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On monotony of γ-dimension

There are several different notions of dimension, two of a few more important are: Menger's "small" inductive dimension "ind" [4] and Čech's "great" inductive dimension "Ind" [1]. Both are defined by induction; it has recently been shown [2] that one can assign to any sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ consisting of 0's and 1's a new notion of dimension, called *γ-dimension* and denoted "γ-dim", in such a way that, vaguely speaking, in a consecutive step of calculating $\gamma\text{-dim}\,X$ for a topological space X one follows "ind" if $\gamma_i = 0$ and "Ind" if $\gamma_i = 1$.

It is known ([3], p. 26) that in the class of all topological spaces "ind" is *monotone*, i.e., if $Y \subset X$, then $\operatorname{ind} Y \leqslant \operatorname{ind} X$. The same proof (with obvious modifications) works to the effect in the same class that "Ind" is *monotone with respect to closed subsets*, i.e., if $\overline{Y} = Y \subset X$, then $\operatorname{Ind} Y \leqslant \operatorname{Ind} X$. Combining these two results we infer that

THEOREM 1. *In the class of all topological spaces γ-dimension is monotone with respect to closed subsets for every sequence γ of* 0*'s and* 1*'s.*

However, using an example of Tychonoff [5] one can prove ([3], p. 154; see also [6]) that in general "Ind" is not monotone. It is the purpose of the present paper to show that

THEOREM 2. *In the class of all topological spaces γ-dimension is monotone if and only if $\gamma = (0, 0, \ldots)$.*

It is unknown, however, whether this theorem is still true if we restrict ourselves to the class of normal spaces or to the class of metric spaces only.

Let X be a topological space and $\{x_1, x_2, \ldots, x_k\}$ a sequence of k distinct points not belonging to X. By X_k we shall mean the space

$$X_k = \{x_1, x_2, \ldots, x_k\} \cup X$$

with the topology defined by the following neighbourhoods: for each x_i the smallest open neighbourhood is the set $\{x_1, \ldots, x_i\}$, and open neighbourhoods of an x in X are of the form $\{x_1, \ldots, x_k\} \cup U$, where $U \subset X$ is an open neighbourhood of x in X. Obviously, X is a closed subset of X_k.

LEMMA. *Let X be a topological space. If $\gamma = (\gamma_1, \gamma_2, \ldots)$ is a sequence consisting of 0's and 1's such that $\gamma_1 = \gamma_2 = \ldots = \gamma_k = 0$, where $k \geqslant 1$, then*

$$\gamma\text{-}\dim X_k = k + (\gamma_{k+1}, \gamma_{k+2}, \ldots)\text{-}\dim X.$$

Proof. Intending to calculate γ-$\dim X_k$ by a direct application of the definition of γ-dimension we have to consider a triple sequence of the form: a point $p_1 \in X_k$, its neighbourhood $U_1 \subset X_k$, the boundary F_1 of U_1; a point $p_2 \in F_1$, its neighbourhood $U_2 \subset F_1$, the boundary F_2 of U_2 taken relative to F_1; a point $p_3 \in F_2$, and so on. If we proceed carefully, that is if we take each time $p_i = x_i$ and $U_i = \{x_i\}$, $i = 1, 2, \ldots, k$, we come — as one can easily check — to $F_k = X$. But if we do not follow this device, then we may come to another $F_k = \bar{Z} = Z \subset X$. Anyway, if we do not impose upon Z any restrictions (except for that of being an admissible set in calculation of γ-dimension), then

$$(1) \qquad \gamma\text{-}\dim X_k = k + \sup (\gamma_{k+1}, \gamma_{k+2}, \ldots)\text{-}\dim Z,$$

where supremum is taken over all admissible Z's.

F_k being a closed subset of X, by Theorem 1 we have

$$(2) \qquad (\gamma_{k+1}, \gamma_{k+2}, \ldots)\text{-}\dim Z \leqslant (\gamma_{k+1}, \gamma_{k+2}, \ldots)\text{-}\dim X$$

for each admissible Z. But obviously X is admissible, and therefore formulas (1) and (2) imply our lemma.

We may now turn to the proof of Theorem 2. We need to construct for each $\gamma = (\gamma_1, \gamma_2, \ldots) \neq (0, 0, \ldots)$ a topological space A together with a subspace $B \subset A$ such that γ-$\dim A < \gamma$-$\dim B$. As our construction will be based upon Tychonoff's example, let us recall it here:

Let ω be the least countable ordinal and let $[0, \omega] = \{n : 0 \leqslant n \leqslant \omega\}$ be provided with the order topology. Similarly, let Ω be the least uncountable ordinal and $[0, \Omega] = \{\alpha : 0 \leqslant \alpha \leqslant \Omega\}$, with the order topology. Let S denote the topological product of $[0, \omega]$ and $[0, \Omega]$, and let T be the complement in S of the single point (ω, Ω).

Proof of Theorem 2. It is known ([3], p. 155) that for any compact space C, if $\operatorname{ind} C = 0$, then also $\operatorname{Ind} C = 0$. Since S is both compact and $\operatorname{ind} S = 0$, then $\operatorname{Ind} S = 0$. However, T is not normal and therefore $\operatorname{Ind} T \neq 0$.

Now let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a sequence of 0's and 1's. If $\gamma_1 = 1$, it suffices to take S itself together with T, because then

$$(3) \qquad \gamma\text{-}\dim S = 0 \quad \text{and} \quad \gamma\text{-}\dim T > 0.$$

If $\gamma_1 = \gamma_2 = \ldots = \gamma_k = 0$ and $\gamma_{k+1} = 1$, where $k \geqslant 1$, then it suffices to take S_k together with T_k, because T_k is a subspace of S_k and, by lemma and (3),

$$\gamma\text{-}\dim S_k = k + (\gamma_{k+1}, \gamma_{k+2}, \ldots)\text{-}\dim S = k,$$
$$\gamma\text{-}\dim T_k = k + (\gamma_{k+1}, \gamma_{k+2}, \ldots)\text{-}\dim T > k.$$

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