ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XI (1968)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XI (1968)

Z. STYPIŃSKI (Poznań)

Steklov means in Orlicz spaces

The aim of this paper is to generalize some results of F. I. Haršiladze concerning spaces L^p of functions (see [4]) replacing spaces L^p by Orlicz spaces $L^{*\varphi}$.

1. We shall introduce convenient notations and give some general remarks. Let $\varphi(u)$ be an even, continuous, convex non-negative function vanishing only at 0, $\varphi(u)/u \to 0$ as $u \to 0$, $\varphi(u)/u \to \infty$ as $u \to \infty$ and let $\varphi^*(v)$ be the function complementary to $\varphi(u)$ in the sense of Young. We denote by $L^{*\varphi}_{0,2\pi}$ the Orlicz space ([3]) of 2π-periodic measurable functions such that

$$(1)\qquad \|f\|_\varphi = \sup\left|\int\limits_0^{2\pi} f(x)g(x)\,dx\right| < \infty,$$

where the supremum is taken over all non-negative periodic measurable functions $g(x)$ such that $\int\limits_0^{2\pi} \varphi^*[g(x)]\,dx \leqslant 1$.

In the sequel we shall assume that the space $L^{*\varphi}_{(0,2\pi)}$ is reflexive. W. Orlicz has shown that in order that the space $L^{*\varphi}$ be reflexive it is necessary and sufficient that the functions $\varphi(u)$ and $\varphi^*(v)$ satisfy the Δ_2-condition for small u ([3]). We denote by $\omega_k(f, t)_\varphi$ the modulus of smoothness of order $k \geqslant 1$ of the function $f(x)$ in the space $L^{*\varphi}$:

$$(2)\qquad \omega_k(f, t) = \sup_{|h|\leqslant t}\|\Delta_h^k f\|_\varphi = \sup_{|h|\leqslant t}\left\|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}f(x+ih)\right\|_\varphi,$$

and by $E_n(f)_\varphi$ the best approximation of the function $f(x)$ by trigonometric polynomials of degree $\leqslant n$ in the $L^{*\varphi}$ sense:

$$(3)\qquad E_n(f)_\varphi = \inf\|f(x)-T_n(x)\|_\varphi,$$

where the infimum is taken over all trigonometric polynomials of degree $\leqslant n$. Denote by $f_h^k(x)$ the Steklov means of order k,

$$(4)\qquad f_h^k(x) = \frac{1}{2h}\int\limits_{-h}^{h}\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}f(x+it)\,dt, \qquad f_h(x) = \frac{1}{2h}\int\limits_{-h}^{h} f(x+t)\,dt.$$

Thus

$$f_h^k(x) = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} \frac{1}{2h} \int_{-h}^{h} f(x+it)\,dt = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} f_{ih}(x),$$

and for $f(x) \in L^{*\varphi}$ we have

(5) $$\|f_h^k(x) - f(x)\|_\varphi \leqslant \sum_{i=1}^{k} \binom{k}{i} \|f_{ih}(x) - f(x)\|_\varphi .$$

2. LEMMA 1. *Let $f(x, y) \geqslant 0$ and let $\int_{-\infty}^{\infty} \|f(x, y)\|_\varphi dx < \infty$ for almost every x, where $\|f(x, y)\|_\varphi$ is the Orlicz norm of $f(x, y)$ as a function of the variable y, x being fixed. Then*

(6) $$\Big\| \int_{-\infty}^{\infty} f(x, y)\,dx \Big\|_\varphi \leqslant \int_{-\infty}^{\infty} \|f(x, y)\|_\varphi dx .$$

Proof. Let E denote an arbitrary set. We set $g(y) = \int_a^b f(x, y)\,dx$. Then, applying Fubini Theorem and Hölder inequality in $L^{*\varphi}_{(0,2\pi)}$ we have

$$\|h(y)\|_\varphi = \sup_{\|g\|_{\varphi^*} \leqslant 1} \Big| \int_E \int_a^b f(x, y) g(y)\,dx dy \Big| \leqslant \sup_{\|g\|_{\varphi^*} \leqslant 1} \Big| \int_a^b \Big\{ \int_E f(x, y) g(y)\,dy \Big\}\,dx \Big|$$

$$\leqslant \sup_{\|g\|_{\varphi^*} \leqslant 1} \Big| \int_a^b \|f(x, y)\|_\varphi \|g\|_{\varphi^*} dx \Big| \leqslant \int_a^b \|f(x, y)\|_\varphi dx .$$

LEMMA 2. *If $f \in L^{*\varphi}_{(0,2\pi)}$ and $f^{(r)} \in L^{*\varphi}_{(0,2\pi)}$, then*

(7) $$\|\Delta^{(r+s)}(f)\|_\varphi \leqslant h^r \|\Delta^s(f^{(r)})\|_\varphi ,$$

where r and s are positive integers.

Proof. If $r = 1$ and $s = 0$, by (6) we have

$$\|\Delta f\|_\varphi = \|f(x+h) - f(x)\|_\varphi = \Big\| \int_x^{x+h} f'(t)\,dt \Big\|_\varphi \leqslant h \|f'\|_\varphi .$$

Thus

$$\|\Delta^{(r+s)}(f)\|_\varphi \leqslant h \|\Delta^{(r+s-1)}(f')\|_\varphi \leqslant \ldots \leqslant h^r \|\Delta^s(f^{(r)})\|_\varphi .$$

LEMMA 3. *Let the function $\varphi(u)$ generating the reflexive space $L^{*\varphi}_{(0,2\pi)}$ be such that $\varphi(u^{1/a})$ is convex for some α, $1 < \alpha \leqslant 2$, and let*

$$E_n(f)_\varphi = O(n^{-\beta}) \quad \text{for} \quad \beta > 0,$$

then

$$\|f_h^k(x) - f(x)\|_\varphi = O(h^{\beta+1/a}) \quad \text{for} \quad k > \beta .$$

Proof. It is known (see [2]) that if $\varphi(u)$ satisfies all the assumptions of this Lemma, then

$$\omega_k\left(f, \frac{1}{n}\right)_\varphi \leqslant \frac{A_{\varphi,k}}{n^k}\left\{\sum_{v=1}^{n} v^{ak-1} E_{v-1}^{a}(f)_\varphi\right\}^{1/a}.$$

Thus, if $E_n(f)_\varphi = O(n^{-\beta})$,

$$\omega_k(f, t)_\varphi = \begin{cases} O(t^\beta) & \text{when} \quad \beta < k, \\ O\left(t^k\left(\log \frac{1}{t}\right)^{1/a}\right) & \text{when} \quad \beta = k, \\ O(t^k) & \text{when} \quad \beta > k. \end{cases} \tag{8}$$

Moreover,

$$\frac{1}{2t}\int_{-t}^{t} \Delta_u^k f(x)\,dn = \frac{1}{2t}\int_{-t}^{t}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+iu)\,du = (-1)^{k+1}\{f_t^k(x)-f(x)\}$$

and

$$f_h^k(x)-f(x) = \frac{(-1)^{k+1}}{2h}\int_{-h}^{h}\Delta_t^k f(x)\,dt. \tag{9}$$

Finally, by (6) we have

$$\|f_h^k(x)-f(x)\|_\varphi \leqslant \frac{1}{2h}\int_{-h}^{h}\|\Delta_t^k f(x)\|_\varphi\,dt = O(h^\beta).$$

LEMMA 4. *Under the assumptions of Lemma 3, if $E_n(f)_\varphi = O(n^{-k})$, where k is an odd number, then*

$$\|f_h^k(x)-f(x)\|_\varphi = O(h^k).$$

Proof. It follows from the equality (8) that if k is an odd number and $E_n(f)_\varphi = O(n^{-k})$, then

$$\omega_{k+1}(f, t)_\varphi = O(t^k).$$

On the other hand (see [4], p. 61),

$$\sum_{v=0}^{k}(-1)^{k-v}\binom{k}{v}\{f(x+vt)-f(x-vt)\} = \sum_{s=0}^{k-1}\Delta_t^{k+1} f(x-kt+st).$$

Then, in virtue of formula (9), we have

$$f_h^k(x)-f(x) = \frac{1}{2h}\int_0^h [\Delta_t^k f(x)+\Delta_t^k f(x)]\,dt = \frac{1}{2h}\int_0^h\sum_{s=0}^{k-1}\Delta_t^{k+1} f(x-kt+st)\,dt.$$

Since

$$\|\Delta_t^{k+1} f(x-kt+st)\|_\varphi \leqslant \omega_{k+1}(f, t)_\varphi,$$

we obtain

$$\|f_h^k(x)-f(x)\|_\varphi = \frac{1}{2h}\left\| \int_0^h \sum_{s=0}^{k-1} \Delta_t^{k+1} f(x-kt+st)\,dt \right\|_\varphi$$

$$\leqslant \frac{k}{2h}\int_0^h \omega_{k+1}(f, t)_\varphi dt = O(h^k).$$

Lemma 5. *Under the assumptions of Lemma* 3, *if*

$$E_n(f)_\varphi = O(n^{-\beta}), \qquad k < \beta < k+1,$$

k is an odd number, then

$$\|f_h^k(x)-f(x)\|_\varphi = O(h^\beta).$$

Proof. First, we observe that $\sum_{v=1}^{\infty} v^{ak-1}E_v^a(f)_\varphi$ is convergent, and by [2], Theorem 2, the function $f(x)$ has an absolutely continuous derivative of order $(r-1)$ such that $f^{(r)} \in L^{*\varphi}$ and the following estimates are true:

$$\omega_k\left(f^{(r)}, \frac{1}{n}\right)_\varphi \leqslant B_{\varphi,r}\left\{\frac{1}{n^k}\Big[\sum_{v=1}^{n} v^{a(r+k)-1}E_{v-1}^a(f)_\varphi\Big]^{1/a} + \Big[\sum_{v=n+1}^{\infty} v^{ar-1}E_v^a(f)_\varphi\Big]^{1/a}\right\}.$$

Thus, if $E_n(f)_\varphi = O(n^{-\beta})$, we have

$$\omega_k(f^{(r)}, t)_\varphi = \begin{cases} O(t^{\beta-r}) & \text{when} \quad \beta-r<k, \\ O\left(t^k\left(\log\dfrac{1}{t}\right)^{1/a}\right) & \text{when} \quad \beta-r=k, \\ O(t^k) & \text{when} \quad \beta-r>k. \end{cases} \tag{10}$$

From (7) we have

$$\omega_{k+1}(f, t)_\varphi \leqslant t^k\omega_1(f^{(k)}, t)_\varphi.$$

Hence and from (10),

$$\omega_{k+1}(f, t)_\varphi = O(t^\beta).$$

Consequently,

$$\|f_h^k(x)-f(x)\|_\varphi \leqslant \frac{k}{2h}\int_0^h \omega_{k+1}(f, t)_\varphi dt = O(h^\beta).$$

Lemma 6. *Let*

$$\|f_h^k-f\|_\varphi = O(h^\beta), \qquad \beta>0.$$

Then

$$E_n(f)_\varphi = O(n^{-\beta}).$$

Proof. First we show that

$$\int_0^\pi t^{\beta+1}|k_n'(t)|\,dt = O(n^{-\beta}).$$

Indeed, as is well known (see e.g.[5]) there exists a sequence $\{k_n(t)\}$ $(n = 0, 1, \ldots)$ of trigonometric polynomials of degree $\leqslant n$ such that

$$(\alpha). \qquad \int_{-\pi}^{\pi} k_n(t)\,dt = 1,$$

$$(\beta) \qquad \int_{-\pi}^{\pi} |k_n(t)|\,dt \leqslant C_1,$$

$$(\gamma) \qquad \int_{-\pi}^{\pi} |t|^k |k_n(t)|\,dt \leqslant C_2(n+1)^{-k}, \qquad n = 0, 1, \ldots$$

Denoting $k_n(t) = b_p(\sin\frac{1}{2}pt/\sin\frac{1}{2}t)^{2k_0}$ where $2k_0 \geqslant \beta+4$ and p is a natural number, and applying Bernstein's inequality for $k_n(t)$:

$$|k_n'(t)| \leqslant n|k_n(t)|,$$

we observe that

$$\int_0^\pi t^{\beta+1+1/\alpha}|k_n'(t)|\,dt \leqslant n\int_0^\pi t^{\beta+1}|k_n(t)|\,dt + n\int_0^\pi t^{\beta+2}|k_n(t)|\,dt$$
$$= O(n^{-\beta}) + O(n^{-\beta-1}) = O(n^{-\beta}).$$

Next, we construct trigonometric polynomials

$$T_n(x) = (-1)^{k+1}\int_{-\pi}^{\pi} k_n(t)\sum_{v=1}^{k}(-1)^{k-v}\binom{k}{v} f(x+vt)\,dt.$$

In account of (α) we have

$$T_n(x) - f(x) = (-1)^{k+1}\int_0^\pi k_n(t)\{\Delta_t^k f(x) + \Delta_{-t}^k f(x)\}\,dt.$$

Differentiating by parts and applying (9) we have

$$|T_n(x) - f(x)| \leqslant O(1)\int_0^\pi t\,|f(x) - f_t^k(x)|\,|k_n'(t)|\,dt,$$

and finally

$$\|T_n(x) - f(x)\|_\varphi \leqslant O(1)\int_0^\pi t^{1+\beta}|k_n'(t)|\,dt = O(n^{-\beta}).$$

From Lemmas 3, 4, 5, 6 we obtain

THEOREM 1. *Let $\beta > 0$. Then the conditions*

$$E_n(f)_\varphi = O(n^{-\beta}) \quad \textit{and} \quad \|f_h^k - f\|_\varphi = O(h^\beta)$$

are equivalent for $\beta < k$ when k is an even number, and for $\beta < k+1$ when k is an odd number.

THEOREM 2. *If*

$$\|f_h^k - f\|_\varphi = \begin{cases} o(h^k) & \text{for } k \text{ even}, \\ o(h^{k+1}) & \text{for } k \text{ odd}, \end{cases}$$

then $f(x) = c$ almost everywhere.

Proof of this Theorem is completely analogous to that given in [4]

THEOREM 3. (a) *Let k be an even number. Then*

$$\|f_h^k - f\|_\varphi = O(h^k)$$

if and only if the following two conditions are satisfied:

(i) *$f(x) = g(x)$ a.e., where $g(x)$ is a function with absolutely continuous derivative of order $(k-2)$ and $g^{(k-1)} \in L^{*\varphi}_{(0,2\pi)}$,*

(ii) $\omega_1(g^{(k-1)}, h)_\varphi = O(h)$.

(b) *Let k be an odd number. Then*

$$\|f_h^k - f\|_\varphi = O(h^{k+1})$$

if and only if the following two conditions are satisfied:

(i′) *$f(x) = g(x)$ a.e., where $g(x)$ has an absolutely continuous derivative of order $(k-1)$ such that $g^{(k)} \in L^{*\varphi}_{(0,2\pi)}$,*

(ii′) $\omega_1(g^{(k)}, h)_\varphi = O(h)$.

Proof. (a) The sufficiency of these conditions follows immediately from some obvious properties of the modulus of smoothness and from (9). Now, we shall prove the necessity of these conditions. Because $\sum_{v=1}^{\infty} v^{ar-1} E_v^a(f)_\varphi$ is convergent for $r = k-1$ we see that $f(x)$ has an absolutely continuous derivative of order $(k-2)$ such that $f^{(k-1)} \in L^{*\varphi}$. In account of (10) we observe that

$$\omega_2(f^{(k-1)}, h)_\varphi = O(h).$$

Next, we put

$$S_h(x) = \frac{[f_h^k(x) - f(x)]}{h^k}, \quad \sigma_n(x) = \sum_{l=0}^{n} \left(1 - \frac{l}{n+1}\right) A_l(x),$$

$$A_m(x) = a_m \cos mx + b_m \sin mx,$$

where a_m, b_m are the Fourier coefficients of $f(x)$, $\sigma_n(v)$ are Fejér means of order $\leqslant n$ for $f(x)$,

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)k_n(t-x)\,dt$$

(see [1], p. 146). In account of (6)

$$\|\sigma_n\|_\varphi \leqslant \|f\|_\varphi, \quad f\in L^{*\varphi}_{(0,2\pi)}.$$

Arguing as in [4] we observe that Fejér means of the series $\sum_{m=1}^{\infty} m^k A_m(x)$ are bounded in $L^{*\varphi}_{(0,2\pi)}$, and k-times differentiating the Fourie series of $f(x)$ we obtain a k-times uniformly convergent series. If $g(x)$ is the sum of the Fourier series of $f(x)$, then from [1], p. 88,

$$\|g^{(k-1)}(x+h)-g^{(k-1)}(x)\|_\varphi = O(h).$$

Proof of (b) is analogous.

References

[1] Н. К. Бари (N. K. Bari), *Тригонометрические ряды*, Москва 1961.

[2] V. Kokilashvili, *On approximations of periodic functions in Orlicz spaces*, Bull. Acad. Polon. Sci. 14 (2) (1966), pp. 77-80.

[3] M. A. Krasnoselskiǐ, Y. B. Rutickiǐ, *Convex functions and Orlicz spaces*, Groningen 1961.

[4] Ф. И. Харшиладзе (F. I. Haršiladze), *Приближение функций средними Стеклова*, Труды Тбилиского Математического Института, 31 (1966). pp. 55-70.

[5] С. Б. Стечкин (S. B. Stečkin), *О порядке наилучших приближений непрерывных функций*, Изв. Акад. Наук СССР 15, (1951), pp. 219- 242.

KATEDRA MATEMATYKI I UNIWERSYTETU A. MICKIEWICZA