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On the maximal planar graphs and the four colour problem

In this paper we consider finite undirected graphs without loops and multiple edges. The necessary and sufficient conditions for a planar graph to be a maximal planar graph (i.e., a planar graph with the maximal number of edges for a given number of vertices) are given in the first part. The second part deals with the four colour problem. This problem is solved for a certain class of planar graphs (Theorem 7).

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1. On the maximal planar graphs. By a *graph* G we will mean a finite simplicial complex of dimension $\leqslant 1$. Two different vertices x and y are called *adjacent* in G, if G contains an edge with end points x and y. The number of vertices adjacent to x in G is called the *degree* of x (in G); it is denoted by $d(x)$.

$[x, y]$ will denote the edge-graph with vertices x and y; if $k \geqslant 3$, then $[x_1, x_2, \ldots, x_k, x_1]$ will denote the circuit with vertices $x_1, x_2, \ldots, x_k$ taken in the cyclic order in which they appear in the circuit.

A graph G which is a triangulation of a polytope $|G|$ in the plane E^2 is said to be *plane*; sometimes we will identify G with $|G|$. Any graph G contains the edges of two kinds. Each edge belonging to a circuit of G is *of the first kind* and each edge contained in none of circuits of G is *of the second kind*. In a plane graph G an edge of the first kind lies on the boundary of two different faces and each of the second kind lies on the boundary of only one face. A face is called a *k-gon* (a *triangle* if $k = 3$) if its boundary is connected and contains k edges where each edge of the second kind is counted twice.

Some results of Fáry [3] yield

Theorem 1. *If a plane graph G with n vertices ($n \geqslant 3$) has a face which is not a triangle, there exists a plane graph with n vertices containing G as a proper subgraph.*

A planar graph G is called *maximal* if it is not a proper subgraph of a planar graph with the same vertices as those of G.

COROLLARY 1. *Each face of a maximal planar graph G with $n \geqslant 3$ vertices is a triangle. Moreover, G is connected; if $n \neq 2$, then G contains no edge of the second kind and if $n \geqslant 4$, then the degree of each vertex of G is greater than 2.*

THEOREM 2 ([7], p. 24). *A planar graph with n vertices ($n \geqslant 3$) has m edges where*

$$m \leqslant 3n-6, \tag{1}$$

and the equality holds if and only if each face is a triangle.

Theorems 1 and 2 imply the following

THEOREM 3. *The following statements are equivalent for $n \geqslant 3$:*

(i) *G is a maximal planar graph with n vertices;*

(ii) *G is a planar graph which contains n vertices and $3n-6$ edges;*

(iii) *G is a planar graph which contains n vertices and each face of which is a triangle.*

A circuit of a graph G is called a circuit *associated* with a vertex x of G if the set of its vertices coincides with the set of all vertices of G adjacent to x; such a circuit is denoted by $C(x)$.

THEOREM 4. *A planar graph G with n vertices ($n \geqslant 4$) is maximal if and only if G is connected and for any vertex x of G there exists a circuit in G associated with x.*

Proof. The necessity is obvious.

Sufficiency. By assumption, G is a planar, connected graph which contains no edge of the second kind. Hence, each face of G is a polygon. Suppose that G is not maximal. Then, by Theorem 3, G has a face R which is a k-gon, $k > 3$. Moreover, G has a bounded face; without loss of generality we may assume that R is a bounded face.

Let D be an unbounded component of $E^2 \setminus \bar{R}$ ($\bar{R}$ is the closure of R). Then $\mathrm{Fr}(D) \subseteq \mathrm{Fr}(R)$ and the boundary $\mathrm{Fr}(D)$ is covered by a circuit L of G, $L = [x_1, x_2, \ldots, x_1]$.

We will show that $E^2 \setminus \bar{D} = R$. Suppose in $E^2 \setminus \bar{D}$ there is a vertex of G. Since G is connected, in $E^2 \setminus \bar{D}$ there is a vertex x adjacent to a vertex of L. Suppose that x is adjacent to x_2. Then a circuit $C(x_2)$ together with $[x_2, x]$ separate $E^2 \setminus \bar{D}$ into a number of domains one of which contains R. Hence either $[x_1, x_2] \not\subset \bar{R}$ or $[x_2, x_3] \not\subset \bar{R}$, a contradiction. Analogously, $E^2 \setminus \bar{D}$ contains no point of any edge of G. Therefore $E^2 \setminus \bar{D} = R$, L covers $\mathrm{Fr}(R)$, and $L = [x_1, x_2, \ldots, x_k, x_1]$, where $k > 3$.

Now, either x_1 and x_3 or x_2 and x_4 are not adjacent, for if x_2 and x_3 are adjacent, then, by the Jordan theorem, x_1 and x_3 cannot be adjacent (Fig. 1). Suppose that x_1 and x_3 are not adjacent. Then $C(x_2)$ is the union of two paths P_1 and P_2, whose only common vertices are x_1 and x_3. Since x_1

and x_3 are not adjacent, there exist two vertices y_1 and y_2 such that $y_i \neq x_1$, $y_i \neq x_3$, $y_i \in P_i$, $i = 1, 2$. Thus P_i is the union of two paths P_{i1} and P_{i3}, where the notation is arranged so that y_i and x_j be the end points of P_{ij} $(i = 1, 2;\ j = 1, 3)$. P_1 together with $[x_2, y_1]$ separate D into a number of domains, two of which (D_1 and D_3) satisfy the conditions

$$[x_1, x_2, y_1] \cup P_{11} = \mathrm{Fr}(D_1) \quad \text{and} \quad [x_3, x_2, y_1] \cup P_{13} = \mathrm{Fr}(D_3).$$

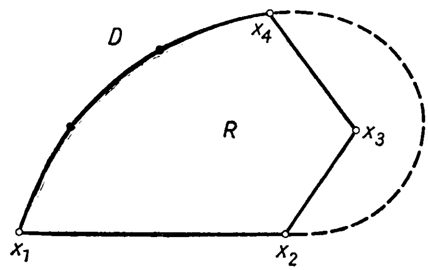

Fig. 1

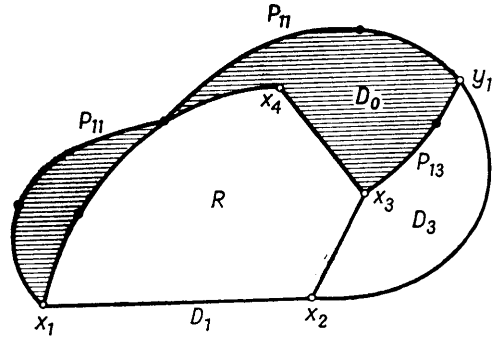

Fig. 2

Let $D_0 = E^2 \setminus (R \cup \bar{D}_1 \cup \bar{D}_3)$. Then y_2 belongs to one of D_0, D_1, D_3 (Fig. 2). If $y_2 \in D_i$ $(i = 1, 3)$ and $\{i, j\} = \{1, 3\}$, then $\mathrm{Fr}(D_i)$ separates y_2 from x_j. Therefore P_{2j} and $C(x_2)$ do not exist, a contradiction. If $y_2 \in D_0$, then G does not contain any edge $[x_2, y_2]$, a contradiction. Thus, the theorem is proved.

For characterizations of the maximal planar graphs see also [1] and [5].

2. On the four colour problem. Since every planar graph is a subgraph of a maximal one, it is enough to consider the four colour problem only for the maximal planar graphs. M_5 will denote the collection of all maximal planar graphs having all vertices of degree at least 5. Observe that it is sufficient to consider the above-mentioned problem only for graphs belonging to M_5. In fact, if the four colour hypothesis were true for graphs belonging to M_5, it would be possible to prove it for all planar graphs using the method similar to that in [2]. One can also prove that it is sufficient to consider the four colour problem for 4-connected graphs belonging to M_5.

Let a graph G have n vertices, m edges, and k_i vertices of degree i. Then

$$n = \sum_{i=0}^{n-1} k_i \quad \text{and} \quad m = \tfrac{1}{2} \sum_{i=1}^{n-1} i k_i. \tag{2}$$

Let $V_r(G)$ denote a set of any r distinct vertices of a graph G $(0 \leqslant r \leqslant n)$ and let q_r be the sum of all their degrees (if $r = 0$, then $q_r = 0$).

THEOREM 5. *Let d be an integer such that $2 \leqslant d \leqslant 5$. If a graph G with n vertices is planar and there is a $V_r(G)$ such that*

$$(3) \qquad q_r > 6n-12-d\cdot(n-r),$$

then G contains a vertex of degree less than d. If G is a maximal planar graph, the condition (3) *is also necessary.*

Proof. Suppose, if possible, that (3) is satisfied and G has no vertex of degree less than d. Then $n \geqslant 3$ and, by (2) and (3), we have

$$m \geqslant \tfrac{1}{2}[d(n-r)+q_r] > 3n-6,$$

contrary to (1). Therefore G contains a vertex of degree less than d.

Now assume that G is a maximal planar graph with n vertices, one of which, say z_0, is of degree $d(z_0)$ less than d. If $n < 3$, then (3) holds for $r = 0$. If $n \geqslant 3$, then from (2) and Theorem 3 it follows that the sum of degrees of all vertices of G different from z_0 is q_{n-1} and $q_{n-1} = 2m-d(z_0) > 6n-12-d\cdot(n-r)$, where $r = n-1$. This concludes the proof of the theorem.

COROLLARY 2. *Every planar graph with n vertices* ($0 < n < 12/(6-d)$, $2 \leqslant d \leqslant 5$) *has a vertex of degree less than d.*

Observe for the proof that (3) is satisfied if $r = 0$.

COROLLARY 3. *If a planar graph G with n vertices contains a vertex x_0 of degree $d(x_0) > n-7$, then it contains a vertex of degree less than* 5.

For the proof it suffices to consider $r = 1$, $V_1(G) = \{x_0\}$, $q_1 = d(x_0)$ and $d = 5$.

Corollary 3, (1), (2) and Theorem 3 imply

THEOREM 6. *If the degree $d(x)$ of any vertex x of a planar graph G with n vertices is $\geqslant 5$, then $k_i = 0$ if $i > n-7$ and*

$$k_5 \geqslant 12+k_7+2k_8+\ldots+(n-13)k_{n-7};$$

the equality holds if and only if $G \in M_5$.

THEOREM 7. *Every planar graph G with n vertices having a vertex x_0 of degree $d(x_0) > n-7$ is 4-chromatic (i.e., its chromatic number is $\leqslant 4$).*

Proof. For $n \leqslant 4$ this is obvious. Let us assume that the theorem is valid for some n ($n \geqslant 4$) and let G be any planar graph with $n+1$ vertices having a vertex x_0 of degree $d(x_0) > (n+1)-7$. Let us assume that x_0 is of maximal degree. From Corollary 3 it follows that G contains a vertex y_0 such that $y_0 \neq x_0$ and $d(y_0) < 5$. Deleting the vertex y_0 and all the edges incident to y_0 we obtain a new graph G_1 with n vertices. The degree of any vertex x in G_1 will be denoted by $d_1(x)$. Observe that $d_1(x_0) \geqslant d(x_0)-1$ and hence $d_1(x_0) > n-7$. By the induction hypothesis, G_1 is 4-chromatic.

Thus, all the vertices of G_1 can be painted with four colours in such a way that the adjacent vertices have different colours and all the vertices which were adjacent to the deleted vertex y_0 are painted with at most three colours. In fact, if these vertices are painted with four colours, one can suitably repaint the vertices of G_1 according to the scheme used in [2]. Consequently, G is 4-chromatic and the proof is complete.

Added in proof. Further research work of the second author stimulated by the results given in Section 1 yielded [8, 9, 10] some new characterizations of the maximal planar graphs in the collection of finite graphs with n ($\geqslant 4$) vertices. These characterizations do not make use of the known criteria of planarity.

Halin obtained [6] the characterization of denumerable maximal planar graphs.

Note in connection with Theorem 6 that Grünbaum and Motzkin [4] answered in the affirmative by a constructive method a Coxeter's problem which in dual formulation concerned the existence of the graphs $G_n^6 \in M_5$ with $k_5 = 12$, $k_6 = n \neq 1$ $(n = 0; 2, 3, \ldots)$, and $k_i = 0$ for $i \neq \neq 5, 6$.

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