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On monotonically normal spaces

In this paper we shall consider some problems concerning continuous monotone functions on preordered monotonically normal topological spaces. The results of the present paper are generalizations of results of Aronszajn and Panitchpakdi [1] and of Seever [4], who considered similar problems for topological spaces without order.

Let X be a preordered topological space (see [3] and [5]). $C^{\uparrow}(X)$ [respectively $C^{\downarrow}(X)$] will denote the space of real-valued continuous increasing [decreasing] functions on X (a real-valued functions f on X is called *increasing* [*decreasing*] if $x \leqslant y$ implies $f(x) \leqslant f(y)$ [$f(x) \geqslant f(y)$], respectively). Since there is an order-preserving homeomorphism from the extended real line onto $[0, 1]$, we may often assume that $0 \leqslant f(x) \leqslant 1$. It is clear that $C^{\uparrow}(X)$ is a cone. The distance in $C^{\uparrow}(X)$ is defined as $\varrho(f, g) = \sup|f(x)-g(x)|$. $K(f, r)$ will denote the closed ball $\{g \in C^{\uparrow}(X)\colon \varrho(f, g) \leqslant r\}$.

A subset A of X is called increasing [decreasing] if its characteristic function χ_A is increasing [decreasing]. $I(A)$ [respectively $D(A)$] will denote the smallest closed increasing [decreasing] set containing A.

Subsets U and V of X will be called *monotonically separated* if there exists an f in $C^{\uparrow}(X)$ such that $f(x) = 0$ for x in U and $f(x) = 1$ for x in V.

We shall write $U < V$ if U and V are decreasing subsets of X and there exists an f in $C^{\downarrow}(X)$ such that $\chi_U \leqslant f \leqslant \chi_V$.

LEMMA 1. *Let U, V be subsets of X and let U be decreasing, and V increasing. Then $U < X \setminus V$ if and only if U and V are monotonically separated.*

Proof. Let us first assume that $U < X\setminus V$. Then there exists an f in $C^{\downarrow}(X)$ such that $\chi_U \leqslant f \leqslant \chi_{X\setminus V}$, $0 \leqslant f \leqslant 1$. Let $g = 1-f$. Then $g \in C^{\uparrow}(X)$. If $x \in U$, then $f(x) = 1$ and $g(x) = 0$. If $x \in V$, then $f(x) = 0$ and $g(x) = 1$. Hence U and V are monotonically separated. Let us now assume that U and V are monotonically separated. Then there exists an f in $C^{\uparrow}(X)$ such that $f(x) = 0$ for x in U and $f(x) = 1$ for x in V. Let $g = 1 \wedge (1-f)^{+}$. Then $0 \leqslant g \leqslant 1$ and $g \in C^{\downarrow}(X)$. If $x \in U$, then $f(x) = 0$ and $g(x) = 1$; if

$x \in V$, then $f(x) = 1$ and $g(x) = 0$. Consequently $\chi_U \leqslant g \leqslant \chi_{X \setminus V}$ and $U < X \setminus V$. This concludes the proof of the lemma.

LEMMA 2. *Let U, V, W be decreasing subsets of X. Then*

(a) $U < W, \quad V < W \Leftrightarrow U \cup V < W,$

(b) $U < V, \quad U < W \Leftrightarrow U < V \cap W,$

(c) $U < V, \quad V \subset W \Rightarrow U < W,$

(d) $U \subset V, \quad V < W \Rightarrow U < W,$

(e) *if $U < V$, then there is a decreasing set W such that $U < W < V$.*

We omit the easy verification.

In the sequel $\mathfrak{m}$ will stand for an infinite cardinal number.

A subset U of X is called $\mathfrak{m}$-*increasing* [$\mathfrak{m}$-*decreasing*] if there exists a subset A of $C^{\uparrow}(X)$ [A of $C^{\downarrow}(X)$] such that $\operatorname{card}(A) < \mathfrak{m}$ and $U = \bigcup_{f \in A} \{x: f(x) > 0\}$.

The space X is called *monotonically normal* if it satisfies the following condition: given a closed decreasing subset F of X and closed increasing subset H of X such that $F \cap H = \emptyset$, there exists a function f in $C^{\uparrow}(X)$ such that $f(x) = 0$ for x in F and $f(x) = 1$ for x in H (see [3] and [5]).

THEOREM. *Let X be a monotonically normal space. Then the following conditions are equivalent:*

(α) *If U, V are disjoint subsets of X, U is $\mathfrak{m}$-decreasing, V is $\mathfrak{m}$-increasing, then U and V are monotonically separated.*

(β) *If A, B are subsets of $C^{\uparrow}(X)$ such that* $\operatorname{card}(A) < \mathfrak{m}$, $\operatorname{card}(B) < \mathfrak{m}$ *and $f \geqslant g$ for all f in A and g in B, then there exists an h in $C^{\uparrow}(X)$ such that $f \geqslant h \geqslant g$ for all f in A, g in B.*

(γ) *If $S = \{K_\lambda: \lambda \in \Lambda\}$ is a set of closed balls in $C^{\uparrow}(X)$ such that* $\operatorname{card}(\Lambda) < \mathfrak{m}$, *and*

(1) $$K_\lambda \cap K_\mu \neq \emptyset \quad \textit{for arbitrary } \lambda, \mu \textit{ in } \Lambda,$$

then $\bigcap K_\lambda \neq \emptyset$.

(δ) *If $\{U_\lambda\}_{\lambda \in \Lambda}$ and $\{V_\mu\}_{\mu \in M}$ are two classes of open subsets of X such that* $\operatorname{card}(\Lambda) < \mathfrak{m}$, $\operatorname{card}(M) < \mathfrak{m}$, *each U_λ is decreasing, each V_μ is increasing, $D(U_{\lambda_0}) \subset U = \bigcup U_\lambda$, $I(V_{\mu_0}) \subset V = \bigcup V_\mu$ for any $\lambda_0 \in \Lambda$, $\mu_0 \in M$, and $U \cap V = \emptyset$, then $D(U) \cap I(V) = \emptyset$.*

Proof. (α)$\Rightarrow$(β). Let A, B be subsets of $C^{\uparrow}(X)$ such that $\operatorname{card}(A) < \mathfrak{m}$, $\operatorname{card}(B) < \mathfrak{m}$, and $1 > f \geqslant g > 0$ for all f in A, g in B. Let Q denote the set of rationals of $[0, 1]$. If $r \in Q$, let

$$V(r) = \bigcup_{g \in B} \{x: g(x) > r\} \quad \text{and} \quad U(r) = \bigcup_{f \in A} \{x: f(x) < r\}.$$

Then $V(r)$ is a $\mathfrak{m}$-increasing set, $U(r)$ is a $\mathfrak{m}$-decreasing set, and

$V(r) \cap U(r) = \emptyset$. Since X has property (α), the sets $V(r)$ and $U(r)$ are monotonically separated and, by Lemma 1,

$$U(r) < X \setminus V(r); \tag{2}$$

furthermore, if $r < s$, then

$$U(r) \subset U(s) \quad \text{and} \quad X \setminus V(r) \subset X \setminus V(s). \tag{3}$$

We shall show that for each r in Q we can find a decreasing set $W(r)$ such that

$$U(r) < W(r) < X \setminus V(r), \tag{4}$$

$$W(r) < W(s) \quad \text{if} \quad r < s. \tag{5}$$

Let $r_1, r_2, \ldots$ be a sequence of all elements of Q such that $r_1 = 0$, $r_2 = 1$, $r_n \neq r_p$ if $n \neq p$. We proceed by induction; we define $W(0) = \emptyset$, $W(1) = X$, and we suppose that decreasing sets $W(r_1), \ldots, W(r_k)$ have been defined and satisfy (4), (5) for r, s in $T_k = \{r_1, \ldots, r_k\}$. We denote

$$p = \sup\{r \colon r \in T_k \;\&\; r < q_{k+1}\}, \qquad p' = \inf\{r \colon r \in T_k \;\&\; r > q_{k+1}\}.$$

Then $W(p) < W(p')$ and, by (3), $U(q_{k+1}) \subset U(p') < W(p')$. Applying Lemma 2, condition (a), we get

$$U(q_{k+1}) \cup W(p) < W(p'). \tag{6}$$

By (2) and (3), $U(q_{k+1}) < X \setminus V(q_{k+1})$ and $W(p) < X \setminus V(p)$; furthermore, $X \setminus V(p) \subset X \setminus V(q_{k+1})$. Therefore, by (c), $W(p) < X \setminus V(q_{k+1})$. Hence, in virtue of (a),

$$U(q_{k+1}) \cup W(p) < X \setminus V(q_{k+1}). \tag{7}$$

By (6), (7), and (b), we have $U(q_{k+1}) \cup W(p) < W(p') \cap \big(X \setminus V(q_{k+1})\big)$. Consequently, by (e), there exists a decreasing set W such that

$$U(q_{k+1}) \cup W(p) < W < W(p') \cap \big(X \setminus V(q_{k+1})\big).$$

This set W will be denoted by $W(q_{k+1})$. It is obvious that conditions (4) and (5) are satisfied for $r \in T_{k+1}$.

Let us now denote $h(x) = \inf\{r \colon r \in Q \;\&\; x \in W(r)\}$. We shall prove the continuity of h. Let $r < h(x) < s$. Then $x \notin W(r)$ and $x \in W(s)$. Therefore $G = W(s) \setminus W(r)$ is a neighborhood of x. If $y \in G$, then $y \in W(s)$, and $y \notin W(r)$; hence $r < h(y) \leqslant s$. Thus, h is continuous. Since the set $W(r)$ is decreasing, we get

$$\{r \colon x \in W(r)\} \subset \{r \colon x' \in W(r)\} \quad \text{for} \quad x' \leqslant x.$$

Consequently, $h(x') \leqslant h(x)$ for $x' \leqslant x$.

All that remains to be shown is that $f(x) \geqslant h(x) \geqslant g(x)$ for all f in A, g in B. Suppose, if possible, that $f(x) < h(x)$ for all f in A. Then there exists an r in Q such that $f(x) < r < h(x)$. Consequently, $x \notin W(r)$ and $x \in U(r)$. This contradicts (4). The proof of the inequality $h(x) \geqslant g(x)$ for all g in B is similar. This implies that $C^{\uparrow}(X)$ has property (β).

(β) ⇒ (γ). Let $f, g \in C^{\uparrow}(X)$. We note that $K(f, r)$ coincides with the interval $[(f-r1)^{+}, f+r1]$ in $C^{\uparrow}(X)$, where 1 is the function identically equal to one. Indeed, $1 \in C^{\uparrow}(X)$ and $f+r1 \in C^{\uparrow}(X)$; $f-r1$ need not be positive, but it is increasing and $(f-r1)^{+}$ belongs to $C^{\uparrow}(X)$.

Let now $K_{\lambda} = K(f_{\lambda}, r_{\lambda})$, where $\lambda \in \Lambda$, $\operatorname{card}(\Lambda) < \mathfrak{m}$, satisfy (1). Thus there exist functions $f_{\lambda\mu}$ in $C^{\uparrow}(X)$ such that $f_{\lambda\mu} \in K_{\lambda} \cap K_{\mu}$ for λ, μ in Λ. Hence

$$(f_{\lambda}-r_{\lambda}1)^{+} \leqslant f_{\lambda\mu} \leqslant f_{\mu}+r_{\mu}1$$

for all λ, μ in Λ. Let us define

$$A = \{(f_{\lambda}-r_{\lambda}1)^{+} : \lambda \in \Lambda\}, \qquad B = \{f_{\mu}+r_{\mu}1 : \mu \in \Lambda\}.$$

Then A and B are subsets of $C^{\uparrow}(X)$ and satisfy (β). Consequently, there exists a function f in $C^{\uparrow}(X)$ such that $(f_{\lambda}-r_{\lambda}1)^{+} \leqslant f \leqslant f_{\mu}+r_{\mu}1$ for all λ, μ in Λ. Hence $\varrho(f_{\lambda}, f) \leqslant r$ for every $\lambda \in \Lambda$ and $f \in \bigcap K_{\lambda}$.

(γ) ⇒ (δ). Let $\{U_{\lambda}\}_{\lambda \in \Lambda}$ and $\{V_{\mu}\}_{\mu \in M}$ be two classes satisfying the assumption of (δ); let $U = \bigcup U_{\lambda}$ and $V = \bigcup V_{\mu}$.

Let us consider the sets $D(U_{\lambda})$ and $X \setminus U$. $D(U_{\lambda})$ is closed and decreasing, $X \setminus U$ is closed and increasing; furthermore, $D(U_{\lambda}) \cap (X \setminus U) = \emptyset$. Since X is monotonically normal, there exists an f_{λ} in $C^{\uparrow}(X)$ such that $f_{\lambda}(x) = 0$ for x in $D(U_{\lambda})$ and $f_{\lambda}(x) = 1$ for x in $X \setminus U$. Similarly, there exists an f_{μ} in $C^{\uparrow}(X)$ such that $f_{\mu}(x) = 0$ for x in $X \setminus V$ and $f_{\mu}(x) = 1$ for x in $I(V_{\mu})$.

Let $f_{\lambda\mu} = \frac{1}{2}(f_{\lambda}+f_{\mu})$. Then $f_{\lambda\mu} \in C^{\uparrow}(X)$, $f_{\lambda\mu}(x) = 0$ for x in $D(U_{\lambda})$, $0 \leqslant f_{\lambda\mu}(x) \leqslant \frac{1}{2}$ for x in U, $f_{\lambda\mu}(x) = \frac{1}{2}$ for x in $X \setminus (U \cup V)$, $\frac{1}{2} \leqslant f_{\lambda\mu}(x) \leqslant 1$ for x in V, and $f_{\lambda\mu}(x) = 1$ for x in $I(V_{\mu})$.

We now consider the set S of closed balls $K(f_{\lambda\mu}, \frac{1}{4})$ in $C^{\uparrow}(X)$, $\lambda \in \Lambda$, $\mu \in M$. It is clear that $\operatorname{card}(\Lambda \times M) < \mathfrak{m}$, and $\varrho(f_{\lambda\mu}, f_{\lambda_0\mu_0}) \leqslant \frac{1}{2} = \frac{1}{4}+\frac{1}{4}$ for arbitrary $\lambda\mu$, $\lambda_0\mu_0$ in $\Lambda \times M$. Thus, S satisfies the assumptions of (γ). Therefore there exists an f belonging to all $K(f_{\lambda\mu}, \frac{1}{4})$. From the above inequalities it follows that $|f(x)| \leqslant \frac{1}{4}$ for x in $D(U_{\lambda})$ and $|f(x)-1| \leqslant \frac{1}{4}$ for x in $I(V_{\mu})$ for all $\lambda \in \Lambda$, $\mu \in M$; hence $|f(x)| \leqslant \frac{1}{4}$ for x in U and $|f(x)-1| \leqslant \frac{1}{4}$ for x in V. Let us define

$$L = \{x : f(x) \leqslant \tfrac{1}{4}\}, \qquad N = \{x : f(x) \geqslant \tfrac{3}{4}\}.$$

Then L is a closed decreasing set, and $U \subset L$, N is a closed increasing set, and $V \subset N$; consequently, $D(U) \subset L$ and $I(V) \subset N$. Since $L \cap N = \emptyset$, we get $D(U) \cap I(V) = \emptyset$. This shows that X has the property (δ).

(δ) ⇒ (α). Let U be $\mathfrak{m}$-decreasing, let V be $\mathfrak{m}$-increasing, and let $U \cap V = \emptyset$. Thus $U = \bigcup_{\lambda \in \Lambda} \{x: g_\lambda(x) > 0\}$, $V = \bigcup_{\mu \in M} \{x: h_\mu(x) > 0\}$, $g_\lambda \in C^\downarrow(X)$, $h_\mu \in C^\uparrow(X)$, $\mathrm{card}(\Lambda) < \mathfrak{m}$ and $\mathrm{card}(M) < \mathfrak{m}$. Let $U_\lambda^n = \{x: g_\lambda(x) > 1/n\}$. Then

$$U_\lambda = \{x: g_\lambda(x) > 0\} = \bigcup_{n=1}^{\infty} U_\lambda^n.$$

Since $\mathfrak{m}$ is infinite, the family $\{U_\lambda^n\}$ is also of cardinal $< \mathfrak{m}$. Moreover, $D(U_\lambda^n) \subset \{x: g_\lambda(x) \geqslant 1/n\} \subset U_\lambda^{n+1}$. Therefore $D(U_{\lambda_0}^n) \subset U_\lambda \subset U$ holds for any λ, λ_0 and n. Similary, $I(V_{\mu_0}^n) \subset V$ for any μ, μ_0 and n.

Thus, in virtue of (δ), $D(U) \cap I(V) = \emptyset$. Since X is monotonically normal, there exists an f in $C^\uparrow(X)$ such that $f(x) = 0$ for x in $D(U)$ and $f(x) = 1$ for x in $I(V)$. Thus, U and V are monotonically separated. This concludes the proof of the theorem.

Let us note that if X is a monotonically normal space with antidiscrete preorder (i.e., $x \leqslant y$ for arbitrary x, y), then the constat functions are the only monotone functions, and the conditions (α)-(δ) are obviously satisfied.

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