ consist of singletons and infinitely many disjoint infinite subsets of A. Then $\mathscr{I}_* \subset \mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{U}}$ and all inclusions are proper by Theorem 1.

Note. By modifying the argument above it may be shown that if $\mathscr{I}_{\mathscr{P}} \subset \mathscr{I}_{\mathscr{U}}$ properly where $\mathscr{P}$ is a partition, then there exists a cover $\mathscr{W}$ such that $\mathscr{I}_{\mathscr{P}} \subset \mathscr{I}_{\mathscr{W}} \subset \mathscr{I}_{\mathscr{U}}$, all inclusions proper.

THEOREM 5. *Under inclusion, the set of cover topologies and the set of standard covers form isomorphic complete, non-complemented, lattices in which all lattice sums are distributive.*

Proof. Isomorphism follows from Corollary 1. The least upper bound $\bigvee \mathscr{I}_{\mathscr{U}_a}$ of the set $\{\mathscr{I}_{\mathscr{U}_a}: a \in I\}$ of cover topologies is given by $\mathscr{I}_{\mathscr{W}}$ where $\mathscr{W} = \bigcup_{a \in I} \mathscr{U}_a$ since the neighborhood system in any upper bound must include $N(A, U)$ for $A \in \mathscr{U}_a$, $a \in I$, and $\mathscr{I}_{\mathscr{W}}$ is an upper bound. Noting that standard covers are precisely those ideals in the lattice of subsets of S whose union is S, the distributivity follows as in Theorem 12.51 of [3] which establishes the corresponding result for the lattice of all ideals in the lattice of subsets of S. Theorem 3 and Theorem 4 imply that if S is infinite the lattice of cover topologies has maximal elements but no atoms and thus is not complemented.

Note. Since a standard cover $\mathcal{U}$ is closed under intersection and finite union we may identify it with the co-topology $\mathcal{U} \cup \{S\}$ on S. This establishes an isomorphism between the lattice of cover topologies on F and the lattice of T_1 topologies on S in which all neighborhoods are open.

3. Partition topologies. If some partition is in the equivalence class of covers generating a given cover topology, the topology is called a *partition topology*. Note that any cover $\mathcal{U}$ which is countably generated (i.e., there is a sequence $\{A_i\}$, $A_i \in \mathcal{U}$, such that any $A \in \mathcal{U}$ is a subset of a finite union from $\{A_i\}$) generates a partition topology. For if $\{A_i\}$ generates a cover $\mathcal{U}$ with topology $\mathcal{I}_{\mathcal{U}}$, then the partition $\{A_i - \bigcup_{k<i} A_k\}_i$ also generates $\mathcal{I}_{\mathcal{U}}$ by Theorem 1. In particular, all countable covers generate partition topologies.

THEOREM 6. *If S is countable and $\mathcal{U}$ a cover of S, a necessary and sufficient condition for $\mathcal{I}_{\mathcal{U}}$ to be a partition topology is that $\mathcal{U}$ be countably generated.*

Proof. If $\mathcal{I}_{\mathcal{U}} = \mathcal{I}_{\mathcal{P}}$ where $\mathcal{P} = \{P_1, P_2, \ldots\}$, then by Theorem 1 $P_i \subset \bigcup_{j=1}^{n_i} A_{ij}$ where $A_{ij} \in \mathcal{U}$ and the countable collection $\{A_{ij}\}$ generates $\mathcal{U}$ since if $A \in U$, A is a subset of a finite union from $\{P_i\}$ and this union is a subset of a finite union from $\{A_{ij}\}$. Sufficiency was shown above for arbitrary S.

EXAMPLE. Not all covers of a countable set are countably generated. For example, let S be the rationals and for each real x let A_x be the trace of a sequence of distinct rationals converging to x, chosen in such a manner that $S = \bigcup \{A_x: x \text{ real}\}$. Since x is the only cluster point of A_x, if $A_x \subset \bigcup_{i=1}^{n} A_{x_i}$, then x must be a cluster point of some A_{x_i} and $x = x_i$. Thus any generating set for the cover $\mathcal{U} = \{A_x: x \text{ real}\}$ must contain each A_x and no subset of $\mathcal{U}$ can generate $\mathcal{U}$.

Note. A sufficient condition, independent of the cardinality of S, that $\mathcal{I}_{\mathcal{U}}$ be a partition topology is that $\mathcal{U}$ be a star refinement of itself (see [1]). This condition is not necessary, for let $S = \{a_1, a_2, \ldots\}$, $A_i = \{a_1, \ldots, a_i\}$ and $\mathcal{U} = \{A_i\}$; $\mathcal{U}$ is not a star refinement of itself, yet $\mathcal{I}_{\mathcal{U}} = \mathcal{I}_{\mathcal{P}}$ where $\mathcal{P}$ is the singleton partition of S.

4. Remarks and questions. In the lattice of standard covers, $\bigvee \mathcal{U}_a$ is $\bigcup \mathcal{U}_a$ standardized and $\bigwedge \mathcal{U}_a$ is $\bigcap \mathcal{U}_a$. In the lattice of cover topologies, $\bigwedge \mathscr{I}_{\mathscr{U}_\alpha}$ is $\bigcap \mathscr{I}_{\mathscr{U}_\alpha}$ if $\bigcap \mathscr{I}_{\mathscr{U}_\alpha}$ is a cover topology; in any case $\bigwedge \mathscr{I}_{\mathscr{U}_\alpha} \subset \bigcap \mathscr{I}_{\mathscr{U}_\alpha}$. Of course $\bigvee \mathscr{I}_{\mathscr{U}_\alpha}$ has $\bigcup \mathscr{I}_{\mathscr{U}_\alpha}$ as a subbase so that $\bigvee \mathscr{I}_{\mathscr{U}_\alpha}$ is the same in the lattice of cover topologies as in the lattice of all topologies between $\mathscr{I}_*$ and $\mathscr{I}^*$.

Not all homogeneous topologies between $\mathscr{I}_*$ and $\mathscr{I}^*$ are cover topologies, nor can they be obtained by intersecting cover topologies. For let R_n be the region of the plane bounded by $(|x|+1)/n$ above and $-(|x|+1)/n$ below and let N_n be the set all functions of E_1 into E_1 whose graphs lie in R_n. Then N_n is a neighborhood base of the 0 function. If neighborhoods of other functions are of the form $f+N_n$, then a homogeneous topology $\mathscr{I}$ between $\mathscr{I}_*$ and $\mathscr{I}^*$ results. For any cover $\mathscr{U}$ of E_1 and unbounded $A \in \mathscr{U}$, no $N_n \subset N(A, U)$ (U a neighborhood of 0 in E_1). So $\mathscr{I} \neq \mathscr{I}_{\mathscr{U}}$ if $\mathscr{U}$ contains unbounded sets. If A is bounded, then for any $N(A, U)$ some N_n lies in $N(A, U)$; but if A is bounded, for all U, $N(A, U) \not\subset N_n$ for any n. In fact if $A \neq E_1$, $N(A, U)$ is contained in no N_n; so $\mathscr{I} \neq \mathscr{I}_{\mathscr{U}}$ where $\mathscr{U}$ is a cover by bounded sets. Thus $\mathscr{I}$ is not a cover topology. If a cover topology $\mathscr{I}_{\mathscr{U}} \neq \mathscr{I}^*$ is comparable with $\mathscr{I}$, $\mathscr{I}_{\mathscr{U}} \subset \mathscr{I}$. (This is the case if $\mathscr{U}$ is a cover by bounded sets for instance.)

Questions remaining are: Is the intersection of cover topologies (or partition topologies) a cover topology (or partition topology)? What is a characterization of covers generating partition topologies? What is a characterization of cover topologies?

In regards to the motivation for this study mentioned in the introduction, it is only fair to note that if F is the space of all functions from the real line into itself, then $(F, \mathscr{I}_*)$ is connected but barely so; $\mathscr{I}_*$ is the only topology in the lattice of cover topologies for which F is connected. If $\mathscr{I}_{\mathscr{U}} \neq \mathscr{I}_*$, some $A \in \mathscr{U}$ is infinite and the subspace of bounded functions on A is open and closed in $(F, \mathscr{I}_{\mathscr{U}})$.

References

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