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# On the maximum correlation between functions

**1. Introduction.** Let $f$ and $g$ be non-negative functions bounded by 1 on $[0,1]$ and zero outside $[0,1]$ and having integrals $\alpha$ and $\beta$ respectively on this interval. Our principal result is a lower bound, in terms of $\alpha$ and $\beta$, for the maximum of correlation function $\int_0^1 f(x)g(x+t)\,dx$.

Our work is motivated in part by a desire to generalize a problem of J. Mycielski and to obtain an improvement of the following theorem of Mycielski announced without proof in [1]:

*Let $X$ and $Y$ be Lebesgue-measurable subsets of $[0,1]$ having measures $\alpha$ and $\beta$ respectively. Then there exists a number $t$ such that if $Y$ is translated through $t$, then the intersection of the translated set $Y_t$ with $X$ has measure at least*

$$1-(1-\alpha\beta)^{1/2} = \frac{1}{2}\alpha\beta+\frac{\alpha^2\beta^2}{8}+\ldots \tag{1.1}$$

Mycielski kindly communicated his proof to us. It involves the use of continued fractions. It will follow from our main result that (1.1) can be replaced by

$$\frac{6}{7}\left(1-(1-7\alpha\beta/6)^{1/2}\right) = \frac{1}{2}\alpha\beta+\frac{7}{6}\cdot\frac{\alpha^2\beta^2}{8}+\ldots \tag{1.2}$$

and although this estimate is not best possible, an example we give shows essentially that the coefficient of $\alpha\beta$ in (1.2) can not be increased.

**2.** Let $\mathscr{F}$ be the family of real-valued functions $f$, Lebesgue-integrable on $[0,1]$ and satisfying the conditions: $0 \leqslant f(x) \leqslant 1$ for $x \in [0,1]$, and $f(x) = 0$ for $x \notin [0,1]$, $\int_0^1 f(x)\,dx = \alpha$.

Let $\mathscr{G}$ be the family of functions satisfying the same conditions with $\beta$ replacing $\alpha$. $\alpha$ and $\beta$ are fixed, but arbitrary. Define

$$\mathscr{M}(t) = \mathscr{M}(f, g; t) = \int_0^1 f(x)g(x+t)\,dx,$$

$$M = M(f, g) = \max_t \mathscr{M}(t), \quad \text{and} \quad \lambda = \inf M,$$

where the infimum is taken over all $f \in \mathscr{F}$ and $g \in \mathscr{G}$. The problem is to find or estimate $\lambda$. The problem of finding $\lambda$ is believed to be an extremely hard one. In this paper, we give lower and upper bounds for $\lambda$ which seem to be quite efficient. For instance, for $\alpha = \beta = 0.5$, our estimates give $0.136 \leqslant \lambda \leqslant 0.166$.

If we further define $\mathscr{N}(t) = \mathscr{M}(t) + \mathscr{M}(-t)$ for $t \geqslant 0$ and $N = \sup\limits_t \mathscr{N}(t)$, it is obvious that $N \leqslant 2M$, $0 \leqslant \mathscr{N}(t) \leqslant 2$ for $t \in [0, 1]$, and $\mathscr{N}(t) = 0$ for $t > 1$. We will also require the following properties of $\mathscr{N}(t)$.

LEMMA. *$\mathscr{N}(t)$ is continuous and hence integrable on $[0, \infty)$.*

Proof. It suffices to show that $\mathscr{M}(t)$ is continuous at each $t$

$$|\mathscr{M}(t+h) - \mathscr{M}(t)| = \left| \int_0^1 f(x)[g(x+t+h) - g(x+t)]\,dx \right| \leqslant \int_0^1 |g(x+t+h) - g(x+t)|\,dx.$$

The last integral tends to zero as $h \to 0$, by the Mean-Continuity property.

LEMMA.

$$\int_0^1 \mathscr{N}(t)\,dt = \alpha\beta.$$

This can be proved by the usual method of interchanging the order of an iterated integral. The change in the order is easy to justify, e.g. by Fubini's Theorem.

LEMMA. *If $0 \leqslant t \leqslant 1$, then $\mathscr{N}(t) \leqslant 2 - 2t$.*

Proof.

$$\mathscr{N}(t) = \int_0^{1-t} f(x)g(x+t)\,dx + \int_t^1 f(x)g(x-t)\,dx \leqslant 1 - t + 1 - t.$$

The next theorem gives an estimate which corresponds to, and implies, that of Mycielski, viz. (1.1).

(2.1) THEOREM.

$$\lambda \geqslant 1-(1-\alpha\beta)^{1/2}.$$

Proof. Using the two preceding Lemmas,

$$\alpha\beta = \int_0^1 \mathcal{N}(t)\,dt \leqslant \int_0^{1-N/2} N\,dt + \int_{1-N/2}^{N} (2-2t)\,dt,$$

which yields the inequality $N^2-4N+4\alpha\beta \leqslant 0$. Solving, we obtain $M \geqslant \frac{1}{2}N \geqslant 1-(1-\alpha\beta)^{1/2}$. Since this inequality holds for all $f$ and $g$, the theorem follows.

Before obtaining improvements on the last theorem, we shall obtain an upper bound for $\lambda$, useful for small values of $\alpha$ and $\beta$.

(2.2) THEOREM. *If* $\frac{1}{2}(\alpha \wedge \beta)+(\alpha \vee \beta) \leqslant 1$, *then*

$$\lambda \leqslant \frac{\alpha\beta}{2-(\alpha \wedge \beta)},$$

*where* $\vee$ *and* $\wedge$ *denote maximum and minimum, respectively.*

Proof. Take $f$ to be the function which is 1 on $[0, \frac{1}{2}\alpha] \cup [1-\frac{1}{2}\alpha, 1]$ and zero elsewhere, and $g$ to be the function which is $\beta/(1-\frac{1}{2}\alpha)$ on $[\frac{1}{4}\alpha, 1-\frac{1}{4}\alpha]$ and zero elsewhere. Here we have supposed, without loss of generality, that $\alpha = \alpha \wedge \beta$.

Expanding the right-hand sides of (2.2) and (2.1) as power series in $\alpha\beta$, we see that the scope for improvement over (2.1) is very slight if any. In particular, we note that it is not possible to improve the leading term.

The estimate in the next theorem is useful only when $\alpha$ and $\beta$ are both 'large', i.e., sufficiently close to 1.

(2.3) THEOREM.

$$\lambda \geqslant \alpha+\beta-1.$$

Proof. Since $\big(1-f(x)\big)\big(1-g(x)\big) \geqslant 0$, we have for all $x$, $f(x)g(x) \geqslant f(x)+g(x)-1$. Hence

$$M \geqslant \mathcal{M}(0) = \int_0^1 f(x)g(x)\,dx \geqslant \int_0^1 \{f(x)+g(x)-1\}\,dx = \alpha+\beta-1.$$

(2.4) LEMMA. *If* $E$ *is any measurable subset of* $[0,1]$, *then*

$$\int_E (f+g) \leqslant \tfrac{1}{2}N + m(E)$$

*where* $m(E)$ *denotes the measure of* $E$.

Proof.

$$N \geqslant \mathcal{N}(0) = 2\int_0^1 f(x)g(x)\,dx \geqslant 2\int_E f(x)g(x)\,dx \geqslant 2\int_E \{f(x)+g(x)-1\}\,dx.$$

LEMMA.

(2.5) $$\mathcal{N}(t) \leqslant \tfrac{1}{4}N+1-t \quad \textit{if} \quad \tfrac{1}{2} \leqslant t \leqslant 1.$$

(2.6) $$\mathcal{N}(t) \leqslant \tfrac{1}{4}N+2-3t \quad \textit{if} \quad 0 \leqslant t \leqslant \tfrac{1}{2}.$$

Proof. Let $\frac{1}{2} \leqslant t \leqslant 1$. Take $E = (0, 1-t) \cup (t, 1)$. Then

$$\mathcal{N}(t) = \int_0^{1-t} f(x)g(x+t)\,dx + \int_t^1 f(x)g(x-t)\,dx \leqslant \int_E f(x)\,dx.$$

Since a similar inequality can also be proved for $g$,

$$\mathcal{N}(t) \leqslant \tfrac{1}{2}\int_E (f+g).$$

Using Lemma (2.4), (2.5) now follows. Similarly, if $0 \leqslant t \leqslant \frac{1}{2}$, we can show that

$$\mathcal{N}(t) \leqslant 2(1-2t) + \int_E f$$

where now $E = (0, t) \cup (1-t, 1)$. From this,

$$\mathcal{N}(t) \leqslant 2(1-2t) + \tfrac{1}{2}\int_E (f+g) \leqslant \tfrac{1}{4}N+2-3t.$$

We are now ready to prove the main theorem.

(2.7) THEOREM.

(i) *If* $\lambda < \frac{1}{3}$, *then*

(2.8) $$\lambda \geqslant \frac{1-\sqrt{1-c\alpha\beta}}{c} \quad \textit{where} \quad c = \frac{5}{4}.$$

(ii) *If* $\frac{1}{3} < \lambda < \frac{1}{2}$, *then*

(2.9) $$\lambda \geqslant \max\left\{\frac{1}{3}, \frac{1-\sqrt{1-d\alpha\beta}}{d}\right\} \quad \textit{where} \quad d = \frac{7}{6}.$$

(iii) *If* $\lambda > \frac{1}{2}$, *then*

$$\lambda \geqslant \max\{\tfrac{1}{2}, \alpha+\beta-1\}.$$

Proof. (i) For any $\delta < \frac{1}{3}-\lambda$, there exist $f_\delta \in \mathcal{F}$ and $g_\delta \in \mathcal{G}$ such that $M(f_\delta, g_\delta) < \lambda + \delta < \frac{1}{3}$. Throughout this proof we shall write $M_\delta$ and $N_\delta$ for $M(f_\delta, g_\delta)$ and $N(f_\delta, g_\delta)$ respectively.

Let $A_1 = (0, 1-\frac{3}{4}N_\delta)$, $A_2 = (1-\frac{3}{4}N_\delta, 1-\frac{1}{4}N_\delta)$ and $A_3 = (1-\frac{1}{4}N_\delta, 1)$. Since $N \leqslant \frac{2}{3}$, if $t \in A_2$, then $t \geqslant \frac{1}{2}$ and hence (2.5) applies. Therefore,

$$a\beta = \int_{A_1} + \int_{A_2} + \int_{A_3} \mathcal{N}(t)\,dt \leqslant N_\delta m(A_1) + \int_{A_2} (\tfrac{1}{4}N_\delta + 1 - t)\,dt + \int_{A_3} (2-2t)\,dt$$

which yields $\frac{5}{16} N_\delta^2 - N_\delta + a\beta \leqslant 0$, i.e.,

$$M_\delta \geqslant \frac{1-\sqrt{1-ca\beta}}{c} = \mu \quad \text{with} \quad c = \frac{5}{4}.$$

Hence, $\lambda > \mu - \delta$. Since $\delta$ is arbitrary, (2.8) holds.

(ii) The proof of (2.9) is similar. In this case, the integral $\int_0^1 \mathcal{N}(t)\,dt$ has to be split over four sub-intervals

$$A_1 = (0, \tfrac{2}{3} - \tfrac{1}{4}N_\delta), \quad A_2 = (\tfrac{2}{3} - \tfrac{1}{4}N_\delta, \tfrac{1}{2}), \quad A_3 = (\tfrac{1}{2}, 1 - \tfrac{1}{4}N_\delta)$$
$$\text{and} \quad A_4 = (1 - \tfrac{1}{4}N_\delta, 1)$$

and then the estimates (2.6) and (2.5) are used over $A_2$ and $A_3$ respectively.

(iii) If $\lambda > \frac{1}{2}$, then $a$ and $\beta$ are both sufficiently large and the estimate of Theorem (2.3) will suffice.

**3.** In conclusion, we observe that our problem is an extension of Mycielski's problem from characteristic tunctions to a much larger family of Lebesgue integrable functions. Also, the constant $\lambda$ in the point-set version of Mycielski is not less than the $\lambda$ defined in § 2. Hence all the lower bounds for the latter, such as Theorems 2.3 and 2.7 automatically hold in the case of Mycielski's problem.

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**Reference**

[1] S. Świerczkowski, *On the intersection of a linear set with the translation of its complement*, Colloq. Math. 5 (1958), pp. 185-197.

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