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# Irregularities of distribution

**1.** If $f$ and $g$ are distributions defined in the one-dimensional interval $(-\infty, +\infty)$, then we write $f \sim g$ to mean that $f-g$ is a smooth function, i.e., $f-g \in C^\infty$.

It is easy to verify that $\sim$ is an equivalence relation.

We denote by $\tilde{f}$ the class of distributions equivalent to $f$. This class is called the *irregularity* of the distribution $f$.

The irregularities preserve only some of the properties of distributions, for example the order, but do not preserve others, for example the growth.

We can differentiate and integrate irregularities. It is interesting that, if we define the integration of irregularities as an inverse operation to differentiation, then this integration is determined uniquely, contrary to the usual case.

It is also possible to define integration of non-integral order, and, as we shall see in the sequel, in two different ways.

THEOREM 1. *Given any positive continuous function $\psi$, and any distribution $f$ of order $k$, there is a smooth function $\omega$ and distribution $g$ such that $f = g+\omega$, where $G^{(k)} = g$ and $|G| < \psi$ in $(-\infty, +\infty)$.*

It is easy to verify that Theorem 1 follows from

LEMMA 1. *Given any positive continuous function $\psi$, and any continuous function $f$, there is a smooth function $\omega$ and a continuous function $g$ such that $f = g+\omega$ and $|g| < \psi$ in $(-\infty, +\infty)$.*

Proof of Lemma. We divide the interval $(-\infty, +\infty)$ into a sequence of non-overlapping intervals $[a_n, b_n]$ such that the variation of $f$ in $[a_n, b_n]$ is less than the maximum of $\psi$ in this interval. Let $\sigma_n$ be a monotonic smooth function such that $\sigma_n(x) = f(a_n)$ for $x \leqslant a_n$ and $\sigma_n(x) = f(b_n)$ for $x \geqslant b_n$. Then the function $\omega(x)$ equal to $\sigma_n(x)$ in $[a_n, b_n]$ for all $n$, has the needed properties.

Lemma 1 follows also from a lemma of Whitney (see [5], p. 79, Lemma 6).

Remark. Theorem 1 says that in the class of equivalent distributions there exist distributions which are in a sense sufficiently small.

We define addition and scalar multiplication in the usual manner:

$$\tilde{f}+\tilde{g} = \widetilde{f+g}, \quad \lambda\tilde{f} = \widetilde{\lambda f},$$

and it is clear that the irregularities form a linear space over the field of real numbers. We put $(\tilde{f})' = \tilde{f'}$.

We denote by $\mathscr{L}^a$ $(a > 0)$, the class of all functions which vanish identically for $x < 0$, are smooth for $x > 0$, equal to $\Gamma^{-1}(a)x^{a-1}$ in a right neighbourhood of the point $x = 0$, and vanish identically outside a right neighbourhood of the point $x = 0$. (These neighbourhoods need not to be the same for different functions.)

Now, let $f$ be a distribution in $(-\infty, +\infty)$ and $\varphi$ an integrable function vanishing for $|x| > x_0 > 0$. Let $(a, b)$ be any given bounded interval. Then there exist an index $k \geqslant 0$ and a continuous function $F$ in $I = (a-x_0, b+x_0)$ such that $F^{(k)} = f$ in $I$. We put

$$\varphi * f = \Big(\int_{-x_0}^{x_0} \varphi(t)F(x-t)\,dt\Big)^{(k)}, \quad x \in (a, b),$$

the derivative being understood in the distributional sense. The consistency of this definition is easy to verify. If we take a subinterval $(c, d)$ of $(a, b)$, then in the respective construction for $(c, d)$ $k$ and $F$ may be taken the same as for $(a, b)$. Thus, the convolution $\varphi * f$ defined on $(c, d)$ coincides in $(c, d)$ with the distribution $\varphi * f$ defined on the whole interval $(a, b)$. Now, if we have two overlapping intervals $(a, b)$ and $(c, d)$, say $a < c < b < d$, and if we define $\varphi * f$ on each of them, then both convolutions coincide in the common part of the intervals, i.e. in $(c, d)$. Now, if we define the convolutions $\varphi * f$ on all bounded intervals, they will coincide in common parts of overlapping intervals. Thus, after a known theorem of Schwartz, there is a unique distribution in $(-\infty, +\infty)$ which coincides with the convolution $\varphi * f$ defined in any bounded interval. This distribution will be considered as convolution $\varphi * f$ in $(-\infty, +\infty)$. (Compare the definition of a convolution in [2].)

The convolution has the following properties:

1° *If $f$ and $g$ are distributions in $(-\infty, +\infty)$, $\varphi$ and $\psi$ are integrable functions with bounded support, then*

$$(\varphi \pm \psi) * f = \varphi * f \pm \psi * f,$$

$$\varphi * (g \pm f) = \varphi * g \pm \varphi * f,$$

$$(\varphi * f)^{(k)} = \varphi * f^{(k)}.$$

2° *If $\varphi$ is an integrable function with bounded support and $F$ is a continuous (smooth) function, then the convolution $\varphi * F$ is a continuous (smooth) function (compare* [2]).

Now, we introduce the definition of the operation $l^a$:

$$\text{(I)} \qquad l^a\tilde{f} = \widetilde{\varphi * f} \quad (\varphi \in \mathcal{L}^a).$$

$l^a f$ does not depend on the choice of $f$ from $\tilde{f}$ and $\varphi$ from $\mathcal{L}^a$. In fact, if $f_1, f_2 \in \tilde{f}$, $\varphi_1, \varphi_2 \in \mathcal{L}^a$, then $\varphi_1 * f_1 \sim \varphi_2 * f_2$, since $\varphi_1 * f_1 - \varphi_2 * f_2 = \varphi_1 * (f_1 - f_2) + (\varphi_1 - \varphi_2) * f_2$ and $\varphi_1 * (f_1 - f_2) \in C^\infty$ by 2° and $(\varphi_1 - \varphi_2) * f_2 \in C^\infty$, in view of the Theorem 16.3, v. [3].

LEMMA 2. *For every positive integer $k$ and positive real $a$, $(l^a\tilde{f})^{(k)} = l^a\tilde{f}^{(k)}$.*

Proof. Let $l^a f^{(k)} \in l^a\tilde{f}^{(k)}$ and $(l^a f)^{(k)} \in (l^a\tilde{f})^{(k)}$. Because $(l^a f)^{(k)} = (\varphi * f)^{(k)} = \varphi * f^{(k)} = l^a f^{(k)}$, therefore $l^a f^{(k)} \sim (l^a f)^{(k)}$.

THEOREM 2. *For any positive real numbers $a$ and $\beta$, $l^a(l^\beta \tilde{f}) = l^{a+\beta}\tilde{f}$.*

Proof. 1° First, we consider the case when $\tilde{f}$ is a class of distributions equivalent to a continuous function. Let $\varphi \in \mathcal{L}^a$, $\psi \in \mathcal{L}^\beta$ and $\varphi(x) = \Gamma^{-1}(a)x^{a-1}$, $\psi(x) = \Gamma^{-1}(\beta)x^{\beta-1}$ for $0 < x < x_0$. Then $h = \varphi * \psi = \int_{-\infty}^{+\infty} \psi(x-t)\varphi(t)\,dt$ belongs to $\mathcal{L}^{a+\beta}$. In fact, if $x < 0$, then $h(x) = 0$. If $x > 0$, then $h(x) \in C^\infty$ and in a right neighbourhood of zero $h(x)$ is of the form $\Gamma^{-1}(a+\beta)x^{a+\beta-1}$, since for $0 < x < x_0$, we have

$$h(x) = \Gamma^{-1}(a)\Gamma^{-1}(\beta) \int_0^x t^{a-1}(x-t)^{\beta-1}\,dt = \Gamma^{-1}(a+\beta)x^{a+\beta-1}.$$

Furthermore, $h$ vanishes identically outside a right neighbourhood of zero since $\varphi$ and $\psi$ have that property.

Now, if $f$ is a continuous function, $\varphi \in \mathcal{L}^a$, $\psi \in \mathcal{L}^\beta$, we have $l^a(l^\beta f) = \varphi * (\psi * f) = (\varphi * \psi) * f = h * f = l^{a+\beta} f$, where $h \in \mathcal{L}^{a+\beta}$. Thus $l^{a+\beta}\tilde{f} = l^a(l^\beta f)$.

2° Now, let $\tilde{f}$ be an equivalence class of an arbitrary distribution $f$. For every bounded interval $(a, b)$ there exist an index $k \geqslant 0$ and a continuous function $F$ in $(-\infty, +\infty)$ such that $F^{(k)} = f$ in $(a, b)$. Applying Lemma 2 and the case of the Theorem just proved, we get

$$l^a(l^a F^{(k)}) = l^a(l^\beta F)^{(k)} = [l^a(l^\beta F)]^{(k)} \sim (l^{a+\beta} F)^{(k)} = l^{a+\beta} F^{(k)}.$$

Since $l^a(l^\beta F^{(k)}) = l^a(l^\beta f)$ in $(a, b)$ and $l^{a+\beta} F^{(k)} = l^{a+\beta} f$ in $(a, b)$, we get $l^{a+\beta} f - l^a(l^\beta f) \in C^\infty$ in $(a, b)$.

If we take a subinterval $(c, d)$ of $(a, b)$, i.e. if $a \leqslant c \leqslant d \leqslant b$, and

$$l^{a+\beta} f - l^a(l^\beta f) = \begin{cases} \omega_1 & \text{in } (a, b), \\ \omega_2 & \text{in } (c, d), \end{cases}$$

then $\omega_1 = \omega_2$ in $(c, d)$. Now, if we have two overlapping intervals $(a, b)$ and $(c, d)$, say $a < c < b < d$, and if

$$l^{a+\beta}f - l^{a}(l^{\beta}f) = \begin{cases} \omega_1 & \text{in } (a, b), \\ \omega_2 & \text{in } (c, d), \end{cases}$$

then $\omega_1 = \omega_2$ in $(c, d)$. Thus the difference $l^{a+\beta}f - l^{a}(l^{\beta}f)$ is a smooth function in every bounded interval. We shall prove that

$$\omega = l^{a+\beta}f - l^{a}(l^{\beta}f) \in C^{\infty} \quad \text{in } (-\infty, +\infty).$$

Suppose the contrary. Then there are a point $x_0$ and an integer $k \geqslant 0$ such that $\omega^{(k)}$ is not continuous at $x_0$. Thus, for each bounded interval $(a, b)$ containing $x_0$, we have $\omega \notin C^{\infty}$ in $(a, b)$, a contradiction.

THOEREM 3. *For every class $\tilde{f}$ and every positive integer $m$*

$$(1) \qquad (l^{m}\tilde{f})^{m} = \tilde{f},$$

$$(2) \qquad l^{m}\tilde{f}^{(m)} = \tilde{f}.$$

Proof of (1). a) First, we consider the case of the class $\tilde{f}$ of distributions equivalent to a continuous function.

Let $m = 1$, $\varphi \in \mathscr{L}^1$, and let $f$ be an element of $\tilde{f}$. By Lemma 1 $f = \omega + g$, where $\omega \in C^{\infty}$ and $g$ is an integrable function satisfying $|g| < e^{-x^2}$.

Since $\varphi = H + \omega_1$, where $H$ is the Heaviside function and $\omega_1 \in C^{\infty}$, we get $l^1 f = (H + \omega_1) * f$. Evidently $l^1 f \sim l^1 g$ and $(l^1 f)' \sim (l^1 g)'$. Using the properties of convolution we get

$$\begin{aligned} (l^1 g)' &= [(H + \omega_1) * g]' \\ &= (H + \omega_1)' * g = (\delta + \omega_1') * g = \delta * g + \omega_1 * g = g + \omega_1' * g. \end{aligned}$$

From $\omega_1' * g \in C^{\infty}$ (see [1] and property 1°), it follows that the difference $(l^1 f)' - g$ is a function of the class $C^{\infty}$. Therefore $(l^1 f)' \sim f$.

To get the general result (for $m > 1$) we apply the induction on $m$.

b) Now, we consider the case when $\tilde{f}$ is the class generated by an arbitrary distribution $f$. For every bounded interval $(a, b)$ there exist an index $k \geqslant 0$ and a continuous function $F$ defined in $(-\infty, +\infty)$ such that $F^{(k)} = f$ in $(a, b)$. Using the result of the first part of the proof we get $(l^m \tilde{F})^{(m)} = \tilde{F}$. Because of $(l^m F^{(k)})^{(m)} = [(l^m F)^{(m)}]^{(k)}$, $[(l^m F)^{(m)}]^{(k)} \sim F^{(k)}$, and $(l^m F^{(k)})^{(m)} = (l^m f)^{(m)}$ in $(a, b)$ we have $(l^m f)^{(m)} - f \in C^{\infty}$ in $(a, b)$. Thus, in a bounded interval, the difference $(l^m f)^{(m)} - f$ belongs to $C^{\infty}$. Hence $(l^m f)^{(m)} - f \in C^{\infty}$ in $(-\infty, +\infty)$ and $(l^m f)^{(m)} \sim f$.

The proof of (2) is similar.

Remark. Theorem 3 implies that the operation $l^1$ is the converse of differentiation. This operation is defined uniquely on the irregularities.

However, this is not surprising, because the difference of two primitive distributions is a constant, i.e. a smooth function.

We introduce the definition:

For every real number $\alpha$ and every irregularity $\tilde{f}$ we put

$$l^{\alpha}\tilde{f} = (l^{\alpha+m}\tilde{f})^{(m)}, \tag{II}$$

where $m$ is a positive integer such that $\alpha+m > 0$.

If, in particular, $\alpha = 0$, then from (II) we get $l^0\tilde{f} = \tilde{f}$.

This definition is consistent, i.e. the right side of (II) does not depend on the choice of the number $m$, viz., for any positive integers $m$ and $\overline{m}$, we have $(l^{\alpha+m}\tilde{f})^{(m)} = (l^{\alpha+\overline{m}}\tilde{f})^{(\overline{m})}$. Indeed, let $\overline{m} > m$. In view of Theorems 2 and 3 we have

$$(l^{\alpha+\overline{m}}\tilde{f})^{(\overline{m})} = [l^{\overline{m}-m}(l^{\alpha+m}\tilde{f})]^{(\overline{m})} = \{[l^{\overline{m}-m}(l^{\alpha+m}\tilde{f})]^{(\overline{m}-m)}\}^{(m)} = (l^{\alpha+m}\tilde{f})^{(m)}.$$

Note that the definitions (I) and (II) are compatible, if $\alpha > 0$. In fact,

$$l^{\alpha}\tilde{f} = (l^{\alpha+m}\tilde{f}) = [l^{m}(l^{\alpha}\tilde{f})]^{(m)} = l^{(\alpha)}\tilde{f}.$$

THEOREM 4. *For every class $\tilde{f}$ and any real numbers $\alpha$, $\beta$, $l^{\alpha+\beta}\tilde{f} = l^{\alpha}(l^{\beta}\tilde{f})$.*

Proof. In view of Definition (II), Theorem 2 and of Lemma 1, we have, for $\alpha+n > 0$ and $\beta+m > 0$ ($m, n$ — positive integers),

$$\begin{aligned} l^{\alpha}(l^{\beta}\tilde{f}) &= l^{\alpha}(l^{\beta+m}\tilde{f})^{(m)} = [l^{\alpha}(l^{\beta+m}\tilde{f})]^{(m)} = [l^{\alpha+n}(l^{\beta+m}\tilde{f})]^{(n+m)} \\ &= (l^{\alpha+\beta+m+n}\tilde{f}) = l^{\alpha+\beta}\tilde{f}. \end{aligned}$$

THEOREM 5. *For every class $\tilde{f}$ and every positive integer $m$, $l^{-m}\tilde{f} = \tilde{f}^{(m)}$.*

Proof. Let $k$ be a positive integer such that $k-m > 0$. Then, by Definition (II), $l^{-m}\tilde{f} = (l^{k-m}\tilde{f})^{(k)}$. If $k = m+n$, then in view of Theorem 3, we have $l^{-m}\tilde{f} = (l^{n}\tilde{f})^{(m+n)} = [(l^{n}\tilde{f})^{(n)}]^{(m)} = \tilde{f}^{(m)}$.

We denote by $\tilde{Z}$ the set of all irregularities $\tilde{f}$ such that $f \in Z$.

THEOREM 6. *If $\tilde{f} \in \tilde{C}$ ($C$ — the class of continuous functions) and $\alpha \geqslant 0$, then $l^{\alpha}\tilde{f} \in \tilde{C}$.*

Proof. For the case $\alpha = 0$ of the Theorem, see the Definition (II). In the case $\alpha > 0$ it suffices to prove that for any function $f \in C$ we have $l^{\alpha}f \in C$. Since $l^{\alpha}f = \varphi * f$, where $\varphi \in \mathscr{L}^{\alpha}$, $f \in C$, we get $l^{\alpha}f \in C$ by the property 2° of the convolution. Thus $l^{\alpha}\tilde{f} \in \tilde{C}$.

The following classical theorem, due to Young, plays in the Theorems 7 and 8 an important role:

If $f \in L^p$, $g \in L^q$ and $\frac{1}{r} = \frac{1}{q} + \frac{1}{p} - 1 > 0$ $(1 \leqslant p, q < \infty)$, then $f * g \in L^r$ (see [4]).

THEOREM 7. *If $\tilde{f} \in \tilde{L}$ ($L$ — the class of integrable functions) and $a \geqslant 0$, then $l^a \tilde{f} \in \tilde{L}$.*

Proof. For the case $a = 0$ of Theorem, see the Definition (II). In the case $a > 0$, it suffices to prove that for any function $f \in L$ also $l^a f \in L$. Since $l^a f = \varphi * f$, where $\varphi \in \mathscr{L}^a$, $f \in L$, we see that $\varphi$ and $f$ satisfy the hypotheses of the Young theorem ($p = q = 1$). Thus, $l^a f \in L$.

THEOREM 8. *If $\tilde{f} \in \tilde{L}^2$ ($L^2$ — the class of 2-integrable functions) and $a \geqslant 0$, then $l^a \tilde{f} \in \tilde{L}^2$.*

Proof. For the case $a = 0$ of Theorem, see the Definition (II). The proof in the case $a > 0$ follows from Young's theorem for $p = 2$, $q = 1$.

**2.** We denote by $\mathscr{L}_a$ ($a > 0$) the class of all functions which vanish identically for $x > 0$, are smooth for $x < 0$, equal to $\Gamma^{-1}(a)\,(-x)^{a-1}$ in a left neighbourhood of the point $x = 0$, and vanish identically outside a left neighbourhood of the point $x = 0$. (These neighbourhood need not to be the same for different functions.)

We formulate the definition of the operation $l_a \tilde{f}$ and the Theorems 2′-8′ and Lemma 1′ in a way quite analogous to that used in the first Section.

THEOREM 9. *For every class $\tilde{f}$ and every integer $a$,*

$$l^a \tilde{f} = l_a \tilde{f}. \tag{3}$$

Proof. For the case $a = 0$ of the Theorem, see Theorems 6 and 6′. In the case when $a$ is a negative integer the Theorem is satisfied, by Theorems 5 and 5′.

Let $a = m$ ($m$ — a positive integer) and let $l_m \tilde{f} = \tilde{G}$, $l^m \tilde{f} = \tilde{F}$. In view of Theorems 3 and 3′ we get $(l^m \tilde{f})^{(m)} = \tilde{F}^{(m)} = \tilde{f}$ and $(l_m \tilde{f})^{(m)} = \tilde{G}^{(m)} = \tilde{f}$. Thus $\tilde{F}^{(m)} = \tilde{G}^{(m)}$. This means that $\tilde{F} = \tilde{G} + P$, where $P$ is a polynomial of degree less than $m$.

Remark. In case when $a$ is a fraction (not integer) the equality (3) need not to be satisfied, since it is possible that $l^a f - l_a f \notin C^\infty$.

EXAMPLE. Let

$$f(x) = \begin{cases} (\sqrt{x})^{-1} & \text{for } x > 0, \\ -(\sqrt{-x})^{-1} & \text{for } x < 0. \end{cases}$$

We shall show that the difference $l^a f - l_a f$ ($a = \frac{1}{2}$) is not continuous at the point $x = 0$.

Let $x > 0$ and suppose $\psi$ is an element of $\mathscr{L}_a$ such that

$$\psi(t) = \begin{cases} [\sqrt{-t}\,\Gamma(\tfrac{1}{2})]^{-1} & \text{for } -\delta \leqslant t < 0, \\ 0 & \text{for } t \leqslant -t_0. \end{cases}$$

Then

$$l_a f = \int_{-\infty}^{+\infty} \psi(x-t)f(t)\,dt = \int_{x}^{x+\delta} F(x,t)\,dt + \int_{x+\delta}^{x+t_0} \psi(x-t)(\sqrt{t})^{-1}\,dt,$$

where $F(x,t) = [\sqrt{t-x}\sqrt{t}\sqrt{\pi}]^{-1}$. Since

$$\int_{x}^{x+\delta} F(x,t)\,dt = \int_{0}^{\delta} [\sqrt{\pi}\sqrt{z(z+x)}]^{-1}\,dz > (\sqrt{\pi})^{-1}\ln|(\delta+x)x^{-1}|$$

and

$$\int_{x+\delta}^{x+t_0} \psi(x-t)(\sqrt{t})^{-1}\,dt = \int_{-t_0}^{-\delta} \psi(z)(\sqrt{x-z})^{-1}\,dz < \int_{-t_0}^{-\delta} \psi(z)(\sqrt{-z})^{-1}\,dz = \Phi(\delta, t_0),$$

we conclude that $l_a f$ tends to $+\infty$ as $x \to 0_+$.

Similarly, let $\varphi \in \mathscr{L}^a$. Then $l^a f = \int_{-\infty}^{+\infty} \varphi(x-t)f(t)\,dt \to -\infty$, as $x \to 0_+$. Therefore, if $x \to 0_+$, then $l_a f - l^a f$ tends to $+\infty$.

Similarly if $x < 0$, then $l_a f - l^a f$ tends to $+\infty$ when $x \to 0_-$.

We introduce the definition:

Let $Z$ denote one of the classes $C$, $L$ or $L^2$. The right-order of the distribution $f$ with respect to $Z$ is the infimum of the set of $a$ for which $l^a\tilde{f} \in \tilde{Z}$. Similarly, the left-order of the distribution $f$ with respect to $Z$ is the infimum of the set of $a$ for which $l_a\tilde{f} \in \tilde{Z}$.

These orders we denote by $Rf$ and $R^*f$, respectively.

From this definition and Theorem 8 we get $|Rf - R^*f| < 1$.

**3.** Note that from Theorem 1 it follows in particular that when defining the Fourier Transform for distributions, it is suficient to take in to account smooth functions only. Then the definition for an arbitrary distribution follows almost automatically.

### References

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