ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XII (1968)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XII (1968)

JAN BOCHENEK (Kraków)

On eigenvalues and eigenfunctions of strongly elliptic systems of differential equations of second order

The purpose of the present paper is to determine eigenvalues and eigenfunctions for the so-called strongly elliptic systems of partial differential equations of second order and to generalize some theorems known for one equation to systems of equations. We shall use the following notation: if $B = \{b_{ij}\}_{i,j=1}^{n}$ is a matrix and $U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_n)$ are vectors, then

$$BU = (b_{11}u_1 + \ldots + b_{n1}u_n, \ldots, b_{1n}u_1 + \ldots + b_{nn}u_n)$$

and $UV = u_1v_1 + \ldots + u_nv_n$.

§ 1. Let G be a bounded Jordan-measurable domain in the space E^m of m variables $X = (x_1, \ldots, x_m)$. G may be approximated by an increasing sequence of domains G_n with regular boundaries (i.e., such that the boundary $F(G_n)$ of G_n is a surface of class C^1_σ; for a definition of a surface of class C^1_σ, see [3] p. 132). We do not require any regularity properties of the boundary of G. Domains with regular boundaries will shortly be called *regular domains*.

We shall consider a system of differential equations of the form

$$L(U) + \mu P(X)U = 0, \tag{1}$$

where

$$L(U) = \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[A_{ij}(X)\frac{\partial U}{\partial x_j}\right] - Q(X)U$$

is a self adjoint differential operator and μ is a real parameter. We make the following assumptions: the coefficients of system (1) are symmetric $n \times n$ matrices, $P(X)$ and $Q(X)$ are positive definite and continuous in $\bar{G}$, $A_{ij}(X) = A_{ji}(X)$ $(i, j = 1, \ldots, m)$ are of class C^1 in $\bar{G}$, and

$$\sum_{i,j=1}^{m} \xi_i A_{ij} \xi_j \geqslant a \sum_{i=1}^{m} |\xi_i|^2$$

for all system of vectors $\xi_i = (\xi_{i1}, \ldots, \xi_{in})$ $(i = 1, \ldots, m)$, $a > 0$ is a real constant, and $U(X) = \big(u_1(X), \ldots, u_n(X)\big)$. We shall also consider a generalized boundary condition (cf. [1]), which in the case where the boundary $F(G)$ is regular may be written in the form

$$(2)\qquad \frac{dU}{d\nu} - K(X)\,U = 0 \quad \text{on } F(G) - \Gamma, \qquad U = 0 \quad \text{on } \Gamma,$$

where Γ denotes an $(m-1)$-dimensional part of $F(G)$ (Γ being connected or not); in extreme cases Γ may be the whole boundary of G, or the empty set. Here $K(X)$ is a symmetric matrix which is continuous and positive definite in $\bar{G}$, $dU/d\nu$ is the transversal derivative of U with respect to system (1), i.e.,

$$(3)\qquad \frac{dU}{d\nu} = \sum_{i,j=1}^{m} A_{ij}(X)\frac{\partial U}{\partial x_i}\cos(n, x_j),$$

n being the internal normal to $F(G)$.

We shall consider the eigenvalues and eigenfunctions corresponding to system (1) and condition (2); we shall shortly say: eigenvalues and eigenfunctions of problem (1) (2). The boundary condition (2) comprises as special cases all boundary conditions useful in applications.

Therefore we shall now explain the meaning of condition (2). Following [1] we introduce some linear function spaces and some bilinear functionals on these spaces. Namely we put

$$(4)\quad D(\Phi,\Psi) = \int_G \Big[\sum_{i,j=1}^{m} \frac{\partial \Phi}{\partial x_i} A_{ij}(X) \frac{\partial \psi}{\partial x_j} + \Phi Q(X)\Psi\Big] dX + \int_{F(G)-\Gamma} \Phi K(X)\Psi dS,$$

$$H(\Phi, \Psi) = \int_G \Phi P(X)\Psi dX,$$

where $\Phi = (\varphi_1, \ldots, \varphi_n)$, $\Psi = (\psi_1, \ldots, \psi_n)$. The functionals (4) and (5) are defined as follows. Given Φ and Ψ, we consider an increasing sequence of regular domains G_n whose closures are contained in G and convergent to G. Suppose that expressions (4) and (5) are defined for Φ and Ψ and every G_n, $n = 1, 2, \ldots$ (In (4) the second integral is taken on $F(G_n) - \Gamma_n$, where Γ_n denotes a part of $F(G_n)$ such that $\Gamma_n \to \Gamma$ for $n \to \infty$.) If these expressions have finite limits for every sequence $\{G_n\}$, these limits are the desired values of D and H, respectively. From (4), (5) and from the assumptions on the coefficients of system (1) it follows that the functionals D and H are symmetric, i.e., $D(\Phi, \Psi) = D(\Psi, \Phi)$ and $H(\Phi, \Psi) = H(\Psi, \Phi)$. Let $D(\Phi) = D(\Phi, \Phi)$ and $H(\Phi) = H(\Phi, \Phi)$. Observe that $D(\Phi) \geqslant 0$ and $H(\Phi) \geqslant 0$ and

$$(6)\qquad D(\alpha\Phi + \beta\Psi) = \alpha^2 D(\Phi) + 2\alpha\beta D(\Phi, \Psi) + \beta^2 D(\Psi),$$

$$(7)\qquad H(\alpha\Phi + \beta\Psi) = \alpha^2 H(\Phi) + 2\alpha\beta H(\Phi, \Psi) + \beta^2 H(\Psi).$$

The equality $H(\Phi) = 0$ may occur only if $\Phi \equiv 0$. (6) and (7) are valid for arbitrary real numbers α and β and for all functions Φ, Ψ for which D and H are defined.

DEFINITIONS. $\mathscr{B}$ denotes the space of all functions Φ of class C_σ^0 in G such that $H(\Phi) < \infty$ (for a definition of a function of class C_σ^n $(n \geqslant 0)$, see [1]). $\mathscr{L}$ denotes the space of all functions Φ of class C_σ^1 in G such that $H(\Phi) < \infty$ and $D(\Phi) < \infty$. $\mathscr{L}_1$ denotes the subspace of $\mathscr{L}$ of functions Φ such that $\Phi(X) \equiv 0$ at all points of G whose distance from Γ is less than or equall to some $\varepsilon > 0$. $\mathscr{L}_2$ denotes the subspace of $\mathscr{L}$ of functions Φ for which there exists a sequence $\Phi_\nu \in \mathscr{L}_1$ such that $H(\Phi_\nu - \Phi) \to 0$ and $D(\Phi_\nu - \Phi) \to 0$ for $\nu \to \infty$.

In the sequel the boundary condition $U = 0$ on Γ will mean that $U \in \mathscr{L}_2$. Let $\mathscr{F}$ denote the subspace of $\mathscr{L}$ consisting of all functions Φ of class C^2 in G such that $L(\Phi) \in \mathscr{B}$.

We want to specify the meaning of the boundary condition $dU/d\nu - K(X)U = 0$ on $F(G) - \Gamma$. We consider a regular domain G_ε whose closure is contained in G, such that the distance between $F(G_\varepsilon)$ and $F(G)$ is less than ε. In a way similar to that in [1] one can prove that

$$(8) \qquad D_\varepsilon(\Phi, \Psi) + H_\varepsilon(P^{-1}L(\Phi), \Psi) + \int_{F(G_\varepsilon) - \Gamma_\varepsilon} \Psi\left(\frac{d\Phi}{d\nu} - K\Phi\right) dS = 0,$$

where $\Phi \in \mathscr{F}$, $\Psi \in \mathscr{L}_1$, and Γ_ε denotes the set of points of $F(G_\varepsilon)$ whose distance from Γ is less than or equal to ε. D_ε and H_ε are defined as functionals D and H, respectively, by integration over the domain G_ε; P^{-1} denotes the inverse matrix of P. Now let $\varepsilon \to 0$ in (8). Since the limits

$$D(\Phi, \Psi) = \lim_{\varepsilon \to 0} D_\varepsilon(\Phi, \Psi) \quad \text{and} \quad H(\Phi, \Psi) = \lim_{\varepsilon \to 0} H_\varepsilon(\Phi, \Psi),$$

exist, there also exists the limit

$$\int_{F(G) - \Gamma} \Psi\left(\frac{d\Phi}{d\nu} - K\Phi\right) dS = \lim_{\varepsilon \to 0} \int_{F(G_\varepsilon) - \Gamma_\varepsilon} \Psi\left(\frac{d\Phi}{d\nu} - K\Phi\right) dS.$$

The involved limits are related by the formula

$$(9) \qquad D(\Phi, \Psi) + H\big(P^{-1}L(\Phi), \Psi\big) + \int_{F(G) - \Gamma} \Psi\left(\frac{d\Phi}{d\nu} - K\Phi\right) dS = 0.$$

One can prove that if (9) is valid for every $\Phi \in \mathscr{F}$ and $\Psi \in \mathscr{L}_1$, then it is also valid for $\Phi \in \mathscr{F}$ and $\Psi \in \mathscr{L}_2$ (compare an analogous theorem for one equation in [2]).

The boundary condition $dU/d\nu - K(X)U = 0$ on $F(G)-\Gamma$ for $U \epsilon \mathscr{F}$ is now defined by the requirement that the equality

$$(10) \qquad \int_{(G)-\Gamma} \Psi\left(\frac{dU}{d\nu} - K(X)U\right) dS = 0$$

be valid for all $\Psi \epsilon \mathscr{L}_2$. The boundary condition (2) is defined by the requirements that $U \epsilon \mathscr{F} \cap \mathscr{L}_2$ and (10) be valid for all $\Psi \epsilon \mathscr{L}_2$. $\mathscr{F}_{K,\Gamma}(G)$ will denote the space of all functions $\Phi \epsilon \mathscr{F}$ satisfying (2) in the above sense.

Remark 1. If the boundary $F(G)$ of G is sufficiently regular (for instance, if $F(G)$ is a surface of class C_σ^1), if $U \epsilon \mathscr{F}$ is of class C^1 in the closure $\bar{G}$ of G, and if U satisfies (2) in the ordinary sense, then $U \epsilon \mathscr{F}_{K,\Gamma}(G)$.

Indeed, it is obvious that U satisfies (10) for all $\Psi \epsilon \mathscr{L}_2$. The proof of the fact that $U \epsilon \mathscr{L}_2$ is analogous to the case of one equation (cf. [1]).

§ 2. Eigenvalues and eigenfunctions of problem (1) (2)

1. We define eigenvalues and eigenfunctions of problem (1) (2) in the following way (variationally): the first eigenvalue λ_1 of problem (1)(2) is

$$(11) \qquad \lambda_1 = \min_{\Phi \epsilon \mathscr{L}_2} \frac{D(\Phi)}{H(\Phi)}$$

and the first eigenfunction U_1 is a function Φ at which the minimum (11) is attained. Having defined eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenfunctions $U_1, \ldots, U_n$ we put

$$(12) \qquad \lambda_{n+1} = \min_{\Phi \epsilon \mathscr{K}_n} \frac{D(\Phi)}{H(\Phi)}$$

where $\mathscr{K}_n$ is the subclass of $\mathscr{L}_2$ consisting of the functions Φ satisfying the orthogonality conditions $H(\Phi, U_i) = 0$ for $i = 1, \ldots, n$; U_{n+1} is a function $\Phi \epsilon \mathscr{K}_n$ at which minimum (12) is attained.

We shall need the following assumption:

HYPOTHESIS Z. *Given* (1) *and* (2), *there exists a sequence of eigenvalues*

$$(13) \qquad 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_2 \leqslant \ldots$$

and a corresponding sequence of eigenfunctions

$$(14) \qquad U_1(X),\ U_2(X),\ U_3(X), \ldots$$

which belong to $\mathscr{F}$.

We do not know whether the hypothesis Z is satisfied under our assumptions on the coefficients of system (1).

2. It is clear that the eigenvalues form an increasing sequence of non-negative numbers. In order to investigate further properties of eigenvalues, we shall now give another definition of eigenvalues of problem (1) (2).

Let $\mathscr{V}_n$ denote a set of n functions $V_1(X), \ldots, V_n(X)$ belonging to the space $\mathscr{B}$ and let

$$d[\mathscr{V}_n] = \min_{U \epsilon \overline{\mathscr{K}}_n} \frac{D(U)}{H(U)}$$

where $\overline{\mathscr{K}}_n$ is the subclass of $\mathscr{L}_2$ consisting of functions U satisfying the orthogonality conditions $H(U, V_i) = 0$ for $i = 1, \ldots, n$.

THEOREM 1. *If hypothesis* Z *and the above assumptions are satisfied, then*

$$\lambda_{n+1} = \sup\{d[\mathscr{V}_n]: \mathscr{V}_n \epsilon \mathscr{B}\}.$$

The proof of this theorem is quite similar to the proof of an analogous theorem in the case of one equation (cf. [2], p. 405 or [4], p. 289). The proofs of the following theorems are also similar to those for one equation and are omitted.

THEOREM 2. *If $\{\mu_n\}$, $\{\lambda_n\}$ and $\{\nu_n\}$ are the sequences of eigenvalues of the system* (1) *with boundary condition $U = 0$ on $F(G)$, with boundary condition* (2), *and with boundary condition $dU/d\nu = 0$ on $F(G)$ (in the generalized sense), respectively, then*

$$\nu_n \leqslant \lambda_n \leqslant \mu_n \quad (n = 1, 2, \ldots).$$

THEOREM 3. *Let $\{\lambda_n^{(1)}\}$ be a sequence of eigenvalues of the system* (1) *with boundary condition $U = 0$ on $F(G_1)$, where $G_1 \subset G$, then*

$$\lambda_n \leqslant \lambda_n^{(1)} \quad (n = 1, 2, \ldots),$$

where $\{\lambda_n\}$ is the sequence of eigenvalues of problem (1) (2).

THEOREM 4. *If the matrix $P_2(X) - P_1(X)$ is positive definite in $\overline{G}$ and if $\{\lambda_n^{(2)}\}$ and $\{\lambda_n^{(1)}\}$ are the sequences of eigenvalues of problem* (1) (2) *where $P(X) = P_2(X)$ and $P(X) = P_1(X)$, respectively, then $\lambda_n^{(2)} \leqslant \lambda_n^{(1)}$ $(n = 1, 2, \ldots)$.*

THEOREM 5. *If the matrix $Q_2(X) - Q_1(X)$ is positive definite in $\overline{G}$ and if $\{\lambda_n^{(2)}\}$ and $\{\lambda_n^{(1)}\}$ are the sequences of eigenvalues of problem* (1) (2), *where $Q(X) = Q_2(X)$ and $Q(X) = Q_1(X)$, respectively, then $\lambda_n^{(2)} \geqslant \lambda_n^{(1)}$ $(n = 1, 2, \ldots)$.*

§ 3. Lagrange's Lemma. *If the function $F(X) = \big(f_1(X), \ldots, f_n(X)\big)$ belongs to the space $\mathscr{B}$ and if*

$$\int_G F(X)\Phi(X)\,dX = 0, \tag{15}$$

where $\Phi(X) = (\varphi_1(X), \ldots, \varphi_n(X))$ *is an arbitrary function of* $\mathscr{L}_2$, *then* $F(X) \equiv 0$ *in domain* G.

Proof. Let $\Phi(X) = (0, \ldots, 0, \varphi_i(X), 0, \ldots, 0)$ belong to $\mathscr{L}_2$. Then (15) may be written in the form

$$\int_G f_i(X)\varphi_i(X)dX = 0. \tag{16}$$

It is known ([4], p. 247) that (16) implies $f_i(X) \equiv 0$ in G.

Theorem 6. *If hypothesis Z is satisfied, then each function* $U_n(X)$ *of sequence* (14) *satisfies equation* (1) *with* $\mu = \lambda_n$ $(n = 1, 2, \ldots)$. *Moreover,* $U_n(X) \epsilon \mathscr{F}_{K,\Gamma}(G)$.

Proof. To begin with, observe that if $\Phi(X) \epsilon \mathscr{F}_{K,\Gamma}(G)$ satisfies (1) for $\mu = t$, then by (9)

$$D(\Phi, \Psi) - tH(\Phi, \Psi) = 0, \tag{17}$$

for every $\Psi \epsilon \mathscr{L}_2$. We shall now show that if (17) holds for a fixed $\Phi(X) \epsilon \mathscr{F}$ and for every $\Psi \epsilon \mathscr{L}_2$, then $\Phi(X)$ satisfies (1) for $\mu = t$ and $\Phi(X)$ satisfies (10) for every $\Psi \epsilon \mathscr{L}_2$. Indeed, by (9) and (17) we have

$$H\big(t\Phi + P^{-1}(X)L(\Phi), \Psi\big) + \int_{F(G)-\Gamma} \Psi\left(\frac{d\Phi}{d\nu} - K(X)\Phi\right) dS = 0.$$

Since Ψ in $\mathscr{L}_2$ is arbitrary, Lagrange's Lemma yields

$$L(\Phi) + tP(X)\Phi = 0 \quad \text{and} \quad \int_{F(G)-\Gamma} \Psi\left(\frac{d\Phi}{d\nu} - K(X)\Phi\right) dS = 0.$$

Therefore all that remains to be shown is that

$$D(U_n, \Psi) - \lambda_n H(U_n, \Psi) = 0, \quad n = 1, 2, \ldots, \tag{18}$$

for any $\Psi \epsilon \mathscr{L}_2$. We shall prove it by induction. If $n = 1$, let Ψ be an arbitrary function of $\mathscr{L}_2$, and let τ be an arbitrary real number. Put $\Phi = U_1 + \tau\Psi$. Then by (11) $D(\Phi) \geqslant \lambda_1 H(\Phi)$. Therefore by (6), (7) and by the definition of U_1 we get

$$2\tau[D(U_1, \Psi) - \lambda_1 H(U_1, \Psi)] + \tau^2[D(\Psi) - \lambda_1 H(\Psi)] \geqslant 0.$$

Since τ is arbitrary, this is possible only if

$$D(U_1, \Psi) - \lambda_1 H(U_1, \Psi) = 0.$$

Let us now assume that (18) holds for $n = 1, \ldots, s$ and for any $\Psi \epsilon \mathscr{L}_2$. Because of (12) we have $D(\Phi) \geqslant \lambda_{s+1} H(\Phi)$, $\Phi = U_{s+1} + \tau W$, τ being an arbitrary real number and W an arbitrary function in $\mathscr{K}_s$. Hence as in 1°,

$$D(U_{s+1}, W) - \lambda_{s+1} H(U_{s+1}, W) = 0.$$

We have to show that this equality holds also for every $\Psi \epsilon \mathscr{L}_2$. Indeed, given $\Psi \epsilon \mathscr{L}_2$, let $a_1, \ldots, a_s$ be real numbers such that $W = \Psi + a_1 U_1 + \ldots \ldots + a_s U_s$ belongs to $\mathscr{K}_s$. This is always possible if we denote $a_i = -H(\Psi, U_i)/H(U_i)$, $i = 1, \ldots, s$. For such W, by the induction assumption and the symmetry of D and H, we get

$$D(U_{s+1}, \Psi) - \lambda_{s+1} H(U_{s+1}, \Psi) = 0, \qquad \text{Q.E.D.}$$

§ 4. Completness of eigenfunctions of problem (1) (2)

THEOREM 7. *If hypothesis* Z *is satisfied, the sequence* (13) *of eigenvalues of problem* (1) (2) *tends to infinity for* $n \to \infty$.

Proof. By Theorem 2 it suffices to prove that the sequence $\{\nu_n\}$ of eigenvalues of Neumann's problem (i.e., with boundary condition $dU/d\nu = 0$ on $F(G)$) tends to $+\infty$. By Theorems 4 and 5, and by the assumption on quadratic form of system (1) we get $D(\Phi)/H(\Phi) \geqslant D_0(\Phi)/H_0(\Phi)$, where

$$D_0(\Phi) = a \int_G \sum_{i=1}^{m} \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_i} dX, \qquad H_0(\Phi) = P \int_G \Phi \Phi \, dX,$$

a is the positive constant appearing in the assumption on quadratic form of system (1); P is a positive constant satisfying the inequality $\Phi P(X) \Phi \leqslant \Phi P \Phi$, and Φ is arbitrary function in $\mathscr{L}$. Let us observe that the functionals D_0 and H_0 correspond to the following system of equations

$$(19) \qquad \Delta \varphi_i + \mu \frac{P}{a} \varphi_i = 0, \qquad i = 1, \ldots, n$$

with the boundary condition $\partial \varphi_i / \partial n = 0$ on $F(G)$, $i = 1, \ldots, n$. Therefore the eigenvalues of Neumann's problem for system (1) are not less than the corresponding eigenvalues of Neumann's problem for the system (19). Consequently every eigenvalue of system (19) is an eigenvalue of a single equation of the form

$$(20) \qquad \Delta u + \mu \frac{P}{a} u = 0$$

with boundary condition $\partial u / \partial n = 0$ on $F(G)$. We see that the sequence of eigenvalues of the system (19) may be received from the sequence of eigenvalues of equation (20) by repeting each eigenvalue of equation (20) a finite number of times, respectively.

It is known ([2], p. 424) that the sequence of eigenvalues of equation (20) tends to $+\infty$ for $n \to \infty$, therefore the sequence of eigenvalues of system (19) also tends to infinity. Denoting by $\{\bar{\nu}_n\}$ the increasing sequence of eigenvalues of system (19) with boundary condition of Neumann's type on $F(G)$, we infer that $\bar{\nu}_n \leqslant \lambda_n$, $n = 1, 2, \ldots$ This yields Theorem 7.

Now we shall use Theorem 7 to prove the following theorem:

THEOREM 8. *The set of eigenfunctions of problem* (1) (2) *form a complete orthogonal system in the domain* G.

Proof. Let $\{U_n(X)\}$ be the sequence of eigenfunctions of problem (1) (2) normalized so that $H(U_n) = 1$, $n = 1, 2, \ldots$ and let $\{\lambda_n\}$ be the corresponding increasing sequence of eigenvalues. Let $F(X)$ be any continuous function in G. Let c_n $(n = 1, 2, \ldots)$ denote the Fourier coefficients of $F(X)$ with respect to $\{U_n(X)\}$ with weight $P(X)$, i.e., let

$$c_n = \int_G F(X)P(X)U_n(X)dX, \qquad n = 1, 2, \ldots$$

We shall prove Parseval's identity:

$$\int F(X)P(X)F(X)dX = \sum_{n=1}^{\infty} c_n^2,$$

or, in another form, $H(F) = \sum c_n^2$. Let $F(X)$ be a function of $\mathscr{L}_1$. Let $\Psi_n = F(X) - \sum_{i=1}^{n} c_i U_i(X)$. We shall prove that $\lim H(\Psi_n) = 0$. If $k \leqslant n$ we have

$$H(\Psi_n, U_k) = H(F, U_k) + \sum_{i=1}^{n} c_i H(U_i, U_k) = c_k - c_k = 0.$$

Hence $H(\Psi_n, U_k) = 0$ for $k \leqslant n$. In virtue of (17), for any function $\Phi(X) \in \mathscr{L}_1$ we have $D(\Phi, U_k) - \lambda_k H(\Phi, U_k) = 0$, $k = 1, 2, \ldots$ Consequently $D(\Psi_n, U_k) = 0$ for $k \leqslant n$. By definition of eigenfunction $U_k(X)$ and by (12) we get

(21) $$H(\Psi_n) \leqslant 1/\lambda_{n+1} D(\Psi_n).$$

On the orher hand

$$D(F) = D\left(\Psi_n + \sum_{i=1}^{n} c_i U_i\right)$$
$$= D(\Psi_n) + 2\sum_{i=1}^{n} c_i D(\Psi_n, U_i) + \sum_{i=1}^{n} c_i^2 D(U_i) = D(\Psi_n) + \sum_{i=1}^{n} c_i^2 \lambda_i.$$

Therefore

(22) $$0 \leqslant D(\Psi_n) = D(F) - \sum_{i=1}^{n} c_i^2 \lambda_i \leqslant D(F),$$

for $\lambda_i \geqslant 0$, $i = 1, 2, \ldots$ From (22) it follows that the sequence $\{D(\Psi_n)\}$ is bounded. By Theorem 7, $\lim \lambda_{n+1} = +\infty$, hence by (21), $\lim H(\Psi_n) = 0$. This equation is equivalent with Parseval's identity for $\{U_n(X)\}$.

Let now $F(X)$ be any continuous function in G. We can approximated it by a function $\Phi(X) \in \mathscr{L}_1$ in such a way that

$$H(F-\Phi) < \frac{\varepsilon}{4}, \varepsilon > 0. \tag{23}$$

For this purpose it is sufficient to use the construction which is given in [4], p. 305, for each component of $\Phi(X)$. Let γ_n be the Fourier coefficient of $\Phi(X)$ to respect $\{U_n(X)\}$ and let

$$\sigma_n = \sum_{k=1}^{n} \gamma_k U_k(X).$$

In virtue of the first part of the proof there exists a number N such that $H(\Phi-\sigma_n) < \varepsilon/4$ for $n > N$. Thus, by (23) it follows

$$H(F-S_n) \leqslant H(F-\sigma_n) < \varepsilon \quad \text{for} \quad n > N,$$

where $S_n = \sum_{i=1}^{n} c_i U_i(X)$. Hence $\lim H(F-S_n) = 0$. This implies Parseval's identity for the function $F(X)$ with respect to the sequence of eigenfunctions of (1) (2).

The following statements are simple consequences of Theorem 8.

COROLLARY 1. *The sequence of eigenvalues of problem* (1) (2) *contains all eigenvalues of this problem.*

COROLLARY 2. *Every eigenvalue of* (1) (2) *has finite multiplicity.*

COROLLARY 3. *Every function $F(X)$ continuous in G can be expanded in a series of eigenfunctions $\{U_n(X)\}$ of problem* (1) (2), *which converges in the mean with weight $P(X)$, i.e.,*

$$\lim_{n\to\infty} H\Big(F(X) - \sum_{k=1}^{n} c_k U_k(X)\Big) = 0. \tag{24}$$

Remark 2. The above expansion (24) may seen to depend on the matrix $P(X)$. Yet, since $P(X)$ is continuous and uniformly positive definite in G, the equality

$$\lim_{n\to\infty} \int_G \Big[F(X) - \sum_{k=1}^{n} c_k U_k(X)\Big] P(X) \Big[F(X) - \sum_{k=1}^{n} c_k U_k(X)\Big]' dX = 0,$$

is possible only if

$$\lim_{n\to\infty} \int_G \Big[F(X) - \sum_{k=1}^{n} c_k U_k(X)\Big] \Big[F(X) - \sum_{k=1}^{n} c_k U_k(X)\Big] dX = 0.$$

Now, let $\varphi_k(X) = a_1 u_k^{(1)}(X) + \ldots + a_n u_k^{(n)}(X)$, where $U_k(X) = \big(u_k^{(1)}(X), \ldots, u_k^{(n)}(X)\big)$, $k = 1, 2, \ldots$, are the eigenfunctions of (1) (2), and $a_1, \ldots, a_n$ are real numbers fulfilling the condition $a_1^2 + \ldots + a_n^2 > 0$.

Let $\{\psi_n(X)\}$ denote a sequence obtained from the $\{\varphi_n(X)\}$ by removing functions which are linearly dependent on remaining functions and orthogonalization.

THEOREM 9. *The sequence $\{\psi_n(X)\}$ is a complete system in the class of continuous functions in domain G.*

Proof. Let $f(X)$ be any function continuous in G, and orthogonal to each function $\psi_n(X)$. Then the function $f(X)$ is also orthogonal to each function $\varphi_n(X)$. The function $F(X) = \big(a_1 f(X), \ldots, a_n f(X)\big)$ is obviously continuous in G and

$$\int_G F(X)\, U_n(X)\, dX = 0, \qquad n = 1, 2, \ldots$$

This means that $F(X)$ is orthogonal to, all eigenfunctions of (1) (2). Since $\{U_n(X)\}$ is a complete system in the class of continuous functions, $F(X) \equiv 0$ in G. Thus $f(X) \equiv 0$ in G. This concludes the proof (cf. [6], p. 164).

§ 5. Multiplicity of eigenvalues of problem (1) (2) and a theorem of Sturm's type. In the case where the system (1) reduces to a single equation, the first eigenvalue of the problem is simple (see [1]). This means that all the eigenfunctions corresponding to first eigenvalue are linearly dependent. Moreover, there are examples (see [2]) of eigenfunctions problems in which all eigenvalues are simple. It is easy to give an example of a problem of the form (1) (2) for which all eigenvalues are n-fold at least. To this purpose let us denote $A_{ij}(X) = a_{ij}(X) E$ $(i, j = 1, \ldots, m)$, $Q(X) = q(X)E$, $P(X) = \varrho(X)E$, $K(X) = k(X)E$, where $a_{ij}(X)$ $(i, j = 1, \ldots, m)$, $q(X)$, $\varrho(X)$, $k(X)$ are the scalar functions sufficiently regular in domain G, and E is the unit matrix of the rank n. Then (1) can be written in the form

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(X)\frac{\partial u_s}{\partial x_j}\right] - q(X)\, u_s + \mu\, \varrho(X)\, u_s = 0 \tag{25}$$

with the boundary condition

$$\frac{du_s}{d\nu} - k(X)\, u_s = 0 \quad \text{on } F(G) - \Gamma, \qquad u_s = 0 \quad \text{on } \Gamma,$$

where $s = 1, \ldots, n$. Let λ be an eigenvalue of the equation

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(X)\frac{\partial u}{\partial x_j}\right] - q(X)\, u + \mu \varrho(X)\, u = 0 \tag{26}$$

with the boundary condition

$$\frac{du}{d\nu} - k(X)\, u = 0 \quad \text{on } F(G) - \Gamma, \qquad u = 0 \quad \text{on } \Gamma, \tag{27}$$

and let $u(X)$ e the corresponding eigenfunction. Then λ is also an eigenvalue of (25) corresponding to the eigenfunctions

$$U_1 = (u, 0, \ldots, 0), \quad U_2 = (0, u, 0, \ldots, 0), \quad \ldots, \quad U_n = (0, \ldots, 0, u)$$

which are linearly independent.

In [1] there are some theorems of Sturm type for the problem (26) (27) which correspond to the following theorem:

THEOREM 10. *If* 1° λ_k, λ_l *are eigenvalues of problem* (1) (2) *and* $\lambda_k < \lambda_l$, 2° *to the eigenvalue* λ_l *there correspond* n *linearly independent functions* $U_{l,i}(X)$ $(i = 1, \ldots, n)$, 3° *the matrix* V *composed of the functions* $U_{l,i}(X)$ $(i = 1, \ldots, n)$ *is not singular and matrices* $\left(\sum_{j=1}^{m} A_{ij} \frac{\partial V}{\partial x_j}\right) V^{-1}$ $(i = 1, \ldots, n)$ *are symmetric, then in every nodal domain of eigenfunction* U_k *there exists a zero-point of* $\det V$.

Proof. Since each function $U_{l,i}(X)$ $(i = 1, \ldots, n)$ satisfies (1) and (2) for $\mu = \lambda_l$ we get

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[A_{ij} \frac{\partial V}{\partial x_j}\right] - QV + \lambda_l PV = 0 \tag{28}$$

and $\int_{F(G)-\Gamma} \Psi\left(\frac{dV}{d\nu} - KV\right) dS = 0$ for any function $\Psi \in \mathscr{L}_2$ and $V \in \mathscr{L}_2$. Similary,

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[A_{ij} \frac{\partial U_k}{\partial x_j}\right] - QU_k + \lambda_k PU_k = 0. \tag{29}$$

Let G_k denote the nodal domain of the function U_k and let $\det V \neq 0$ in the closure $\bar{G}_k$ of G_k. Then the assumption 3° of Theorem 10 is satisfied in domain G_k. Multiplying both sides of the equality (28) by $V^{-1}U_k$ from the right and by U_k from the left, multiplying the equality (29) by U_k from the left, and combining the resulting the equations we obtain

$$\sum_{i,j=1}^{m} \left\{U_k \frac{\partial}{\partial x_i}\left[A_{ij} \frac{\partial U_k}{\partial x_j}\right] - U_k \frac{\partial}{\partial x_i}\left[A_{ij} \frac{\partial V}{\partial x_j}\right] V^{-1} U_k\right\} + (\lambda_k - \lambda_l) U_k P U_k = 0.$$

By assumption 3° of Theorem 10, we get

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[U_k \left(A_{ij} \frac{\partial U_k}{\partial x_j} - A_{ij} \frac{\partial V}{\partial x_i} V^{-1} U_k\right)\right] -$$
$$- \sum_{i,j=1}^{m} \left(\frac{\partial U_k}{\partial x_i} - \frac{\partial V}{\partial x_i} V^{-1} U_k\right) A_{ij} \left(\frac{\partial U_k}{\partial x_j} - \frac{\partial V}{\partial x_j} V^{-1} U_k\right) +$$
$$+ (\lambda_k - \lambda_l) U_k P U_k = 0.$$

Integrating both sides of this equation over G_k and applying the boundary conditions for the function U_k and for the matrix V we obtain the equality

$$-\int_{G_k}\sum_{i,j=1}^{m}\left(\frac{\partial U_k}{\partial x_i}-\frac{\partial V}{\partial x_i}V^{-1}U_k\right)A_{ij}\left(\frac{\partial U_k}{\partial x_j}-\frac{\partial V}{\partial x_j}V^{-1}U_k\right)dX+$$
$$+(\lambda_k-\lambda_l)\int_{G_k}U_kPU_k dX = 0.$$

The left-hand side of this equation being negative, we get contradiction. Therefore there exist a zero-point of the $\det V$ in G_k.

Remark 3. In the proof of Theorem 10 we applied a method of L. M. Kuks [5], who used it in a proof of a similar theorem for a more special strongly elliptic system with boundary condition of Dirichlet's type.

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