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On approximation of generalized almost periodic functions

1. R. A. Abbasow [1] discussed the pecularities of the basic theorems of the constructive theory of functions in the class of uniformly almost periodic functions. The purpose of this note is to investigate a constructive characterization of generalized almost periodic functions; the set of these functions will be denoted by $\mathscr{B}$. (For definitions and basic properties of $\mathscr{B}$-spaces see [2].)

For readers convenience we give only the definition of a norm which we shall use in the sequel. Let $f(\cdot)$ denote a representant of the element $\sigma \in \mathscr{B}$, then (see [2])

$$(*) \qquad \|f\|_{\overline{\mathfrak{m}}} = \sup_{E_{\overline{N}}} \left\{ \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |f\cdot g|\, dx \right\},$$

where $E_{\overline{N}} = \{g(\cdot);\ g(\cdot) \in \overline{N};\ \overline{S}_N(|g|) \leqslant 1\}$, and $\overline{N}$ denote the set of all real-valued or complex-valued functions $g(t)$, defined on $-\infty < t < \infty$, N — summable on every finite interval $(-T, T)$, such that

$$\overline{S}_N(|g|) \equiv \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} N[|g(t)|]\, dt < \infty,$$

here N denoted the complementary functions (in the sense of Young) to a given functions $M(\cdot)$ which generates $\mathscr{B}$ (see [2], [3]). Elements of $\mathscr{B}$ are called *$\mathscr{B}$-almost periodic functions*. By the *Bochner-Fejér kernel* we understand the function

$$K_\mu(t) = k_{n_1}(\beta_1 t) \ldots k_{n_p}(\beta_p t),$$

$-\infty < t < \infty$, $k_{n_i}(\beta_i t) = n_i^{-1}\left(\sin\frac{n_i\beta_i t}{2} \Big/ \sin\frac{\beta_i t}{2}\right)^2$ and $\beta_1, \beta_2, \ldots, \beta_p$ are arbitrary real rationaly linearly independent numbers. This kernel is non-negative for every t and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} K_\mu(t)\, dt = 1.$$

2. Let $f(\cdot)\in\mathscr{B}$. we denote by f_μ^k and $f_{\mu,\nu}$ the Bochner-Fejér polynomials:

$$f_\mu^k(x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sum_{\nu=1}^{k}(-1)^{\nu-1}\binom{k}{\nu}f(x+\nu u)K_\mu(u)\,du,$$

$$f_{\mu,\nu}(x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+\nu u)K_\mu(u)\,du,$$

by $\omega_k(f;t)_{\overline{\mathrm{m}}}$ the modulus of smoothness of degree $k\geqslant 1$,

$$\omega_k(f,t)_{\overline{\mathrm{m}}} = \sup_{|h|\leqslant t}\Big\|\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}f(x+\nu h)\Big\|_{\overline{\mathrm{m}}} = \sup_{|h|\leqslant t}\|\Delta_h^k f\|_{\overline{\mathrm{m}}}$$

and by $E_\mu(f)_{\overline{\mathrm{m}}}$ the best approximation by Bochner-Fejér polynomials

$$E_\mu(f)_{\overline{\mathrm{m}}} = \inf\|f-f_\mu^k\|_{\overline{\mathrm{m}}}$$

where the infimum is taken over all these Bochner-Fejér polynomials of degree $\leqslant\mu$.

THEOREM 1. *Let $f(x,y)\geqslant 0$ for all real $x, y, f(x,\cdot)\in\mathscr{B}$ and let $\|f(x,\cdot)\|_{\overline{\mathrm{m}}}\in\mathscr{B}$, where $\|f(x,\cdot)\|_{\overline{\mathrm{m}}}$ is the norm $(*)$ of $f(x,y)$ as a function of y. Then*

$$\Big\|\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}f(x,y)K_\mu(x)\,dx\Big\|_{\overline{\mathrm{m}}} \leqslant \lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}\|f(x,\cdot)\|_{\overline{\mathrm{m}}}K_\mu(x)\,dx.$$

Proof of this theorems is completely analogous to that given in [2] (Theorem 4.8, p. 51).

From the above it is obvious that the following theorems holds:

THEOREM 2. *If $f(\cdot)\in\mathscr{B}$, then*

(i) $$f_{\mu,\nu}(\cdot)\in\mathscr{B},\quad f_\mu^k(\cdot)\in\mathscr{B},$$

(ii) $$\|f_\mu^k\|_{\overline{\mathrm{m}}} \leqslant \sum_{\nu=1}^{k}\binom{k}{\nu}\|f_{\mu,\nu}\|_{\overline{\mathrm{m}}} \leqslant (2^k-1)\|f\|_{\overline{\mathrm{m}}},$$

(iii) $$\|f_\mu^k-f\|_{\overline{\mathrm{m}}} \leqslant \sum_{\nu=1}^{k}\binom{k}{\nu}\|f_{\mu,\nu}-f\|_{\overline{\mathrm{m}}}.$$

THEOREM 3. *If*

$$f(\cdot)\in\mathscr{B}\quad and\quad \lim_{T\to\infty}\int_{-T}^{T}|u|^kK_\mu(u)\,du = O(\mu^{-k}),$$

then

$$E_\mu(f)_{\overline{\mathrm{m}}} \leqslant c_k(\mu)\,\omega_k(f,\mu^{-1})_{\overline{\mathrm{m}}}\quad for\quad k\geqslant 1.$$

Proof. In account of Theorem 1 we have

$$\begin{aligned} E_\mu(f)_{\overline{\mathrm{m}}} &\leqslant \|f_\mu^k - f\|_{\overline{\mathrm{m}}} \\ &= \left\| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{\nu=0}^{k} (-1)^{\nu-1} \binom{k}{\nu} f(x+\nu\cdot u) K_\mu(u)\,du \right\|_{\overline{\mathrm{m}}} \\ &\leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\Delta_u^k f\|_{\overline{\mathrm{m}}} K_\mu(u)\,du \\ &\leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega_k(f, u)_{\overline{\mathrm{m}}} K_\mu(u)\,du. \end{aligned}$$

Next, in view of the properties of the modulus of smoothness we obtain

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega_k(f, u)_{\overline{\mathrm{m}}} K_\mu(u)\,du & \\ &\leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [1+\mu|u|]^k \omega_k(f, \mu^{-1})_{\overline{\mathrm{m}}} K_\mu(u)\,du \\ &\leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2^k \mu^k |u|^k \omega_k(f, \mu^{-1})_{\overline{\mathrm{m}}} K_\mu(u)\,du + \\ &\qquad + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2^k \omega_k(f, \mu^{-1})_{\overline{\mathrm{m}}} K_\mu(u)\,du \\ &\leqslant c_k(\mu)\, \omega_k(f, \mu^{-1})_{\overline{\mathrm{m}}}. \end{aligned}$$

COROLLARY 1. *If*

$$\mu^k \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |u|^k K_\mu(u)\,du = O(1), \quad \textit{for } \mu \to \infty,$$

then for arbitrary function $f \in \mathscr{B}$,

$$E_\mu(f)_{\overline{\mathrm{m}}} \leqslant C_k \omega_k(f, \mu^{-1})_{\overline{\mathrm{m}}}$$

where c_k *is independent of* μ.

We say that $f \in \mathscr{B}^{(p)}$ if $f \in \mathscr{B}$ and $f(\cdot)$ possesses p absolutely continuous derivatives ($\mathscr{B}^{(0)} \equiv \mathscr{B}$).

THEOREM 4. *By all assumptions of Theorems 3, for an arbitrary function* $f(\cdot) \in \mathscr{B}^{(p)}$ *we have*

$$E_\mu(f)_{\overline{\mathrm{m}}} \leqslant C_{k+p} \mu^{-p} \omega_k(f^{(p)}, \mu^{-1})_{\overline{\mathrm{m}}}.$$

Proof. In virtue of Corollary 1, $E_\mu(f)_{\overline{\mathfrak{m}}} \leqslant c_{k+p}\omega_{k+p}(f, \mu^{-1})_{\overline{\mathfrak{m}}}$ for $f \in \mathscr{B}$. Thus from obvious properties of the modulus of smoothness we obtain

$$E_\mu(f)_{\overline{\mathfrak{m}}} \leqslant c_{k+p}\mu^{-p}\omega_k(f^{(p)}, \mu^{-1})_{\overline{\mathfrak{m}}}.$$

COROLLARY 2. *It is seen that Theorems 3 and 4 generalize theorems on best approximation of a 2π-periodic function as elements of Orlicz space L^{*M} by trigonometric polynomials T_n.*

Indeed, it is sufficient to apply Theorems 3 and 4 to $E_n(f)_n = \inf\|f-T_n\|_M$ in place of $E_\mu(f)_{\overline{\mathfrak{m}}}$, where $\|\cdot\|_M$ is the norm in the Orlicz space L^{*M} of 2π-periodic functions ([3], § 9), and the infimum is taken over all trigonometric polynomials T_n of degree $\leqslant n$. Then $E_\mu(f)_{\overline{\mathfrak{m}}} = E_n(f)_M$ and putting $\mu = n$ we obtain

$$E_n(f)_M \leqslant c_k\omega_k(f, n^{-1})_M.$$

References

[1] Р. А. Аббасов, Изв. Акад. Наук А. ССР, No 5, 1966, pp. 3-7.

[2] J. Albrycht, *The theory of Marcinkiewicz-Orlicz spaces*, Rozprawy Mat. 27 (1962), pp. 1-55.

[3] M. A. Krasnosel'skiĭ, Y. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Groningen 1962.