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# Properties of nodal lines of eigenfunctions of a certain system of partial differential equations of second order

Let $G$ be a bounded domain in the $m$-dimensional Euclidean space $E^m$ of real-valued $X = (x_1, \ldots, x_m)$, measurable in the sense of Jordan. We assume that the domain $G$ can be approximated from inside by an increasing sequence of domains $G_n$ with regular boundaries ($G = \sum_n G_n$); it means that the boundaries are surfaces of the class $C^1_\sigma$, where the definition of such surfaces can be found in [4], p. 132. We make no assumption converning the boundary $F(G)$ of the domain $G$.

We shall consider the system of differential equations of the form

$$(1)\qquad \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(X)\frac{\partial u_p}{\partial x_j}\right] + \sum_{k=1}^{n} [\mu\varrho_{pk}(X) - q_{pk}(X)]u_k = 0$$

with the boundary condition of the form

$$(2)\qquad \frac{du_p}{d\nu} = \sum_{l=1}^{n} k_{pl}(X)u_l = 0 \text{ on } F(G) - \Gamma, \qquad u_p = 0 \text{ on } \Gamma$$

$(p = 1, \ldots, n)$ where

$$(3)\qquad \frac{du_p}{d\nu} = \sum_{i,j=1}^{m} a_{ij}(X)\frac{\partial u_p}{\partial x_j}\cos(n, x_i);$$

$n$ denotes here the interior normal to the boundary $F(G)$ of $G$. We make the following assumptions concerning the coefficients appearing in the system (1) and boundary condition (2) (which we shall for simplicity call problem (1)(2)): $a_{ij}(X) = a_{ji}(X)$ $(i, j = 1, \ldots, m)$ belong to the class $C^1$ in $\bar{G}$, $\varrho_{pl}(X), q_{pl}(X), k_{pl}(X)$ $(p, l = 1, \ldots, n)$ are defined and continuous in the domain $\bar{G}$. Besides, we assume that the matrices $\{q_{pl}(X)\}$ and $\{k_{pl}(X)\}$ are positive-definite in $\bar{G}$, and the matrices $\{a_{ij}(X)\}$ and

$\{\varrho_{pl}(X)\}$ are uniformly positive-definite in $\bar{G}$. In the boundary condition (2), $\Gamma$ denotes a certain $(m-1)$-dimensional part of the boundary $F(G)$, not necessarily connected. In particular, $\Gamma$ may consist of the whole boundary $F(G)$ or it may be empty. On the other hand, the boundary condition should be understood in the generalized sense, as presented in papers [1] and [3]. The problem (1)(2) is thus a particular case of the problem in paper [3] for more general system of differential equations.

The object of this paper is to study some properties of nodal lines of eigenfunctions connected with system (1) and with boundary condition (2). The definition of eigenvalues and eigenfunctions of problem (1)(2) is given in paper [3].

Let $U(X) = \big(u_1(X), \ldots, u_n(X)\big)$ be an eigenfunction of problem (1) (2). Using the results proved in [2] we shall prove the following

LEMMA 1. *If* $1^\circ$ $U(X_0) = 0$ *for* $X_0 \in \bar{G}$; $2^\circ$ *there exists a ball* $K \subset G$ *such that* $X_0 \in F(K)$; $3^\circ$ $U(X)$ *belongs to the class* $C^1$ *in the closure* $\bar{K}$ *of the ball* $K$; $4^\circ$ $U(X) \neq 0$ *for* $X \in K$; *then* $\sum_{i=1}^{n} \operatorname{grad}^2 u_i > 0$ *at the point* $X_0$.

Proof. Denote $R(X) = [u_1^2(X)+\ldots+u_n^2(X)]^{1/2}$, and denote by $e(X)$ the unit vector parallel to, and of the same direction as, the vector $U(X)$ for $U(X) \neq 0$. For $X \in K$, by assumption $4^\circ$ of the lemma, we have $R(X) = U(X)e^*(X)$, where $e^*(X)$ is the transpose to vector $e(X)$. Assume that $\sum_{i=1}^{n} \operatorname{grad}^2 u_i = 0$ at the point $X_0$. Since $U(X)$ belongs to the class $C^1$ at $X_0$, we have $dU/dl = 0$ for every direction $l$ starting from $X_0$ and entering the ball $K$, or

$$\lim_{X\to X_0} \frac{U(X)-U(X_0)}{\overline{XX_0}} = 0, \tag{4}$$

where $X \in l$. On the other hand, since $R(X_0) = 0$ and $U(X_0) = 0$ we have

$$[R(X)-R(X_0)]/\overline{XX_0} = [U(X)-U(X_0)]e^*(X)/\overline{XX_0}$$

which implies, together with (4) that

$$\lim_{X\to X_0} \frac{R(X)-R(X_0)}{\overline{XX_0}} = 0 \qquad (X \in l). \tag{5}$$

Equality (5) contradicts the assertion of Theorem 2 from paper [2].

Let us denote by $Z$ the set of zeros of $U(X)$ in the domain $G$ (nodal lines of function $U(X)$), and by $Z_p$ the set of zeros of the function $u_p(X)$ $(p = 1, \ldots, n)$, $Z = \bigcap_{p=1}^{n} Z_p$.

THEOREM 1. *If* $X_0$ *is an arbitrary point of the set* $Z$, *then in every neighbourhood of the point* $X_0$ *there are points from at least one of the sets* $Z_p$.

Proof. Suppose that there exists a neighbourhood $O(X_0)$ of $X_0$ such that for every $p = 1, \ldots, n$, we have $u_p(X) \neq 0$ for all $X \epsilon O(X_0)$ and $X \neq X_0$. It follows that every function $u_p(X)$ assumes its local extremum at the point $X_0$, hence $\sum_{i=1}^{n} \operatorname{grad}^2 u_i = 0$ at $X_0$. However, this equality contradicts the fact that $U(X) \neq 0$ for $X \epsilon O(X_0)$ and $X \neq X_0$.

THEOREM 2. *If $X_0 \epsilon Z$ and $\sum_{i=1}^{n} \operatorname{grad}^2 u_i \neq 0$ at the point $X_0$, then:* 1° *at least one of the functions $u_p(X)$ changes its sign in every neighbourhood of $X_0$;* 2° *at least one component of the set $Z_p$ which cuts the set $G$ passes through $X_0$.*

Proof. 1° Suppose that there exists a neighbourhood $O(X_0)$ of the point $X_0$ such that for every $p = 1, \ldots, n$ the function $u_p(X)$ preserves its sign for $X \epsilon O(X_0)$. Since $u_p(X_0) = 0$ for every $p = 1, \ldots, n$ every function $u_p(X)$ assumes its local extremum at $X_0$, which contradicts the assumptions of theorem 2.

2° The second assertion of Theorem 2 follows directly from the first part and from the continuity of function $u_p(X)$ in $G$.

THEOREM 3. *If $X_0 \epsilon Z$ and $\sum_{i=1}^{n} \operatorname{grad}^2 u_i = 0$ at $X_0$, then the point $X_0$ is a "branching" point of the set $Z$ in the sense that every ball contained in $G$, whose surface contains $X_0$, contains points of the set $Z$.*

Proof of Theorem 3 follows directly from Lemma 1.

To study further properties of nodal lines of eigenfunctions of problem (1) (2) we make further assumptions concerning the coefficients. We shall assume that $q_{pl}(X) = \delta_{pl} q(X)$, $k_{pl}(X) = \delta_{pl} k(X)$ $(p, l = 1, \ldots, n)$, and the elements of the matrix $\{\varrho_{pl}(X)\}$ are constants.

Since the matrix $P = \{\varrho_{pl}\}$ is, by assumption, symmetric and positive-definite, there exists an orthogonal transformation (see [5])

$$W(X) = VU(X) \tag{6}$$

such that the matrix $VPV^{-1} = \overline{P} = \{\delta_{pl}\varrho_p\}$, where $\varrho_p > 0$ $(p = 1, \ldots, n)$. We see easily that the transformation (6) reduces the system (1) to a system of $n$ independent equations, and the boundary conditions (2) to $n$ independent conditions; it can be written in the form

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(X)\frac{\partial w_p}{\partial x_j}\right] - q(X)w_p + \mu\varrho_p w_p = 0, \tag{7}$$

$$\frac{dw_p}{d\nu} - k(X)w_p = 0 \text{ on } F(G) - \Gamma, \qquad w_p = 0 \text{ on } \Gamma \ (p = 1, \ldots, n). \tag{8}$$

Problem (7) (8) can be treated as $n$ independent problems separately for every $p = 1, \ldots, n$. Each of these problems coincides with the problem treated in paper [1]. Using the theorems of this paper we give some properties of eigenfunctions of problem (7) (8). Let us denote by

$$\lambda_{1p}, \lambda_{2p}, \lambda_{3p}, \ldots \qquad (p = 1, \ldots, n) \tag{9}$$

the sequence of eigenvalues of problem (7) (8), and the corresponding sequence of eigenfunctions by

$$w_{1p}(X), w_{2p}(X), w_{3p}(X), \ldots \qquad (p = 1, \ldots, n). \tag{10}$$

The scalar functions of sequence (10) are eigenfunctions for one equation considered in paper [1], corresponding to eigenvalues of the sequence (9). On the other hand, to the eigenvalue $\lambda_{kp}$ there corresponds a vector eigenfunction $W_{kp}(X) = (0, \ldots, w_{kp}(X), \ldots, 0)$, with all components except the $p$-th equal to zero. Under transformation (6), to the function $W_{kp}(X)$ there corresponds the function $U_{kp}(X) = V^{-1}W_{kp}(X)$. We see easily that the function $U_{kp}(X)$ may be written in the form $U_{kp}(X) = A_p w_{kp}(X)$, where $A_p$ denotes the vector, whose components are elements of the $p$-th row of matrix $V^{-1}$.

Using the mentioned theorems of paper [1] we obtain the following properties of the sequence of functions $U_{kp}(X)$:

1° Every function $U_{1p}(X)$ $(p = 1, \ldots, n)$ does not vanish in the domain $G$;

2° The function $U_{kp}(X)$ $(p = 1, \ldots, n)$ vanishes along the nodal lines which cut the domain $G$ into at most $k$ nodal domains;

3° In every nodal domain of the eigenfunction $U_{kp}(X)$ there are nodal lines of all eigenfunctions $U_{lp}(X)$ for $l > k$;

4° If to the same eigenvalue $\lambda_{kp}$ there correspond two eigenfunctions $U_{kp}^{(1)}(X)$ and $U_{kp}^{(2)}(X)$ which are linearly independent, then the zeros of these functions cross each other.

## References

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