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Inverse monotone functions and their applications

In [6] the basic lemmas concerning properties of inverse monotone functions are formulated and applied in the theory of entire functions. In this note we shall give the generalization of these lemmas and the applications in some problems of the regularization of functions and sequences, in the theory of quasi-analytic classes of functions and in Bernstein's problem of approximation.

1. Inverse functions. Let N_∞^a denote the class of all real-valued functions, non-decreasing for $x \geqslant a$ and tending to ∞ as $x \to \infty$, and let N_∞ denote the class of all functions $p(x) \in N_\infty^a$ for some a depending on p.

In [6] the inverse function of the function $p \in N_\infty^a$ was defined by the formula: $q(u) = \sup\{x\colon p(x) \leqslant u\}$. In this note we shall use the following modified definition: $\overline{p}(x) = \inf\{u \geqslant a\colon p(u) > x\}$. Evidently, we have $\overline{p}(x) = q(x)$ for $x \geqslant p(a)$ and $\overline{p}(x) = a$ for $x < p(a)$. Hence, using Lemma 1 and 2 in [6] we obtain:

1.1. *If $p(x) \in N_\infty^a$, then $\overline{p}(x) \in N_\infty^a$ and*

(a) $\overline{p}(x+0) = \overline{p}(x)$, $\overline{p}(x-0) = \inf\{u \geqslant a,\ p(u) \geqslant x\}$, $\overline{\overline{p}}(x) = p(x)$ *at each point of right-hand continuity of p.*

(b) $\overline{p}[p(x)] = \sigma_p(x) \geqslant x$; *if* $\sigma_p(x) > x$, *then* $p(t) = p(x)$ *in* $\langle x, \sigma_p(x))$, $p(t) > p(x)$ *for* $t > \sigma_p(x)$.

(c) *If* $p[\overline{p}(x)] \neq x$, *then* $\overline{p}(t) = \overline{p}(x)$ *in the intervals* $\langle x, p[\overline{p}(x)])$ *or* $\langle p[\overline{p}(x)], x\rangle$.

Now we shall prove the following properties of the inverse functions.

1.2. *If $p(x) = a$ for $\alpha \leqslant \alpha' < x < \beta$, then $\overline{p}(a+0) \geqslant \beta$, $\overline{p}(a-0) \leqslant \alpha'$. If $p(a-0) \leqslant t < p(a+0)$, then $\overline{p}(t) = a$.*

Indeed, it follows from 1.1 (a) that $\overline{p}(a+0) = \overline{p}(a) = \inf\{u\colon p(u) > a\} \geqslant \beta$ and $\overline{p}(a-0) = \inf\{u\colon p(u) \geqslant a\} \leqslant a$. Denote $U_t = \{u\colon p(u) > t\}$ for $t \in \langle p(a-0), p(a+0))$. If $u > a$, then $u \in U_t$; if $u < a$, then $u \notin U_t$. Hence $a = \inf U_t = \overline{p}(x)$.

1.3. *If* $g(x) = p[\tau(x)]$, $p(x) \in N_\infty^a$, $\tau(x) \in N_\infty$, $\tau(x) \geqslant a$, *then* $\bar{g}(x) \leqslant \bar{\tau}[\bar{p}(x)]$; *if* $p(x)$ *is left-hand continuous in* (a, ∞), *then* $\bar{g}(x) = \bar{\tau}[\bar{p}(x)]$ *for all* $x \geqslant p(a)$.

Proof. Let us suppose $\bar{\tau}[\bar{p}(x)] < t < \bar{g}(x)$, then

$$g(t) = p[\tau(t)] \leqslant x \quad \text{and} \quad \tau(t) > \bar{p}(x).$$

Hence $p[\tau(t)] > x$ and we get a contradiction. In a similar way we get a contradiction in case when $\bar{g}(x) < t < \bar{\tau}[\bar{p}(x)]$ and $p(x-0) = p(x)$ in (a, ∞).

2. Regularization of functions and sequences. Let A be a set of real numbers, and let $\mathfrak{a}$ be a given class of real functions defined on A. The operation $f(\mathfrak{a} \to \mathfrak{a})$ is called *regularization*, and the function $a^*(t) = f(a)$ the *regularized function* if the following conditions (R) are satisfied:

(R_1) If $a_1 \leqslant a_2$ in A, then $f(a_1) \leqslant f(a_2)$,

(R_2) $f(a) \leqslant a$ in A,

(R_3) $f[f(a)] = f(a)$.

Let $\mathfrak{L}$ denote the range of the operation f. The following property of regularized function is an immediate consequence of the definition of the regularization.

2.1. *The regularized function is the greatest minorant of the function* a *in* $\mathfrak{L}$.

Indeed, if the function $b(t) \in \mathfrak{L}$ and the inequality $f(a) \leqslant b \leqslant a$ holds in A, then $b = f(a_1)$ and $f(a) \leqslant f(a_1) \leqslant f(a)$. Consequently, $b = f(a)$.

There are many examples of regularizations in connection with various problems of mathematics. We shall consider only a generalization of the "convex regularization" of Mandelbrojt [1].

Suppose that $\psi \in N_\infty^\beta$ and $\mathfrak{a} = \mathfrak{a}_{\varphi,T}$ is the class of functions $a(t)$ defined for $t \geqslant T \geqslant \beta$, satisfying the following conditions (K):

(K_1) $\inf a(t) > -\infty$ in each finite interval,

(K_2) $\lim_{t\to\infty} a(t)/\psi(t) = \infty$,

(K_3) there exists a sequence $t_n \to \infty$ (depending on a) such that $a(t_n) < \infty$ (the value ∞ for $t \neq t_n$ is not excluded).

2.2. *For each function* $\varphi \in N_\infty^a$ *the following operation is defined in* $\mathfrak{a}$:

$$F_{\varphi,\psi}(a) = \sup_{t \geqslant T} \{\varphi(x)\psi(t) - a(t)\} = A(x) \quad (x \geqslant a) \text{ [2]}.$$

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(1) Various methods of regularizations of sequences are given by Mandelbrojt in [11], the general theory of the regularization of functions is given in [10].

(2) Mandelbrojt [10] assumes that $\varphi(x) = x$, $\psi(t) = t$. By means of the same operation the complementary φ-function is defined in [12].

Indeed, (K_2) implies that

$$\varphi(x)\psi(t)-a(t) = \psi(t)[\varphi(x)-a(t)/\psi(t)] \to -\infty \quad \text{for} \quad t\to\infty.$$

Therefore, according to (K_1)

$$\sup_{t\geqslant T}\{\varphi(x)\psi(t)-a(t)\} = \sup_{T\leqslant t\leqslant T_1}\{\varphi(x)\psi(t)-a(t)\} < \infty.$$

To investigate the properties of this operation we shall apply the following symbols:

$$\Phi(x,t) = \varphi(x)\psi(t)-a(t),$$

$$\mathfrak{T}_x = [\{t_n\}\colon t_n \geqslant T\ (n=1,2,\ldots),\ \lim_{n\to\infty}\Phi(x,t_n) = A(x)],$$

$$(2.1)\qquad p(x) = \sup_{\mathfrak{T}}\{\limsup_{n\to\infty} t_n\},$$

$$b(x) = \sup_{\mathfrak{T}}\{\limsup_{n\to\infty}\psi(t_n)\},$$

$$I_\varepsilon = \big(p(x)-\varepsilon,\ p(x)+\varepsilon\big).$$

From these definitions it follows immediately that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(2.2)\qquad A(x) = \sup_{I_\varepsilon}\Phi(x,t) \geqslant \Phi(x,t)+\delta \quad \text{for} \quad t\geqslant p(x)+\varepsilon.$$

The basic properties of the operation $F_{\varphi,\psi}$ yield the following theorem:

2.3. *If* $A(x) = F_{\varphi,\psi}(a)$; $p(x), b(x)$ *are defined by* (2.1) *then* $A(x)$, $p(x), b(x)\in N_\infty$ *and* $\lim_{x\to\infty}A(x)/\varphi(x) = \infty$. *If* $\varphi(x)$ *is continuous, then*

$$(2.3)\qquad A(x) = A(\xi)+\int_\xi^x b(u)\,d\varphi(u) \qquad (a\leqslant\xi\leqslant x).$$

Moreover, if $\psi(x)$ *is continuous, then*

$$(2.4)\qquad A(x) = A(\xi)+\int_\xi^x \psi[p(u)]\,d\varphi(u).$$

Proof. Suppose that $x_2 > x_1 \geqslant a$, $\{t_n'\}\in\mathfrak{T}_{x_1}$, $\{t_n''\}\in\mathfrak{T}_{x_2}$, $\delta_n' = A(x_1)-\Phi(x_1,t_n')$, $\delta_n'' = A(x_2)-\Phi(x_2,t_n'')$. Then

$$\lim_{n\to\infty}\delta_n' = \lim_{n\to\infty}\delta_n'' = 0$$

and

$$\Phi(x_2,t_n')-\Phi(x_1,t_n')-\delta_n' = [\varphi(x_2)-\varphi(x_1)]\psi(t_n')-\delta_n'$$
$$\leqslant A(x_2)-A(x_1) \leqslant \Phi(x_2,t_n'')+\delta_n''-\Phi(x_1,t_n'') = [\varphi(x_2)-\varphi(x_1)]\psi(t_n'')+\delta_n''.$$

Passing to the limit we get the inequality

$$(*)\qquad [\varphi(x_2)-\varphi(x_1)]b(x_1) \leqslant A(x_2)-A(x_1) \leqslant [\varphi(x_2)-\varphi(x_1)]b(x_2).$$

If $\varphi(x_2) = \varphi(x_1)$ then $A(x_2) = A(x_1)$, $\mathfrak{x}_{x_2} = \mathfrak{x}_{x_1}$ and consequently, $b(x_2) = b(x_1)$. If $\varphi(x_2) \neq \varphi(x_1)$ then $b(x_1) \leqslant b(x_2)$. Suppose that $b(x) \leqslant C$, then for fixed x_1 and $x > x_1$ there is

$$\varphi(x)\psi(t)-a(t) \leqslant A(x) \leqslant A(x_1)+C[\varphi(x)-\varphi(x_1)].$$

Therefore

$$a(t) \geqslant \varphi(x)[\psi(t)-C]-A(x_1)+C\varphi(x_1) \to \infty$$

for all sufficiently large t and $x \to \infty$. The contradiction with the assumption (K_3) proves that $b \in N_\infty$. Now, from $(*)$ it follows immediately that $A(x_2) \geqslant A(x_1)$ for $x_2 > x_1 \geqslant X$ and $\lim\limits_{x\to\infty} A(x)/\varphi(x) = \infty$.

According to (2.2) the assumption $p(x_2) < p(x_1)$ for $x_2 > x_1 \geqslant X$ implies the inequality

$$(**)\qquad A(x_2) - A(x_1) \geqslant \delta + [\varphi(x_2) - \varphi(x_1)]\, b(x_1).$$

Evidently, $\psi[p(x)-0] \leqslant b(x) \leqslant \psi[p(x)+0]$; consequently, $b(x_2) = b(x_1)$ and $(**)$ contradicts $(*)$.

Given any partition of the interval $\langle x_0, x\rangle$, summing up the inequalities $(*)$ and passing to the limit yields (2.3). If $\psi(x)$ is continuous then $b(x) = \psi[p(x)]$ and we obtain (2.4).

Now we give an example. Suppose the functions φ, ψ be continuous, $\psi \in N_\infty^\beta$, $\varphi \in N_\infty^a$, $\gamma(t) \in N_\infty^T$, $\tau \geqslant T \geqslant \beta$, $a(t) = \int_\tau^t \gamma(u)\,d\psi$. Then

$$\Phi(x, t) = \varphi(x)\psi(\tau) + \int_\tau^t [\varphi(x)-\gamma(u)]\,d\psi(u),$$

and it follows from the definition of tbe inverse function that $p(x) = \sup\{y\colon \psi(y) = \bar{\gamma}[\varphi(x)]\}$. Denote $\delta(x) = \bar{\gamma}[\varphi(x)]$. Since $\Phi(x, t)$ is continuous with respect to t, we obtain from (2.4)

$$(2.5)\qquad S(x) = \sup_{t\geqslant T}\{\varphi(x)\psi(t) - \int_\tau^t \gamma(u)\,d\psi\}$$

$$= \varphi(x)\psi[\delta(x)] - \int_\tau^{\delta(x)} \gamma(u)\,d\psi = S(\xi) + \int_\xi^x \psi[\delta(u)]\,d\varphi$$

for $x \geqslant a$.

Now we apply the operation $F_{\varphi,\psi}$ to define a regularization in $\mathfrak{a}$.

2.4. *If $a \in \mathfrak{a}$ and if $A(x) = F_{\varphi,\psi}(a)$, then the operation*

$$f_{\varphi,\psi}(a) = \sup_{x \geqslant X \geqslant a} \{\varphi(x)\psi(t) - A(x)\} = a^*(t)\ (^3)$$

is a regularization. If the functions φ, ψ are continuous, then the regularized function $a^*(t)$ can be represented in the form

$$a^*(t) = a^*(\tau) + \int_\tau^t \varphi[\bar{p}(u)]d\psi \qquad (\tau \geqslant T), \tag{2.6}$$

where $\bar{p}(u)$ is the inverse function of $p(x)$ defined by the formulas (2.1) *for $x \geqslant X$.*

Proof. It follows from 2.3 that $f = f(\mathfrak{a} \to \mathfrak{a})$. Denoting $A^*(x) = F(a^*)$ and applying the definition of F and f we obtain the formulas

$$a^*(t) = \sup_{x \geqslant X} \{\varphi(x)\psi(t) - \sup_{u \geqslant T} [\varphi(x)\psi(u) - a(u)]\}, \tag{a}$$

$$A^*(x) = \sup_{t \geqslant T} \{\varphi(x)\psi(t) - \sup_{u \geqslant X} [\varphi(u)\psi(t) - A(u)]\}. \tag{b}$$

Putting $u = t$ in (a), according to (b) we have $a^*(t) \leqslant a(t)$ and $A^*(x) \geqslant A(x)$. Therefore, setting $u = x$ in (b) we obtain $A^*(x) = A(x)$ and consequently, $a^{**}(t) = a^*(t)$. If $a_1(t) \leqslant a(t)$ for $t \geqslant T$, then $F(a_1) \geqslant F(a)$; thus $a_1^*(t) \leqslant a^*(t)$. To prove (2.6) we assume that $q(u) \in N_\infty^T$, $q(u) \geqslant a$ for $u \geqslant T$ and we put $\gamma(u) = \varphi[q(u)]$ in (2.5). Since $\varphi(u)$ is constant in the intervals where $\sigma_\varphi(u) \neq u$, it follows from 1.3 and 1.1 that

$$\delta(x) = \bar{q}[\sigma_\varphi(x)] \quad \text{and} \quad \int_\xi^x \psi[\delta(u)]d\varphi = \int_\xi^x \psi[\bar{q}(u)]d\varphi.$$

Moreover, if $\bar{q}[\sigma_\varphi(x)] > u > \bar{q}(x)$, then $\sigma_\varphi(x) \geqslant q(u) > x$ and by virtue of 1.1 and the continuity of φ we have

$$\varphi(x) \leqslant \varphi[q(u)] \leqslant \varphi[\sigma_\varphi(x)] = \varphi(x)$$

and

$$\int_{\bar{q}(x)}^{\delta(x)} \varphi[q(u)]d\psi = \varphi(x)\{\psi[\delta(x)] - \psi[\bar{q}(x)]\}.$$

Now, formula (2.5) yields

$$\begin{aligned} S_1(x) &= \sup_{t \geqslant T} \{\varphi(x)\psi(t) - \int_\tau \varphi[q(u)]d\psi\} \\ &= \varphi(x)\psi[\bar{q}(x)] - \int_\tau^{\bar{q}(x)} \varphi[q(u)]d\psi = S_1(\xi) + \int_\xi^x \psi[\bar{q}(u)]d\varphi. \end{aligned} \tag{2.7}$$

(3) Where no misunderstanding may arise, we shall omit the indices φ, ψ.

Changing in this formula the role of φ and ψ and putting p instead of q we obtain (2.6), according to (2.4).

Remark. From (2.2) it follows that the change of T implies the change of values of $A(x)$ in a finite interval only. This fact implies that if $a_1(t) = a_2(t)$ for $t \geqslant t_0$, then $a_1^*(t) = a_2^*(t)$ for t sufficiently large.

2.5. *Suppose that φ, ψ are continuous functions, $\varphi \in N_\infty^\alpha$ and φ is strictly increasing, $\psi \in N_\infty^\beta$. If f is the regularization defined in 2.4 ($T \geqslant \beta, X \geqslant \alpha$), then the equality $f(a) = a$ holds if and only if the function $a \in \mathfrak{a}$ can be written in the form*

$$(*) \qquad a(t) = C + \int_\tau^t \gamma(u)\,d\psi \qquad (t \geqslant \tau \geqslant T),$$

where C is an arbitrary constant, $\gamma(u) \in N_\infty^T$ and $\gamma(u) \geqslant \varphi(X)$ for $u \geqslant T$.

Proof. By 2.4, the necessity of $(*)$ is evident; here $p(x)$ is considered for $x \geqslant X$ and therefore $\bar{p}(x) \geqslant X$. Without loss of generality it can be supposed that $\gamma(u)$ is right-hand continuous. If $a(t)$ is of the form $(*)$, applying (2.5) and (2.7) we obtain

$$a^*(t) = C_1 + \int_\tau^t \varphi[\bar{\delta}(u)]\,d\psi,$$

where $\delta(u) = \bar{\gamma}[\varphi(x)]$. By 1.1 (a) and 1.3, $\varphi[\bar{\delta}(u)] = \gamma(u)$; hence $a^*(t) = C_2 + a(t)$. The application of the definition of f and the condition (R_3) shows that $a^* = f(a^*) = C_2 + a^*$. Therefore $C_2 = 0$ and $a^*(t) = a(t)$.

Corollary 1. *If the assumptions of 2.5 are satisfied, then the range $\mathfrak{L}$ of the operation f is equal to the set of functions defined by the formula $(*)$.*

Corollary 2. *If the assumptions of 2.5 are satisfied and if $\mathfrak{L}$ is the set of functions defined by $(*)$, then $a^*(t)$ is the greatest minorant of a in $\mathfrak{L}$.*

Now we shall study sequences. Mandelbrojt has shown that if the function $a(t)$ can be ∞ for some t, then putting $a(t) = a_n$ for $t = n$ $(n = 1, 2, \ldots)$ and $a(t) = \infty$ for $t \neq n$ we can apply the methods of regularization of functions to sequences.

Now we shall apply Mandelbrojt's method to the regularization defined in 2.4.

Let $\{c_n\}$ be non-decreasing and $\lim_{n\to\infty} c_n = \infty$. Let $\{a_n\}$ satisfy the condition $\lim_{n\to\infty} a_n/c_n = \infty$. Suppose that $\psi(x) \in N_\infty^1$ is a continuous function and $\psi(n) = c_n$. If $\varphi(x) \in N_\infty^1$ and φ is continuous, then putting $a(t) = a_n$ for $t = n$ $(n = 1, 2, \ldots)$ and $a(t) = \infty$ for $t \neq n$ we obtain

$$A(x) = \sup_{t\geqslant 1}\{\varphi(x)\psi(t) - a(t)\} = \max_{n\geqslant 1}\{\varphi(x)c_n - a_n\} = F_{\varphi,\psi}(\{a_n\}).$$

If $\nu(x)$ is the largest integer n such that the maximum is attained, then $p(x) = \nu(x)$, and formula (2.3) implies

$$A(x) = \varphi(x)c_{\nu(x)} - a_{\nu(x)} = A(\xi) + \int_{\xi}^{x} c_{\nu(u)}\,d\varphi. \tag{2.8}$$

This formula is proved in [6] independently of the theory of the regularization of functions and is used in some problems of the theory of entire functions.

Applying (2.6) we obtain the regularized sequence in the form

$$a_n^* = a^*(n) = \sup_{x \geqslant x_0}\{\varphi(x)c_n - A(x)\} = a_1^* + \int_{1}^{n} \varphi[\bar{\nu}(u)]\,d\psi. \tag{2.9}$$

Putting $n = \nu(\xi)$ we have:

$$a_{\nu(\xi)}^* = \sup_{x \geqslant x_0}\{\varphi(x)c_{\nu(\xi)} - \varphi(x)c_{\nu(x)} + a_{\nu(x)}\} \geqslant a_{\nu(\xi)}.$$

From this inequality and from the properties of regularization proved in 2.4 we obtain the following properties of the above regularization of sequences, immediately:

2.6. *If $\{a_n^*\}$ is the regularized sequence defined by formula* (2.9), *then $a_n^* \leqslant a_n$, $a_{\nu(x)}^* = a_{\nu(x)}$, $A^*(x) = A(x)$, $a_n^{**} = a_n$. If $a_n = b_n$ for $n > N_1$, then $a_n^* = b_n^*$ for $n > N$.*

In particular, if we put for $q \in N_\infty^1$

$$a(t) = C + \int_{\tau}^{t} q(u)\,d\psi, \qquad a_n = a(n), \qquad c_n = \psi(n), \tag{$*$}$$

and write $A_\alpha(x) = F_{\varphi,\psi}(a)$, $A_a(x) = F_{\varphi,\psi}(\{a_n\})$, $\delta(x) = \bar{q}[\varphi(x)]$, then, applying the first part of (2.5), we obtain

$$A_\alpha(x) = \varphi(x)\psi[\delta(x)] - a[\delta(x)] \geqslant A_a(x) \geqslant \varphi(x)\psi[n_\delta(x)] - a[n_\delta(x)],$$

where $n_\delta(x)$ is the integral part of $\delta(x)$. Thus we have

$$A_\alpha(x) \geqslant A_a(x) \geqslant \varphi(x)\{\psi[n_\delta(x)] - \psi[\delta(x)]\} + A_\alpha(x). \tag{2.10}$$

Accordingly:

2.7. *If the representation* ($*$) *of the sequence $\{a_n\}$ such that the function $\bar{q}(x)$ is integer-valued exists, the regularized sequence $\{a_n^*\}$ is equal to $\{a_n\}$.*

Remark. The operation $f_{x,t}$ is called *convex regularization*, the operation $f_{\ln x,t}$ — *logarythmic convex regularization*. The operation $(m_n \to m_n^c)$ defined by formulas

$$T(r) = \max_{n \geqslant 1}\{r^n/m_n\}, \qquad m_n^c = \sup_{r \geqslant 1}\{r^n/T(r)\} \tag{2.11}$$

is also of great importance. Since $\ln T(r) = \max_{n\geqslant 1}[n\ln r - \ln m_n]$, $\ln m_n^c = \sup_{r\geqslant 1}[\bar{n}\ln r - \ln T(r)]$, it is a convex logarythmic regularization with respect to $\ln m_n$ and $\ln m_n^c$. Therefore according to (2.8) and (2.9) the integral representation

$$(2.12)\quad T(r) = T(r_0)\exp\Big(\int_{r_0}^{r} \nu(u)u^{-1}du\Big),\quad m_n^c = m_1^c\exp\Big(\int_1^n \ln\bar{\nu}(u)\,du\Big)$$

holds and the integer-valued function $\nu(x)\in N_\infty$ (4).

3. Applications.

3.1. The second part of (2.7) can be written for $\tau = \bar{q}(\xi)$ in the form

$$(3.1)\quad \int_{(q)\xi}^{\bar{q}(x)} \varphi[q(u)]d\psi = \varphi(x)\psi[\bar{q}(x)] - \varphi(\xi)\psi[\bar{q}(\xi)] - \int_{\xi}^{x} \psi[\bar{q}(u)]d\varphi.$$

This formula is also valid in case when $\varphi(x)$ (or ψ) is non-increasing; consequently, it is a generalization of Lemma 3 in [6].

Suppose that $F(x) > 0$ is a continuous, non-decreasing function, $q(x)\in N_\infty$ and $a(x) = C + \int_{\xi}^{x} \psi[\bar{q}(u)]F^{-1}(u)dF$. Applying (3.1) we can prove in the same way as in [6] that the integrals

$$(3.2)\quad \int^{\infty} \frac{d\psi}{F[q(u)]},\quad \int^{\infty} \frac{\psi[\bar{q}(u)]}{F^2(u)}dF,\quad \int^{\infty} \frac{a(u)}{F^2(u)}dF$$

are all convergent or all divergent. Thus we have obtained a generalization of Lemma 4 in [6].

Denoting

$$\beta(x) = \frac{1}{x}\int_a^x q(u)\,du,\qquad q(x)\in N_\infty^a$$

we have the inequality

$$(*)\qquad q(x) \geqslant \beta(x) \geqslant \tfrac{1}{2}\beta(x/2) + \tfrac{1}{2}q(x/2)\qquad (x/2 > a).$$

If $\int^{\infty} \exp\big(-q(x)\big)dx < \infty$, then applying Schwarz inequality we obtain

$$(**)\qquad \Big(\int_{u_0}^{u} e^{-\beta(x)}dx\Big)^2 \leqslant \int_{u_0}^{2u} e^{-\beta(x/2)}dx \int_{u_0}^{u} e^{-q(x/2)}dx \leqslant C\int_{u_0/2}^{u} e^{-\beta(x)}dx.$$

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(4) Mandelbrojt has obtained the first of formulae (2.12) applying the convex regularization and the change of variables.

Putting $\psi(x) = x$ and $F(x) = e^x$ in (3.2), according to (*) and (**) we infer that the integrals

$$(3.3) \qquad \int^{\infty} e^{-q(x)}dx, \qquad \int^{\infty} \bar{q}(x)e^{-x}dx, \qquad \int^{\infty} e^{-\beta(x)}dx$$

are convergent or divergent, simultaneously.

3.2. Let $F(r)$ be a positive, increasing, continuous function defined for $r \geqslant r_0 > 0$ satisfying the following conditions

(a) $\ln F(r)$ is a convex function of $\ln r$,

(b) $\lim\limits_{r\to\infty} r^x/F(r) = 0$ for each $x > 0$.

It is well-known that $F(r)$ can be represented in the form

$$(*) \qquad \ln F(r) = \ln F(r_0) + \int_{r_0}^{r} \gamma(u)u^{-1}du, \quad \text{where} \quad \gamma(r) \in N_{\infty}^{r_0} \text{ and } \gamma(r_0) \geqslant 0.$$

Denote $\ln G(x) = \sup\limits_{r\geqslant r_0}[x\ln r - \ln F(r)]$ $(x \geqslant 0)$. Putting $\psi(t) = \ln t$, $\varphi(x) = x$, $T = r_0$ $X = 0$, $a(t) = \ln F(t)$ in 2.5 we obtain $\sup\limits_{x\geqslant 0}[x\ln r - \ln G(x)] = \ln F(r)$. This particular case of 2.5 is proved by Fenshel [4]. Another proof is due to Horvath [5] who supposed $r_0 = 1$ and $x \geqslant 1$. The simple example $\ln F(r) = ar$ $(a < 1)$ shows that the second assumption is incorrect since in this example $\ln G(x) = x\ln r - ar$ and if $1 \leqslant r < 1/a$, then the function $\Phi(x, r) = x\ln r - \ln G(x)$ attains its maximum at the point $x_r = ar < 1$. Therefore $\max\limits_{x\geqslant 1}\Phi(x, r) = \Phi(1, r) \neq \Phi(x_r, r) = ar = \ln F(r)$.

3.3. Let W be the class of functions which can be represented in the form

$$p(x) = \exp\Big(C + \int_{x_0}^{x} q(u)u^{-1}du\Big)$$

for $x \geqslant x_0$ where x_0 and C are constants depending on, p, $q(x) \in N_{\infty}^{x_0}$ and $q(x+0) = q(x)$. If $p(x) \in W$, then

$$x^a p(\gamma x) = \exp\Big(C_1 + \int_{x_1}^{x} [q(\gamma u) + a]u^{-1}du\Big).$$

Consequently, $p(ax) \in W$ and $x^a p(x) \in W$.

Let K be an arbitrary constant. Choose n_0 so that $\bar{q}(n_0) > 0$ and define the sequence $\{c_n\}$ and the function $f(z)$ by formulas

$$(*)\qquad c_n = \begin{cases} \exp\left(-K - \int\limits_{n_0}^{n} \ln q(u)\,du\right) & \text{for} \quad n \geqslant n_0, \\ c_{n_0} & \text{for} \quad n < n_0 \end{cases}$$

$$f(z) = \sum_{n=1}^{n} c_{2n} z^{2n}.$$

Since $\lim\limits_{n\to\infty} \ln c_n/n = -\infty$, $f(z)$ is entire. It follows from (2.7) for $\psi(t) = t$ and $\varphi(x) = \ln x$ that the constant K can be chosen so that

$$(**)\qquad \ln \mu_f(x) \leqslant \sup_{t\geqslant n_0} \left[t \ln x - \int\limits_{n_0}^{t} \ln \bar{q}(u)\,du\right] - K = \ln p(x).$$

From (2.10) for $\varphi(x) = \ln x$, $\psi(t) = 2t$, $a(t) = \int\limits_{n_0}^{2t} \ln \bar{q}(u)\,du$ we obtain

$$\ln \mu_f(x) \geqslant 2\ln x + \sup_{t\geqslant n_0} \left[t \ln x - \int\limits_{n_0}^{t} \ln \bar{q}(u)\,du\right] - K \geqslant C + 2\ln x + \ln p(x).$$

We shall apply the above estimations of $\ln \mu_f(x)$ to prove the following theorem:

(M) *Let $F(x)$ be a positive even function defined for all x. If there exist positive constants x_0, δ and a function $p(x) \in W$ such that $p(x) \leqslant F(x)$ for $x \geqslant x_0$, $F(x) \geqslant \delta$ for $0 \leqslant x \leqslant x_0$ and $\int^{\infty} \ln p(x) x^{-2} dx = \infty$, then $F(x)$ possesses an entire minorant of the form $\sum\limits_{n=0}^{\infty} a_n x^{2n}$ $(a_n > 0)$ and of a positive genus* ([5]).

Proof. For $x \geqslant x_0$ and arbitrary $c_0 > 0$, $a > 0$, the function $g(z) = c_0 + f(az)$ satisfies the inequality

$$g(x) \leqslant c_0 + a^2/(1-a^2)\mu_f(x) \leqslant c_0 + a^2/(1-a^2) F(x).$$

If c_0 and a are sufficiently small, this inequality holds for all x. Since $f(x) \geqslant \mu_f(x)$, the integral $\int^{\infty} \ln g(x) x^{-2} dx = \infty$ and $g(z)$ is of positive genus. Therefore $g(x)$ is the desired minorant.

The above theorem is a generalization of the following theorem of Achiezer [1]:

([5]) If $f(z) = \exp Q(z) \prod\limits_{n=1}^{\infty} E(z/z_n, p)$, then the number $\gamma = \min(p, q)$, where q is the degree of the polynomial Q, is called the *genus* of $f(z)$.

(A) *Let W_0 be the class of functions $M(x)$ satisfying the following conditions*:

(a) *$M(x)$ is even and non-decreasing for $x > 0$, $M(0) > 0$,*

(b) $\ln M(x)$ *is a convex function of* $\ln x$.

Let M_Φ be the greatest minorant in the class W_0 of a given function Φ. The function M_Φ possesses an entire minorant of the form

$$\omega(x) = \sum_{n=0}^{\infty} a_n x^{2n} \qquad (a_0 > 0,\ a_n \geqslant 0 \text{ for } n = 1, 2, \ldots)$$

and of positive genus if and only if the integral

$$\int_{-\infty}^{\infty} \frac{\ln M_\Phi(x)}{1+x^2}\,dx = \infty.$$

The necessity is evident, since $\omega(x) \in W_0$ and it is of positive genus. Thus $\int_{-\infty}^{\infty} \frac{\ln \omega(x)}{1+x^2}\,dx = \infty$. It is well-known that $W_0 \subset W$. Consequently, according to (M) the desired minorant exists for each function $M \in W_0$ satisfying the condition $\int_{-\infty}^{\infty} \frac{\ln M(x)}{1+x^2}\,dx = \infty$. In particular, it exists for the function $M_\Phi(x)$. In this demonstration (contrary to the proof of Achiezer) it is not necessary to make use of the exact form of the minorant.

Theorem (M) can be also applied to generalize the following theorem of Bernstein ([2], p. 63):

(B) *Let $F(x)$ be a continuous positive function defined for all x. Let $f(x)$ be an arbitrary continuous function defined in $(-\infty, \infty)$ such that $\lim_{x\to\pm\infty} f(x) = 0$. Finally, let $\mathfrak{M}$ be the class of all polynomials $P(x)$. If there exists an entire function $F_1(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ of positive genus such that $a_0 > 0$, $a_n \geqslant 0$ for $n = 1, 2, \ldots$ and $F_1(x) \leqslant F(x)$ for all x, then*

$$(*) \qquad \inf_{\mathfrak{M}}\{\sup_{(-\infty,\infty)} |f(x) - P(x)/F(x)|\} = 0.$$

If $(*)$ holds for each continuous function $f(x)$ satisfying the condition $\lim_{x\to\pm\infty} f(x) = 0$, then $F(x)$ is called a *weight function.*

Applying (M) we immediately conclude that the following generalization of (B) is also true:

(B′) *Let $F(x)$ be a continuous positive even function defined for all x. If there exists a function $p(x) \in W$ such that*

$$(**) \quad \int^{\infty} \ln p(x) x^{-2} dx = \infty \quad \textit{and} \quad p(x) \leqslant F(x) \quad \textit{for all} \quad x \geqslant X,$$

then $F(x)$ is a weight function. In particular, if $p(x) \in W$ and $(**)$ holds, then $p(x)$ can be extended to a weight function.

This theorem is also a consequence of the Denjoy-Carleman theorem in the theory of quasi-analytic classes of functions (cf. [11], p. 184), but the proof by application of Bernstein's theorem is more elementary. Moreover, we shall show that conversely, applying (B′) the Denjoy-Carleman theorem can be proved in a simple way.

3.4. Denote by C_I^∞ the class of real functions $f(x)$, infinitely derivable in an interval I and let $\{m_n\}$ be an arbitrary sequence of positive numbers. We say that $f \in C\{m_n\} \subset C_I^\infty$, if there exists a constant $A = A_f$ such that

$$\sup_I |f^{(n)}(x)| \leqslant A^n m_n \quad \text{for} \quad n = 0, 1, 2, \ldots$$

The class $C\{m_n\}$ will be said to be *quasi-analytic* (q. analytic) if the conditions $f \in C\{m_n\}$ and $f^{(n)}(x_0) = 0$, where $x_0 \in I$, $n = 0, 1, \ldots$, imply $f(x) \equiv 0$. Without loss of generality we may suppose that $I = \langle 0, a \rangle$. To formulate Ostrowski's condition of q. analicity we shall consider the function $f(u)$ defined in $I = \langle 0, a \rangle$ satisfying the following conditions:

$$(\mathrm{Q}) \quad f^{(n)}(0) = 0 \text{ for } n = 0, 1, \ldots; \quad \sup_I |f^{(n)}(x)| = M_n < \infty, \quad f(u) \not\equiv 0.$$

From Taylor's formula we have for such a function

$$(*) \qquad f^{(n-k)}(u) = \frac{u^k}{k!} f^{(n)}(\xi) \quad (\xi, u \in I) \quad \text{and} \quad M_{n-k} \leqslant \frac{a^k}{k!} M_n.$$

Let $C\{m_n\}$ be not q. analytic and $\liminf_{n\to\infty} \sqrt[n]{m_n} < \infty$. Therefore there exists a function $f(u)$ satisfying conditions (Q) and a subsequence $\{n_k\}$ of indices such that $M_{n_k} \leqslant K^{n_k} A^{n_k}$. According to $(*)$ $M_n \leqslant C_1 (AK)^n$ for $n = 0, 1, \ldots$, and $f(u)$ can be expended in Taylor's series, but this contradicts (Q). Thus we can suppose that $\lim_{n\to\infty} \sqrt[n]{m_n} = \infty$ and the function $T(r) = \max_{n\geqslant 1} r^n/m_n$ is defined for all $r > 0$. The divergence of the integral $\int^{\infty} \ln T(r) r^{-2} dr$ is a necessary and sufficient condition for the class $C\{m_n\}$ to be q. analytic (Ostrowski's form of Denjoy-Carleman theorem). We shall prove its sufficiency.

Let $f(u)\in C_I^\infty$ and $\varphi(x) = f(x^2-x)$. We show by induction

$$\varphi^{(n)}(x) = f^{(n)}(u)(2x-1)^n + \sum_{k=1}^{[n/2]} f^{(n-k)}(u)(2x-1)^{n-k}\frac{n(n-1)\dots(n-2k+1)}{k!},$$

where $u = x^2-x$. Denoting

$$M_n^f = \sup_{\langle -1/4,0\rangle} |f^{(n)}(u)|, \qquad M_n^\varphi = \sup_{\langle 0,1\rangle} |\varphi^{(n)}(x)|$$

and applying (*) we obtain

$$M_n^\varphi \leqslant M_n^f\Big[1+\sum_{k=1}^{[n/2]}\binom{n}{k}\binom{n-k}{k}\Big] \leqslant M_n^f\sum_{k=0}^{n}\binom{n}{k}^2 \leqslant 4^n M_n^f.$$

Thus if $f(x)$ satisfies conditions (Q) in $I = \langle -\frac{1}{4}, 0\rangle$, then $\varphi(x)$ satisfies the following conditions in $\langle 0, 1\rangle$:

(Q′) $\quad \varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$ for $n = 0, 1, \dots$; $\quad \varphi(x)\not\equiv 0, \quad M_n^\varphi \leqslant 4^n M_n^f$.

Let us suppose $\int^\infty \ln T(r) r^{-2} dr = \infty$ and let $C\{m_n\}$ be not q. analytic $(I = \langle -1/4, 0\rangle)$. Thus there exists a function $\varphi(x)\in C\{m_n\}$ in $\langle 0, 1\rangle$ satisfying conditions (Q′). Let $\psi(x) = \varphi(x)$ for $x\in\langle 0, 1\rangle$ and $\psi(x) = 0$ for $x\notin\langle 0, 1\rangle$. Moreover, let

$$g(x) = g_1(x)+ig_2(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(t)e^{-2\pi itx}dt = \frac{1}{\sqrt{2\pi}}\int_0^1\varphi(t)e^{-2\pi itx}dt.$$

By repeated integration by parts we obtain for $|x| \geqslant |x_0| > 0$

$$|g(x)| \leqslant B^n m_n |x|^{-n} = \Big[\Big(\frac{|x|}{B}\Big)^n / m_n\Big]^{-1}.$$

Since $\max\limits_{n\geqslant 1}(x/B)^n/m_n = T(x/B)$, we have

$$(\dot{*}) \qquad |g(x)| \leqslant T^{-1}\Big(\frac{x}{B}\Big).$$

Repeated differentiation of the inversion formula

$$\psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{2\pi itx}dx$$

gives for $t = 0$

$$(\dot{**}) \qquad \int_{-\infty}^{\infty} x^n g(x)dx = \int_{-\infty}^{\infty} x^n g_1(x)dx + i\int_{-\infty}^{\infty} x^n g_2(x)dx = 0 \quad (n = 0, 1, \dots).$$

Let $F(x)$ be a positive, continuous even extension of $x^{-2}T(x/B)$ to the whole straight line. Since $\int^{\infty}\log T(r)r^{-2}dr = \infty$, it follows from 3.3 and (B′) that $F(x)$ is a weight-function.

Since $\lim\limits_{x\to\pm\infty} g_\nu(x) = 0$ $(\nu = 1, 2)$, given any $\varepsilon > 0$ there exist polynomials $P_\nu(x)$ such that

$$|g_\nu(x)F(x) - P_\nu(x)| < \varepsilon F(x) \quad \text{in} \quad (-\infty, \infty).$$

Multiplying both sides by $|g_\nu(x)|$ and integrating from $-\infty$ to ∞ we obtain

$$\left|\int\limits_{-\infty}^{\infty}[g_\nu^2(x)F(x) - g_\nu(x)P(x)]\,dx\right| \leqslant \varepsilon \int\limits_{-\infty}^{\infty}|g_\nu(x)|F(x)\,dx \leqslant K\varepsilon.$$

Since ε was arbitrary, according to (**) we have

$$\int\limits_{-\infty}^{\infty} g_\nu^2(x)F(x)\,dx = 0$$

and consequently, $\varphi(x) \equiv 0$. The contradiction shows that $C\{m_n\}$ is q. analytic (6).

The necessity in Denjoy-Carleman theorem can be easily proved applying Mandelbrojt-Bang condition of q. analiticity, and the equivalence of Ostrowski's and Mandelbrojt's conditions is an immediate consequence of the simultaneous convergence of integrals shown in 3.1. In fact, applying (2.12) we have

$$[\nu(n+1)]^{-a} \leqslant (m_n^c/m_{n+1}^c)^a \leqslant [\nu(n)]^{-a},$$

$$(m_n^c)^{-\frac{a}{n}} = (m_1^c)^{-\frac{a}{n}} \cdot \exp\left\{-\frac{a}{n}\int^{n}\ln\bar{\nu}(u)\,du\right\}.$$

Therefore from (3.3) and (3.2) for $\psi(t) = t$, $F(x) = x^a$, it follows that the integral $\int^{\infty}\ln T(r)r^{-a-1}dr$ and the series $\sum\limits_{n=1}^{\infty}(m_n^c/m_{n+1}^c)^a$, $\sum\limits_{n=1}^{\infty}(\sqrt[n]{m_n^c})^{-a}$ diverge simultaneously. Since the divergence of $\sum\limits_{1}^{\infty} m_n^c/m_{n+1}^c$ or $\sum\limits_{1}^{\infty}(\sqrt[n]{m_n^c})^{-1}$ are just Mandelbrojt-Bang conditions, the desired equivalence is proved.

For the sake of completeness we give still the demonstration of Carleman's condition $\sum\limits_{1}^{\infty} 1/\beta_n = \infty$ where $\beta_n = \inf\limits_{k\geqslant n}\sqrt[k]{m_k}$ (see [3]).

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(6) Ostrowski's proof (cf. [13] or [9], p. 52) is based on a different principle.

Denote $T_\beta(r) = \max_{n\geqslant 1} r^n/\beta_n^n$, $\sigma(x) = \beta_n$ for $n \leqslant x < n+1$, $a_n = \exp =\big(\int_1^n \ln\sigma(u)\,du\big)$, $T_a(r) = \max r^n/a_n$. Since the sequence $\sqrt[n]{m_n^c}$ is non-decreasing and $\sqrt[n]{a_n} \leqslant \beta_n$, we have

$$(*)\qquad \sqrt[n]{m_n} \geqslant \beta_n \geqslant \sqrt[n]{m_n^c} \quad \text{and} \quad T_a(r) \geqslant T_\beta(r) = T(r).$$

It is clear that $\bar\sigma(x)$ is integer-valued. Therefore according to 2.7 and (2.12) we have $a_n^c = a_n$, $\ln T_a(r) = \ln T_a(r_0) + \int_{r_0}^{r} \bar\sigma(u)u^{-1}du$ and the integrals $\int^\infty [\sigma(u)]^{-a}du$, $\int^\infty \ln T_a(r) r^{-a-1}dr$ are convergent or divergent simultaneously. Application of $(*)$ completes the proof.

For $a \neq 1$ the simultaneous convergence or divergence of the expressions

$$\sum_1^\infty \frac{1}{\beta_n^a}, \qquad \sum_1^\infty (\sqrt[n]{m_n^c})^{-a}, \qquad \sum_1^\infty \left(\frac{m_n^c}{m_{n+1}^c}\right)^{-a}, \qquad \int^\infty \frac{\ln T(r)}{r^{a+1}}\,dr$$

is also of great importance (cf. [11], p. 49 and [8]).

3.5. Let $\{\varphi_k(x)\}$ $(k = 1, 2, \ldots)$ be a given system of functions such that

$$(*)\qquad \varphi_k(x) \in C_I^\infty, \qquad \sup_I |\varphi_k^{(n)}(x)| = \sigma_{nk} < \infty \text{ for } n = 0, 1, \ldots;\ k = 1, 2, \ldots$$

Let us consider the expansions of the form

$$(**)\qquad f(x) = \sum_{n=1}^\infty a_n\varphi_n(x).$$

Clearly, if the coefficients a_n satisfy suitable conditions, then $f(x)$ belongs to a q. analytic class. Such conditions were introduced by de la Valleé Poussin [14] and Mandelbrojt [9] in case when $\{\varphi_k(x)\}$ is the trigonometric system. More general case was considered in [7]. The basic result in [7] can be generalized. Namely, the following theorem is true:

Let the conditions $(*)$ and $(**)$ be satisfied. Let us assume that there exist a constant K, a sequence $\{\gamma_n\}$, $\inf\gamma_n > 0$ and an increasing convex function $\varphi(x)$ such that $\sigma_{nk} \leqslant A^k[\gamma_n]^{\varphi(k)}$. If there exists a function $q \in N_\infty^\gamma$ $(\gamma = \inf_n \gamma_n)$ such that

$$(***)\qquad \int^\infty \{\bar q[\varphi(u)]\}^{-\varphi'(u)}du = \infty \quad \textit{and} \quad \sum_{n=1}^\infty |a_n|\,a(\gamma_n) < \infty,$$

where $a(t) = \exp\big(C + \int_\gamma^t q(u)u^{-1}du\big)$, then $f(x)$ belongs to a q. analytic class.

Proof. In the same way as in [7] we obtain

$$\sup_I |f^{(k)}(x)| \leqslant CA^k \sup_{n\geqslant 1} \exp[\varphi(k)\ln\gamma_n - \ln\alpha(\gamma_n)] = CA^k m_k.$$

By (2.5),

$$A(x) = \sup_{t\geqslant\gamma}[\varphi(x)\ln t - \ln\alpha(t)] = A(x_0) + \int_{x_0}^{x} \ln\bar{q}[\varphi(x)]d\varphi.$$

Hence, denoting $p(x) = \{\bar{q}[\varphi(x)]\}^{\varphi'(x)} \epsilon N_\infty$ and applying (2.7) we have

$$B(t) = \sup_{x\geqslant x_0}\{x\ln t - A(t)\} = B(\gamma) + \int_{\gamma}^{t} \bar{p}(u)u^{-1}du.$$

Let $T(r)$ be the Ostrowski's function of the sequence $\exp A(n)$. It follows from (2.10) and (3.2) that $\ln T(r) \geqslant B(r) - \ln r$ and the divergence of (***) implies the divergence of $\int^{\infty} \frac{\ln T(r)}{\gamma^2} dr$. Since $m_n \leqslant \exp A(n)$, it follows from Ostrowski's condition that the class $C\{m_n\}$ is q. analytic.

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