ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XI (1967)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XI (1967)

E. ŚLIWIŃSKI (Kraków)

On the Neumann problem in an n-dimensional half-space

In the present paper we give a solution of the problem of Neumann in the n-dimensional half-space. The assumptions concerning the density of $f(s)$ are more general than in the paper [1] (see [1], p. 185-192).

Let ϱ denote the distance of the points $X = (x_1, \ldots, x_n)$ and $S = (s_1, \ldots, s_{n-1})$ and let $dS = ds_1 \ldots ds_{n-1}$.

Let $f(S)$ be a continuous function defined in the whole $(n-1)$-dimensional Eucliden space E_{n-1}, $T = (t_1, \ldots, t_{n-1}, 1)$, $|T|^n = (t_1^2 + \ldots + t_{n-1}^2 + 1)^{n/2}$ and $dT = dt_1 \ldots dt_{n-1}$. We shall prove that, under some assumptions on this function, the solution of Neumann problem in half-space is given by the formula

$$(1) \qquad u(X) = \frac{1}{(2-n)\alpha_n} \iint_{E_{n-1}} \frac{f(S)\,dS}{[(x_1-s_1)^2 + \ldots + (x_{n-1}-s_{n-1})^2 + x_n^2]^{\frac{1}{2}n-1}}$$

$$= \frac{1}{(2-n)\alpha_n} \iint_{E_{n-1}} \frac{f(S)\,dS}{\varrho^{n-2}},$$

where $\alpha_n = \iint_{E_{n-1}} \frac{dT}{|T|^n}$. That means that if $x_n > 0$ then the function $u(X)$ satisfies the conditions 1° $u(X) \in C^2$, 2° $\Delta u(X) = 0$, 3° if $X \to (x_1^0, \ldots, x_{n-1}^0, 0)$ then we have $\lim u'_{x_n}(X) = f(x_1^0, \ldots, x_{n-1}^0)$.

We begin with two definitions and two lemmas. We shall consider two classes of functions, K and L.

DEFINITION 1. K is the class of functions $f(X)$ defined in E_{n-1} and satisfying the following conditions: 1° $f(X)$ is continuous in E_{n-1}, 2° there exist a positive constant r_1 and a function $\omega(r)$, where $r = (s_1^2 + \ldots + s_{n-1}^2)^{1/2} \geqslant r_1$, such that if r is outside $C(K_{r_1}) = \{S\colon s_1^2 + \ldots + s_{n-1}^2 = r_1\}$, then $|f(S)| < \omega(r)$, 3° $\omega(r)$ is a nonnegative nondecreasing function such that the integral $\int_{r_1}^{\infty} r^{-2}\omega(r)\,dr$ is finite.

DEFINITION 2. L is the class of functions defined for $S \in E_{n-1}$ and satisfying the following conditions: 1° $f(S)$ is continuous in E_{n-1}, 2° there exists a nonnegative nondecreasing, continuous function $\Omega(r)$ defined for $r \geqslant 0$ such that for every pair of points $(x_1, \ldots, x_{n-1}), (y_1, \ldots, y_{n-1}) \in E_{n-1}$ and $r = |XY|$ the inequality

$$|f(x_1, \ldots, x_{n-1}) - f(y_1, \ldots, y_{n-1})| \leqslant \Omega(r)$$

is satisfied, 3° the integral $\int_1^\infty r^{-2}\Omega(r)\,dr$ is finite, 4° there exists a continuous nonnegative function $\varphi(x)$ defined for $x \geqslant 0$ such that $\varphi(0) = 0$ and $\Omega(a \cdot b) \leqslant \varphi(a)\Omega(b)$ for $a \geqslant 0$, $b \geqslant 0$.

LEMMA 1. *If $f(S) \in K$, then the integrals*

$$J_1 = x_n^i \iint\limits_{E_{n-1}} \frac{f(S)}{\varrho^{n+2j}}\,dS, \qquad J_2 = \iint\limits_{E_{n-1}} \frac{(x_i - s_i)^j f(S)}{\varrho^{n+2j}}\,dS$$

converge uniformly in any cylinder $x_1^2 + \ldots + x_{n-1}^2 \leqslant B^2$, $\eta < x_n < A$, where η, A and B are arbitrary positive numbers, and $i = 1, \ldots, n-1$, $j = 0, 1$.

Proof. Let $r = (s_1^2 + \ldots + s_{n-1}^2)^{1/2}$. We choose two numbers r_0 and B such that $r_0 \geqslant 2B$ and the inequality

$$f(S) < \omega(r) \quad \text{if} \quad r \geqslant r_0, \tag{2}$$

is satisfied. If $r \geqslant r_0$, then we have

$$\varrho = [(x_1 - s_1)^2 + \ldots + (x_{n-1} - s_{n-1})^2 + x_n^2]^{1/2} \geqslant r - B \geqslant r - \tfrac{1}{2}r = \tfrac{1}{2}r,$$

and, in view of (2), we obtain

$$\left| x_n^i \iint\limits_{C(K_r)} \frac{f(S)}{\varrho^{n+2j}}\,dS \right| \leqslant 4C \iint\limits_{C(K_r)} \frac{\omega(r)}{r^{n+2j}}\,dS,$$

C being a positive number. Let us introduce spherical coordinates

$$s_1 = r\cos\varphi_1 \ldots \cos\varphi_{n-1}\sin\varphi_{n-1}, \quad \ldots, \quad s_{n-1} = r\sin\varphi_1 \tag{T}$$

in the right-hand integral which is transformed into the following

$$\iint\limits_{C(K_R)} \frac{\omega(r)}{r^{n+2j}}\,dS = C_1 \int_R^\infty \frac{\omega(r) r^{n-2}}{r^{n+2j}}\,dr = C_1 \int_R^\infty \frac{\omega(r)}{r^{n+2j}}\,dr,$$

C_1 being a positive constant. We have assumed that $\int_R^\infty r^{-2}\omega(r)\,dr < \infty$. Then for every $\varepsilon > 0$ there exists $r_1(\varepsilon)$ such that if $R > r_1(\varepsilon)$, then

$$\left| x_n^i \iint\limits_{C(K_{r_1})} \frac{f(S)}{\varrho^{n+2j}}\,dS \right| < \varepsilon.$$

In order to prove the uniform convergence of the integral J_2 we consider the inequality $|x_i - s_i| \leqslant r + (x_1^2 + \ldots + x_{n-1}^2)^{1/2} \leqslant r + B \leqslant \frac{1}{2}3r$. Hence if $r \geqslant r_0$ we obtain

$$|J_2'| = \left| \mathop{\mathsf{SS}}_{C(K_R)} \frac{f(S)(x_i - s_i)^j}{\varrho^{n+2j}} dS \right| \leqslant C \mathop{\mathsf{SS}}_{C(K_R)} \frac{\omega(r) r^j}{r^{n+2j}} dS$$

and, by introducing the transformation (T) we obtain finally

$$|J_2'| \leqslant C_1 R^{-j} \int_R^\infty \frac{\omega(r)}{r^2} dr < \varepsilon.$$

LEMMA 2. *If $f(S)$ is continuous in E_{n-1} and $f(S) \in K \cap L$, then the function*

$$v(X) = \frac{x_n}{a_n} \mathop{\mathsf{SS}}_{E_{n-1}} \frac{f(S)}{\varrho^n} d\tau \tag{3}$$

converges to $f(x_1^0, \ldots, x_{n-1}^0)$ when $(x_1, \ldots, x_n) \to (x_1^0, \ldots, x_{n-1}^0, 0)$.

Proof. Lemma 1 implies the continuity of $u(X)$ in the half-space $x_n > 0$. Moreover, we have

$$f(x_1^0, \ldots, x_{n-1}^0) = \frac{1}{a_n} \mathop{\mathsf{SS}}_{E_{n-1}} \frac{f(x_1^0, \ldots, x_{n-1}^0)}{|T|^n} dT.$$

We transform the integral (3) and obtain substituting $s_1 - x_1 = t_1 x_n, \ldots, s_{n-1} - x_{n-1} = t_{n-1} x_n$,

$$v(X) = \frac{1}{a_n} \mathop{\mathsf{SS}}_{E_{n-1}} \frac{f(x_1 + t_1 x_n, \ldots, x_{n-1} + t_{n-1} x_n)}{|T|^n} dT.$$

Then

$$\begin{aligned} g(X) &= v(X) - f(x_1^0, \ldots, x_{n-1}^0) \\ &= \frac{1}{a_n} \mathop{\mathsf{SS}}_{E_{n-1}} \frac{[f(x_1 + t_1 x_n, \ldots, x_{n-1} + t_{n-1} x_n) - f(x_1^0, \ldots, x_{n-1}^0)]}{|T|^n} dT \end{aligned}$$

and

$$\begin{aligned} g(X) &\leqslant \frac{1}{a_n} \mathop{\mathsf{SS}}_{E_{n-1}} \frac{\Omega\big([(x_1 - x_1^0 + t_1 x_n)^2 + \ldots + (x_{n-1} - x_{n-1}^0 + t_{n-1} x_n)^2]^{1/2}\big)}{|T|^n} dT \\ &\leqslant \frac{1}{a_n} \mathop{\mathsf{SS}}_{E_{n-1}} \frac{\Omega\big([(|x_1 - x_1^0| + |t_1| x_n)^2 + \ldots + (|x_{n-1} - x_{n-1}^0| + |t_{n-1}| x_n)^2]^{1/2}\big)}{|T|^n} dT. \end{aligned}$$

Let ε be an arbitrary positive number. If the coordinates of a point X satisfy the inequalities $|x_1-x_1^0|<\varepsilon, \ldots, |x_{n-1}-x_{n-1}^0|<\varepsilon$, $0<x_n<\varepsilon$, then we have

$$\Omega\big([(|x_1-x_1^0|+|t_1|x_n)^2+\ldots+(|x_{n-1}-x_{n-1}^0|+|t_{n-1}|x_n)^2]^{1/2}\big) \leqslant \Omega(n\varepsilon|T|)\leqslant\varphi(n\varepsilon)\,\Omega(|T|).$$

In view of the convergence of the integral

$$J=\mathop{\mathsf{SS}}_{E_{n-1}}\frac{\Omega(|T|)}{|T|^n}\,dT$$

we obtain the estimation

$$|g(X)|\leqslant\frac{\varphi(n\varepsilon)}{a_n}\mathop{\mathsf{SS}}_{E_{n-1}}\frac{\Omega(|T|)}{|T|^n}\,dT=C_1\frac{\varphi(n\varepsilon)}{a_n},$$

where C_1 is a positive constant. In fact, if we apply the transformation (T), then we have

$$J=C_2\int_0^\infty\frac{\Omega\big((r^2+1)^{1/2}\big)r^{n-2}}{(r^2+1)^{n/2}}\,dr,\qquad C_2=\text{const},$$

$$\int_R^\infty\frac{\Omega\big((r^2+1)^{1/2}\big)r^{n-2}}{(r^2+1)^{n/2}}\,dr\leqslant\int_R^\infty\frac{\Omega\big((r^2+1)^{1/2}\big)(r^2+1)^{(n-2)/2}}{(r^2+1)^{n/2}}\,dr=\int_R^\infty\frac{\Omega\big((r^2+1)^{1/2}\big)}{r^2+1}\,dr$$

$$\leqslant\varphi(2)\int_R^\infty\frac{\Omega(r)}{r^2}\,dr\qquad(R>1).$$

So we conclude that $g(X)\to 0$ if $X\to(x_1^0,\ldots,x_{n-1}^0,0)$.

THEOREM. *If $f(X)$ is continuous and belongs to $K\cap L$ for $S\in E_{n-1}$ then the function $u(X)$ defined by formula* (1) *is a solution of the problem of Neumann in the half space $x_n>0$.*

This follows from Lemma 1 (since the function ϱ^{2-n} is harmonic).

References

[1] О. Л. Соболев, *Уравнения математической физики*, Москва 1954.