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A simplified characterization of an open m-arrangement

The axioms which define an m-arrangement are given in [1], 3.1-3.9, and the concept of an open m-arrangement is discussed in [1], chapter IV. The purpose of this paper is to give a substantially simpler characterization of an open m-arrangement than is given in [1]. Specifically, we shall prove

THEOREM. *Let X be a topological space with geometry G of length $m-1 \geqslant 0$. Suppose*

(i) $F^0 = \{\{x\} \mid x \in X\}$.

(ii) *If f is a $(k-1)$-flat and g is a k-flat with $f \subset g$, then f disconnects g into two convex components which are open in g, $1 \leqslant k \leqslant m$.*

(iii) *Each 1-flat is connected.*

(iv) *If f is an $(m-1)$-flat, then we call the components of $X-f$ half-spaces of X. The collecton of half-spaces of X forms a subbasis for the topology of X.*

Then X and G form an open m-arrangement. Moreover, if X is a space with geometry G such that X and G form an open m-arrangement, then X and G satisfy properties (i)-(iv).

Proof. We first prove that if a space X together with geometry G on X of length $m-1$ satisfies (i)-(iv), then X and G form an open m-arrangement. (i) is 3.1. Since every point of X is a cut point of any 1-flat which contains it, if X and G do form an m-arrangement, then this m-arrangement must be open. (iii) is 3.3. The proof that the other axioms in the definition of an m-arrangement are satisfied is broken down into propositions and lemmas each of which refers to the situation cited in the theorem.

PROPOSITION 1. *Each flat is closed.*

Proof. From (ii), any $(m-1)$-flat is closed. Any i-flat, $0 \leqslant i \leqslant m-2$, is the intersection of finitely many $(m-1)$-flats and hence is closed. Of course X and $\emptyset$ are also closed.

Let x and y be distinct points of X. Then $f_1(x, y)-\{x\} = A_x \cup B_x$, where A_x and B_x are disjoint, non-empty, connected subsets of $f_1(x, y)$ which are open in $f_1(x, y)$. Similarly, $f_1(x, y)-\{y\} = A_y \cup B_y$. Suppose $y \in A_x$ and $x \in A_y$. Define $|xy| = \{x, y\} \cup (A_x \cap A_y) = \mathrm{Cl}A_x \cap \mathrm{Cl}A_y$. Then $|xy| \subset f_1(x, y)$.

PROPOSITION 2. *$|xy|$ is closed and connected.*

Proof. $|xy|$ is closed since it is the intersection of two closed sets. Suppose $B_x \cap B_y \neq \emptyset$. Then $B_x \cup \mathrm{Cl}B_y$ is connected, and hence must be a subset of B_x. This contradicts the assumption that $y \in A_x$. Therefore $B_x \cap B_y = \emptyset$. Suppose $|xy|$ is not connected. Then $|xy| = F \cup F'$, where F and F' are closed, disjoint, non-empty sets. If $\{x, y\} \subset F$, then $f_1(x, y) = (\mathrm{Cl}B_x \cup F) \cup (\mathrm{Cl}B_y \cup F')$, hence is not connected. If $x \in F$ and $y \in F'$, then $f_1(x, y) = (\mathrm{Cl}B_x \cup F) \cup (\mathrm{Cl}B_y \cup F')$, hence is also disconnected a contradiction in either case. Therefore $|xy|$ is connected.

PROPOSITION 3. *$f_1(x, y) = B_x \cup |xy| \cup B_y$; moreover, B_x, $|xy|$, and B_y are pairwise disjoint.*

Proof. Suppose $z \in f_1(x, y)-(B_x \cup B_y)$. Since $z \notin B_x$, $z \in A_x \cup \{x\}$, and since $z \notin B_y$, $z \in A_y \cup \{y\}$. Therefore $z \in |xy|$. From the proof of proposition 2, we have $B_x \cap B_y = \emptyset$. It follows easily from the definition of $|xy|$ that $|xy| \cap B_x = \emptyset = |xy| \cap B_y$.

COROLLARY 1. $A_x = (|xy|-\{x\}) \cup B_y$ *and* $A_y = (|xy|-\{y\}) \cup B_x$.

Proof. We show that $A_x = (|xy|-\{x\}) \cup B_y$; the proof for A_y is analogous. From proposition 3 we have $f_1(x, y)-\{x\} = A_x \cup B_x = \big((|xy|-\{x\}) \cup B_y\big) \cup B_x$. Since $A_x \cap B_x = \emptyset = \big((|xy|-\{x\}) \cup B_y\big) \cap B_x$, it follows that $A_x = (|xy|-\{x\}) \cup B_y$.

COROLLARY 2. *$|xy|$ is irreducibly connected between x and y.*

Proof. If $t \in |xy|-\{x, y\}$ and $|xy|-\{t\}$ is connected, then $f_1(x, y)-\{t\} = \mathrm{Cl}B_x \cup (|xy|-\{t\}) \cup \mathrm{Cl}B_y$ is connected, contradicting (ii).

Let f be any 1-flat and $x_0 \in f$. We define an ordering $<$ on f as follows: $f-\{x_0\} = A \cup B$, where A and B are disjoint connected subsets of f. $x_0 < y$ for any $y \in A$. $z < y$ for any $z \in B$ and $y \in \mathrm{Cl}A$. For any $w \in f-\{x_0\}$, let C_w be the component of $f-\{w\}$ which contains x_0 and D_w be the other component of $f-\{w\}$. For $z, z' \in B$, $z < z'$ if $z' \in C_z$. For $y, y' \in A$, $y < y'$ if $y' \in D_y$.

PROPOSITION 4. *$<$ is a total ordering on f.*

Proof. Let x and y be distinct points of f and suppose $x \nless y$; we show that $y < x$. Since $x \nless y$, it is impossible to have $x \in B$ and $y \in A$. If $x \in A$ and $y \in B$, then $y < x$; therefore we have only to consider the cases when x and y are either both in A or both in B. Suppose x and y are both in A; the case when x and y are both in B is analogous. Since $x \nless y$, $y \notin D_x$, hence we must show $x \in D_y$. Now if $x \in C_y$, then from Proposition 3, Corollary 1, we have $C_x = (|xy| - \{y\}) \cup D_y$. However, since $x_0 \notin D_y$, it follows that $x_0 \in |xy|$ and hence is a cut point of $|xy|$ by Proposition 3, Corollary 2. This implies then that x and y could not both be in B, a contradiction. Therefore $x \in D_y$, hence $y < x$. Suppose $x < y$ and $y < x$ with both x and y in A. Then $y \in D_x$ and $x \in D_y$, hence $f_1(x, y) = C_x \cup |xy| \cup C_y$ with $C_x \cap C_y = \emptyset$. But $x_0 \in C_x \cap C_y$, a contradiction. Thus if x and y are both in A, it is impossible to have $x < y$ and $y < x$ simultaneously; moreover, it is easily seen that if $x < y$ or $y < x$, $x \neq y$. The same conclusions can be drawn in an analogous manner if x and y are both in B.

It remains to be shown that $<$ is transitive. Suppose $x < y$ and $y < z$. The only cases of consequence are when x, y and z are either all in A, or all in B. Assume x, y and z are all in A; the proof for B is analogous. Since $x < y$, $y \in D_x$ and $y < z$ imply $z \in D_y$. Since $y \in D_x$, $x \in C_y$; for if $x \in D_y$, then this would mean that $x < y$ and $y < x$, a contradiction. Then by Proposition 3, Corollary 1, $f = C_x \cup |xy| \cup D_y$ with $C_x \cap D_y = \emptyset$. If $z \in D_y$, then $z \notin C_x$, hence $z \in D_x$; thus $x < z$.

PROPOSITION 5. *f with the ordering as described above has the order topology.*

Proof. If $x \in f$, it is easily verified that if $x \in B$, then $C_x = \{w | x < w\}$. If $x = x_0$, then $A = \{w | x_0 < w\}$, and if $x \in A$, then $C_x = \{w | w < x\}$. D_x can be characterized in an analogous manner by means of the ordering. This leads at once to $|xy| - \{x, y\} = \{w | x < w < y\}$ (assuming $x < y$), hence $|xy| - \{x, y\}$ is open in both the induced and order topologies on f. On the other hand, because of (iv), the collection of such subsets of f forms a basis for the topology on f.

PROPOSITION 6. *A subset W of X is convex if and only if given any two points x and y of W, $|xy| \subset W$.*

Proof. Suppose $|xy| \subset W$ for any $x, y \in W$; let f be any 1-flat and $w, z \in f \cap W$. Then $|wz| \subset f \cap W$ and $|wz|$ is connected, hence $f \cap W$ is connected. Therefore W is convex. On the other hand, if $x, y \in W$ and $|xy| \not\subset W$, then $f_1(x, y) \cap W$ is not connected, hence W could not be convex.

COROLLARY. *The intersection of any family of convex subsets of X is convex.*

Therefore because of Proposition 1, G is a topological geometry on X X (3.2).

PROPOSITION 7. *X is locally convex* (3.4).

Proof. Proposition 7 follows at once from Proposition 6, Corollary, and (iv).

PROPOSITION 8. *If x, y and z are points of a 1-flat f, then $|xy| \cup |yz| = |xy|$, $|yz|$, or $|xz|$.*

(Once it has been established in Proposition 10 above that $|xy| = xy$, this proposition becomes 3.5.)

Proof. Proposition 8 follows from the total ordering of f together with the fact that if $x < y$, then $|xy| = \{w | x \leqslant w \leqslant y\}$.

Let $S = \{x_0, \ldots, x_k\}$ be a linearly independent subset of X. Set $S_i = S - \{x_i\}$. By (ii), $f_{k-1}(S_i)$ disconnects $f_k(S)$ into convex components A_i and B_i. We shall assume $x_i \in A_i$.

PROPOSITION 9. a) *$f_{k-1}(S_i)$ is a minimal disconnecting subset of $f_k(S)$.*
b) $\mathrm{Fr} A_i = \mathrm{Fr} B_i = f_{k-1}(S_i)$, *hence* $\mathrm{Cl} A_i = A_i \cup f_{k-1}(S_i)$.
c) $\mathrm{Cl} A_i$ *is convex.*

Proof. a) Suppose $w \in A_i$, $z \in B_i$ and $y \in f_{k-1}(S_i)$. Then $|wy| \cup |yz|$ is connected, hence $f_k(S) - \big(f_{k-1}(S_i) - \{y\}\big)$ is connected. Therefore $f_{k-1}(S_i)$ is a minimal disconnecting subset of $f_k(S)$.

b) Let $w \in f_{k-1}(S_i)$. If $w \notin \mathrm{Fr} A_i$, then some neighborhood U of w either does not meet A_i, or does not meet B_i. Suppose U does not meet A_i. Choose $z \in B_i$. Then $f_1(w, z) \cap A_i \neq \emptyset$, or B_i would not be convex. Then $f_1(w, z) = \big((f \cup U) \cup (f \cap B_i)\big) \cup (f \cap A_i)$, hence $f_1(w, z)$ is not connected, a contradiction. Each neighborhood of w can thus be shown to meet both A_i and B_i. Since A_i and B_i are open, $f_{k-1}(S_i) = \mathrm{Fr} A_i = \mathrm{Fr} B_i$.

c) Suppose $x, y \in \mathrm{Cl} A_i$. If $x, y \in f_{k-1}(S_i)$, then $|xy| \subset f_1(x, y) \subset f_{k-1}(S_i)$. If $x, y \in A_i$, then $|xy| \subset A_i$ since A_i is convex (Proposition 6). Suppose $x \in f_{k-1}(S_i)$ and $y \in A_i$. If $|xy| \not\subset \mathrm{Cl} A_i$, then there is $w \in |xy| \cap B_i$. But then $|wy| \cap f_{k-1}(S_i) \neq \emptyset$; this implies that $|xy|$ intersects $f_{k-1}(S_i)$ in two distinct points (since $f_{k-1}(S_i)$ would disconnect $|xy|$) and hence $|xy| \subset f_{k-1}(S_i)$, a contradiction. Therefore in all cases, $|xy| \subset \mathrm{Cl} A_i$, hence by Proposition 6, $\mathrm{Cl} A_i$ is convex.

We continue to let $S = \{x_0, \ldots, x_k\}$ be a linearly independent subset of X and $S_i = S - \{x_i\}$; A_i and B_i will be as previously defined. Let $Y = \bigcap_{i=0}^{k} \mathrm{Cl} A_i$. Set $I(Y) = \bigcap_{i=0}^{k} A_i$, $E^i Y = f_{k-1}(S_i) \cap Y$, and $B(Y) = \bigcup_{i=0}^{k} E^i Y$.

LEMMA 1. $I(Y) = Y - B(Y)$.

Proof. Since Y is the intersection of closed sets, $\mathrm{Cl} Y = Y$. From Proposition 9, b), we have $B(Y) \subset \mathrm{Fr} Y$. But $I(Y)$ is an open subset of Y, hence $I(Y) \subset Y^0$, where Y^0, denotes the topological interior of Y. Thus $I(Y) = Y - B(Y)$ follows from the equality $Y^0 = \mathrm{Cl} Y - \mathrm{Fr} Y$.

PROPOSITION 10. a) $C(S) = Y$.

b) $Y = \bigcup \{\overline{x_i y} |\ y \in E^i Y\}$, $0 \leqslant i \leqslant k$.

Proof. We first prove a) and b) for $k = 1$ and $k = 2$. If $k = 1$, then $Y = |x_0 x_1|$, which is irreducibly connected between x_0 and x_1. It follows then that $|x_0 x_1| = C(\{x_0, x_1\}) = \overline{x_0 x_1}$. b) is trivially true for $k = 1$.

Suppose $k = 2$. For any k, $C(S) \subset Y$ since Y is a convex set (since Y is the intersection of convex sets) which contains S. Suppose $w \in Y - C(S)$. Since $f_1(x^0, w)$ disconnects $f_2(S)$, $f_1(x_0, w) \cap E^0 Y \neq \emptyset$. We prove this last statement as follows:

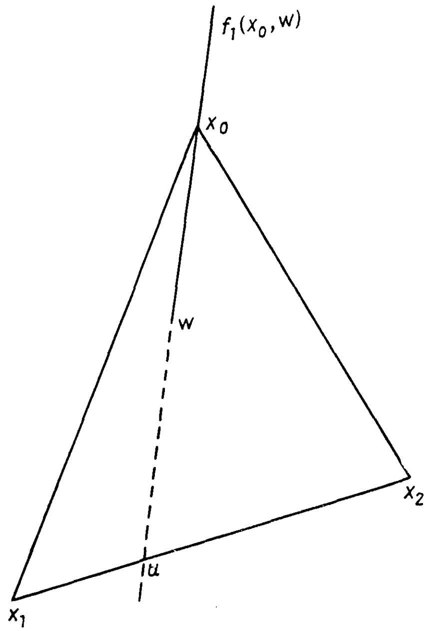

Fig. 1

$E^0 Y = f_1(x_1, x_2) \cap Y = f_1(x_1, x_2) \cap (\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2)$ (since $f_1(x_1, x_2) \subset \mathrm{Cl}A_0$). From this it follows easily that $E^0 Y = |x_1 x_2| = \overline{x_1 x_2}$. From the fact that $f_1(x_1, x_2)$ disconnects $f_2(S)$, it can be shown that $f_1(x_1, x_2) \cap \cap (\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2) = \overline{x_1 x_2}$ disconnects $\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$. For if $\overline{x_1 x_2}$ did not disconnect $\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$, then it follows that $f_1(x_1, x_2)$ would not disconnect $f_2(S)$. The components of $(\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2) - \overline{x_1 x_2}$ are $Y - \overline{x_1 x_2}$ and $B_0 \cap \mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$. Both components are in fact convex.

Let h be that component of $f_1(x_0, w)$ which contains w. Then $h = f_1(x_0, w) \cap (\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2)$. If $h \cap \overline{x_1 x_2} = \emptyset$, then $\overline{x_1 x_2}$ does not disconnect h; hence $h \subset Y - \overline{x_1 x_2}$. But it can be shown that h disconnects $\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$ and h cannot disconnect $\mathrm{Cl}A_1 \cap \mathrm{Cl}A_2$ if $h \subset Y$. Therefore we have $h \cap \overline{x_1 x_2} = f_1(x_0, w) \cap \overline{x_1 x_2} \neq \emptyset$. Suppose $u \in f_1(x_0, w) \cap E^0 Y$. Then $w \in \overline{x_0 u}$. On the other hand, $E^0 Y = \overline{x_1 x_2} = F^0 C(S)$. Therefore $w \in \overline{x_0 u} \subset C(S)$, a contradiction. Similarly, if $w \in Y$, then there is $u \in E^0 Y = F^0 C(S)$ such that $w \in \overline{x_0 u}$; given any $u \in E^0 Y$, $\overline{x_0 u} \subset Y$ since Y is convex.

Thus $Y = \bigcup \{\overline{x_0 u} | u \in F^0 C(S) = E^0 Y\}$. Similarly, $Y = \bigcup \{\overline{x_i u} | u \in E^i Y\}$, $i = 1, 2$.

We now assume that Proposition 10 has been proved for $1 \leqslant j \leqslant k-1$. In order to complete the proof of Proposition 10, several lemmas will be used. These all refer to the situation described in Proposition 10 and its preceding remarks.

LEMMA 2. $E^i Y = F^i C(S) = C(S_i)$, $0 \leqslant i \leqslant k$.

Proof. Lemma 2 is trivially true for $k = 1$. If f is any k-flat, then the subspace f with geometry G_f (of length $k-1$) satisfies (i)-(iv). Now $S_i \subset f_{k-1}(S_i)$, hence by the preceding observation, the definition of $E^i Y$, and the induction assumption, $E^i Y = C(S_i) = F^i C(S)$.

LEMMA 3. *Suppose $w \in I(Y)$ and f is any $(k-1)$-flat which contains w. Then f disconnects Y.*

Proof. f disconnects $f_k(S)$ into convex components A and B. If f does not disconnect Y, then $Y - f \subset A$, or $Y - f \subset B$; assume the former. Since $X - B(Y) = I(Y) \cup (X - Y)$ and $X - Y$ is non-empty (since each B_i is non-empty) and open, $B(Y)$ disconnects X. Choose $z \in B$. Then $\overline{zw} \cap f = \{w\}$. But $\overline{zw} \cap B(Y) \neq \emptyset$ since $B(Y)$ disconnects $\overline{zw}$. Therefore $w \in B(Y)$, a contradiction.

LEMMA 4. If $f \cap \operatorname{Int} F^i C(S) \neq \emptyset$, *then $f \cap f_{k-1}(S_i)$ is a $(k-2)$-flat which disconnects $E^i Y = F^i C(S)$.*

Proof. $f_k(S) - f = A \cup B$ as usual. Then $Y - f = (A \cap Y) \cup \cup (B \cap Y)$. If $E^i Y - f$ is connected, we may suppose it to be a subset of A. Let g be a $(k-2)$-flat in $f_{k-1}(S_i)$ with $f \cap f_{k-1}(S_i) \subset g$. Choose $z \in (g - f) \cap E^i Y$; this intersection is non-empty since g disconnects $E^i Y$ by Lemma 3 while f does not. Choose $u \in B \cap Y$. Then $\overline{uz} \cap g = \{z\}$, but $\overline{uz} \cap f \neq \emptyset$ since f disconnects $\overline{uz}$. Therefore $\overline{uz} \subset g$, a contradiction since $z \in B \cap Y$, but $g \cap Y \subset A \cap Y$.

Since a $(k-2)$-flat is a minimal disconnecting subset of a $(k-1)$-flat, $f \cap f_{k-1}(S_i)$ is a $(k-2)$-flat; and if a $(k-2)$-flat disconnects $E^i Y = C(S_i)$, it can be shown to be a minimal disconnecting subset.

LEMMA 5. *Suppose f is a $(k-1)$-flat, $F^1(F^0 C(S)) \subset f$ and $f \cap I(Y) \neq \emptyset$. Then $f \cap \overline{x_0 x_1}$ consists of exactly one point z and $f \cap Y = C(\{x_2, x_3, \ldots \ldots, x_k, z\})$.*

Proof. The lemma has already been proved for $k = 2$. Assume it is true for $k - 1 \geqslant 2$. If $f \cap \overline{x_0 x_1} \neq \emptyset$, then it can be shown that f does not disconnect Y, contradicting Lemma 3. Since $\overline{x_0 x_1} \not\subset f$, the intersection must consist of a single point z. Because of the induction assumption, $Y \cap f = C(\{x_2, x_3, \ldots, x_k, z\})$ (cf. the proof of Lemma 2).

Proof of Proposition 10 completed: It follows from Lemma 5 that $Y = \bigcup \{C(\{x_2, \ldots, x_k, z\}) | z \in \overline{x_0 x_1}\}$. By the induction assumption b), for any $z \in x_0 x_1$, $C(\{x_1, \ldots, x_k, z\}) = \bigcup \{\overline{x_2 v} | v \in F^2 C(\{x_2, \ldots, x_k, z\})$. We already have $E^i Y = C(S_i) \subset C(S)$, $0 \leqslant i \leqslant k$. Now $E^2 Y = F^2 C(S) = \bigcup \{F^2 C(\{x_2, \ldots, x_k, z\}) | z \in \overline{x_0 x_1}\}$, hence $Y = \bigcup \{\overline{x_2 v} | v \in E^2 Y = F^2 C(S)\} \subset C(S)$. An analogous proof could be used to show $Y = \bigcup \{\overline{x_i v} | v \in F^i C(S)\}$, $0 \leqslant i \leqslant k$. Therefore a) and b) hold for k and the proof of Proposition 10 is complete.

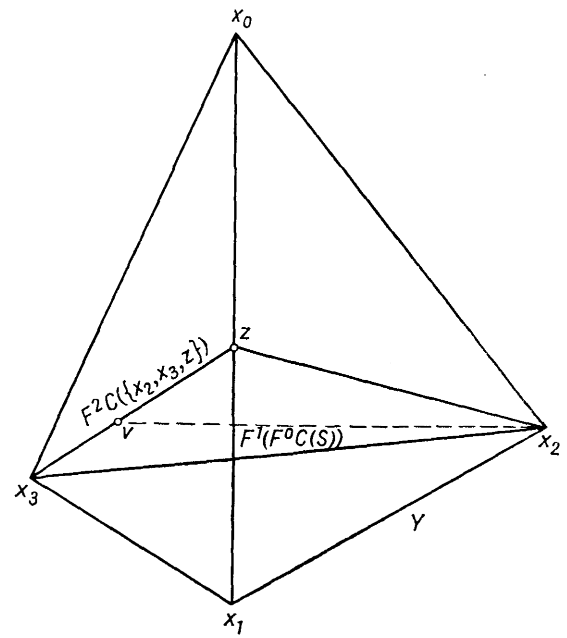

Fig. 2

The following results follow immediately from what has been done so far.

PROPOSITION 11. a) $\mathrm{Fr}\, Y = B(Y) = \mathrm{Bd}\, C(S)$ (3.8).

b) *If f is a 1-flat in $f_k(S)$ such that $f \cap \mathrm{Int} F^0 C(S) \neq \emptyset$, then $f \cap \mathrm{Int} C(S) = f \cap I(Y) \neq \emptyset$. (For if not, then one component of $f_k(S) - f$ would not be convex.)* (3.7).

c) *If f is a $(k-1)$-flat contained in a k-flat f' and $C(S)$ is a 2-simplex in f' such that f intersects the interior of one face of $C(S)$ in a single point, then f intersects another face of $C(S)$ also.*

Using the results of [2] and c) in Proposition 11, we have that X and G satisfy 3.1-3.9, hence X and G form an m-arrangement.

Suppose now that X and G form an open m-arrangement. Then (i) is 3.1 and (iii) is 3.3. (ii) follows from 3.25, 4.1, and 4.4.4 of [1], while (iv) follows from 4.6 and the following proposition.

PROPOSITION 12. *If* $S = \{x_0, \ldots, x_m\}$ *is a linearly independent subset of* X, *let* A_i *be that component of* $X - f(S - \{x_i\})$ *which contains* x_i. *Then* $\bigcap_{i=0}^{m} A_i = \operatorname{Int} C(S)$.

Proof. If $z \in \operatorname{Int} C(S)$, then it is easily shown that $\overline{x_i z} \subset C(S) - f_i(S - \{x_i\}) \subset A_i$; thus $\operatorname{Int} C(S) \subset \bigcap_{i=0}^{m} A_i$. If $y \in \bigcap_{i=0}^{m} A_i - \operatorname{Int} C(S)$, then some neighborhood V of y must lie entirely in $\bigcap_{i=0}^{m} A_i - \operatorname{Int} C(S)$. For if not, then some net of points of $\operatorname{Int} C(S)$ must converge to y. Then $y \in \operatorname{Cl}(\operatorname{Int} C(S)) = C(S)$. But then $y \in \operatorname{Bd} C(S)$. However, since any face $F^i C(S)$ of $C(S)$ is in $f_{m-1}(S - \{x_i\})$, y could not be in A_i. Therefore $\bigcap_{i=0}^{m} A_i - \operatorname{Int} C(S)$ is open as is $\operatorname{Int} C(S)$. Then $\bigcap_{i=0}^{m} A_i$ is not connected, contradicting the fact that it is convex. Therefore $\bigcap_{i=0}^{m} A_i - \operatorname{Int} C(S) = \emptyset$, hence $\bigcap_{i=0}^{m} A_i \subset \operatorname{Int} C(S)$.

This completes the proof of the main theorem.

The following examples illustrate the independence of (i)-(iv). It is of course realized that (ii) is really several axioms. No attempt is made here to fully analyze all its parts or the independence thereof.

Independence of (i). The usual spherical geometry on the 2-sphere, i.e. the 0-flats being pairs of antipodal points and the 1-flats being great circles.

Independence of (ii). Let X be the union of the x and y-axes in the coordinate plane with the usual topology. Let $G = \{F^{-1}, F^0\}$, where $F^0 = \{\{x\} \mid x \in X\}$. Then each $x \in X$ disconnects X into at least two components each of which is convex; moreover, the collection of these components forms a subbasis for the topology on X. Of course $(0,0)$ disconnects X into four components rather than two.

Let R^2 be the coordinate plane with the usual geometry, but with the coarsest topology which makes each line a closed set. Then each line does not disconnect R^2, but the collection of sets of the from $R^2 - f$, where f is a line, forms a subbasis for the topology on R^2.

Independence of (iii). Let $X = \{1, 2, 3\}$ with the discrete topology and $F^0 = \{\{x\} \mid x \in X\}$.

Independence of (iv). Let R be the set of real numbers with $F^0 = \{\{x\} \mid x \in R\}$. Let a subbasis for a topology on R consist of the open intervals together with $\{x \mid x$ is a rational number in $(0,1)\}$.

References

[1] M. Gemignani, *Topological geometries and a new characterization of R^m*, Notre Dame Journal of Formal Logic, Vol. VII, No. 1 (Jan., 1966), pp. 57-100.

[2] — *On eliminating an unwanted axiom from the characterization of R^m by means of topological geometries*, ibidem, Vol. VII, No. 4 (Oct., 1966), pp. 365-366.