On the Neumann problem in an \( n \)-dimensional half-space

In the present paper we give a solution of the problem of Neumann in the \( n \)-dimensional half-space. The assumptions concerning the density of \( f(s) \) are more general than in the paper \([1]\) (see \([1]\), p. 185-192).

Let \( q \) denote the distance of the points \( X = (x_1, \ldots, x_n) \) and \( S = (s_1, \ldots, s_{n-1}) \) and let \( dS = ds_1 \ldots ds_{n-1} \).

Let \( f(S) \) be a continuous function defined in the whole \((n-1)\)-dimensional Euclidean space \( E_{n-1}, T = (t_1, \ldots, t_{n-1}, 1), |T|^n = (t_1^2 + \ldots + t_{n-1}^2 + 1)^{n/2} \) and \( dT = dt_1 \ldots dt_{n-1} \). We shall prove that, under some assumptions on this function, the solution of Neumann problem in half-space is given by the formula

\[
(1) \quad u(X) = \frac{1}{(2-n)a_n} \iint_{E_{n-1}} f(S) dS \left( \frac{1}{2} - \frac{n-1}{2} \right)
\]

where \( a_n = \iiint_{|T|^n} dT \). That means that if \( x_n > 0 \) then the function \( u(X) \) satisfies the conditions 1° \( u(X) \in C^2, 2° \Delta u(X) = 0, 3° \) if \( X \to (x_1^0, \ldots, x_{n-1}^0, 0) \) then we have \( \lim u'(X) = f(x_1^0, \ldots, x_{n-1}^0) \).

We begin with two definitions and two lemmas. We shall consider two classes of functions, \( K \) and \( L \).

**Definition 1.** \( K \) is the class of functions \( f(X) \) defined in \( E_{n-1} \) and satisfying the following conditions: 1° \( f(X) \) is continuous in \( E_{n-1}, 2° \) there exist a positive constant \( r_1 \) and a function \( \omega(r) \), where \( r = (s_1^2 + \ldots + s_{n-1}^2)^{1/2} \geq r_1 \), such that if \( r \) is outside \( C(K_{r_1}) = \{S: s_1^2 + \ldots + s_{n-1}^2 = r_1\} \), then \( |f(S)| < \omega(r) \), 3° \( \omega(r) \) is a nonnegative nondecreasing function such that the integral \( \int_{r_1}^\infty r^{-2} \omega(r) dr \) is finite.
Definition 2. \( L \) is the class of functions defined for \( S \in E_{n-1} \) and satisfying the following conditions: 1° \( f(S) \) is continuous in \( E_{n-1} \), 2° there exists a nonnegative nondecreasing, continuous function \( \Omega(r) \) defined for \( r \geq 0 \) such that for every pair of points \( (x_1, \ldots, x_{n-1}), (y_1, \ldots, y_{n-1}) \in E_{n-1} \) and \( r = |XY| \) the inequality
\[
|f(x_1, \ldots, x_{n-1}) - f(y_1, \ldots, y_{n-1})| \leq \Omega(r)
\]
is satisfied, 3° the integral \( \int_1^\infty r^{-2} \Omega(r) \, dr \) is finite, 4° there exists a continuous nonnegative function \( \varphi(x) \) defined for \( x \geq 0 \) such that \( \varphi(0) = 0 \) and \( \Omega(a \cdot b) \leq \varphi(a) \Omega(b) \) for \( a \geq 0, b \geq 0 \).

Lemma 1. If \( f(S) \in K \), then the integrals
\[
J_1 = \int_{E_{n-1}} f(S) \, dS, \quad J_2 = \int_{E_{n-1}} \frac{(x_1 - s_1)^2 + \ldots + (x_{n-1} - s_{n-1})^2 + x_n^2}{r^{n+2j}} \, dS
\]
converge uniformly in any cylinder \( x_1^2 + \ldots + x_{n-1}^2 \leq B^2, \eta < x_n < A \), where \( \eta, A \) and \( B \) are arbitrary positive numbers, and \( i = 1, \ldots, n-1, \) \( j = 0, 1 \).

Proof. Let \( r = (s_1^2 + \ldots + s_{n-1}^2)^{1/2} \). We choose two numbers \( r_0 \) and \( B \) such that \( r_0 \geq 2B \) and the inequality
\[
f(S) < \omega(r) \quad \text{if} \quad r \geq r_0,
\]
is satisfied. If \( r \geq r_0 \), then we have
\[
\rho = [(x_1 - s_1)^2 + \ldots + (x_{n-1} - s_{n-1})^2 + x_n^2]^{1/2} \geq r - B \geq r - \frac{1}{2}r = \frac{1}{2}r,
\]
and, in view of (2), we obtain
\[
\int_{E_{n-1}} f(S) \, dS \leq 4C \int_{E_{n-1}} \omega(r) \, dS,
\]
where \( C \) is a positive number. Let us introduce spherical coordinates
\[
(T) \quad s_1 = r \cos \varphi_1 \ldots \cos \varphi_{n-1} \sin \varphi_{n-1}, \ldots, \quad s_{n-1} = r \sin \varphi_1
\]
in the right-hand integral which is transformed into the following
\[
\int_{E_{n-1}} \rho^{-2} \omega(r) \, dS = C_1 \int_0^\infty \omega(r) r^{n-2} \, dr = C_1 \int_0^\infty \frac{\omega(r)}{r^{n+2j}} \, dr,
\]
where \( C_1 \) is a positive constant. We have assumed that \( \int_0^\infty \frac{\omega(r)}{r^{n+2j}} \, dr < \infty \).

Then for every \( \varepsilon > 0 \) there exists \( r_1(\varepsilon) \) such that if \( R > r_1(\varepsilon) \), then
\[
\int_{E_{n-1}} f(S) \, dS < \varepsilon.
\]
In order to prove the uniform convergence of the integral $J_2$ we consider the inequality 

$$|x_i - s_i| \leq r + (x_1^2 + \ldots + x_{n-1}^2)^{1/2} \leq r + B \leq \frac{1}{3}r.$$ 

Hence if $r \geq r_0$ we obtain

$$|J_2| = \left| \sum_{S \in C(K)} f(S) \frac{|x_i - s_i|^i}{\frac{1}{2} + j} dS \right| \leq C \sum_{S \in C(K)} \frac{\omega(r)^i}{r^{n+2j}} dS$$

and, by introducing the transformation (T) we obtain finally

$$|J_2| \leq C_1 R^{-j} \int \frac{\omega(r)}{r^2} dr < \varepsilon.$$

**Lemma 2.** If $f(S)$ is continuous in $E_{n-1}$ and $f(S) \in K \subset L$, then the function

$$(3) \quad v(X) = \frac{x_n}{a_n} \sum_{S \in E_{n-1}} f(S) \frac{|x_i - s_i|^i}{\frac{1}{2} + j} dS$$

converges to $f(x_1^0, \ldots, x_{n-1}^0)$ when $(x_1, \ldots, x_n) \rightarrow (x_1^0, \ldots, x_{n-1}^0, 0)$.

**Proof.** Lemma 1 implies the continuity of $u(X)$ in the half-space $x_n > 0$. Moreover, we have

$$f(x_1^0, \ldots, x_{n-1}^0) = \frac{1}{a_n} \sum_{S \in E_{n-1}} f(x_1^0, \ldots, x_{n-1}^0) dT.$$ 

We transform the integral (3) and obtain substituting $s_1 - x_1 = t_1 x_n, \ldots, s_{n-1} - x_{n-1} = t_{n-1} x_n$,

$$v(X) = \frac{1}{a_n} \sum_{S \in E_{n-1}} f(x_1 + t_1 x_n, \ldots, x_{n-1} + t_{n-1} x_n) \frac{|x_i - s_i|^i}{|T|^n} dT.$$ 

Then

$$g(X) = v(X) - f(x_1^0, \ldots, x_{n-1}^0)$$

$$= \frac{1}{a_n} \sum_{S \in E_{n-1}} \frac{[f(x_1 + t_1 x_n, \ldots, x_{n-1} + t_{n-1} x_n) - f(x_1^0, \ldots, x_{n-1}^0)]}{|T|^n} dT$$

and

$$g(X) \leq \frac{1}{a_n} \sum_{S \in E_{n-1}} \frac{\Omega\left(\sum_{i=1}^{n-1} (|x_i - x_i^0 + t_i x_n|^2)^{1/2}\right)}{|T|^n} dT$$

$$\leq \frac{1}{a_n} \sum_{S \in E_{n-1}} \frac{\Omega\left(\sum_{i=1}^{n-1} (|x_i - x_i^0| + |t_i x_n|^2)^{1/2}\right)}{|T|^n} dT.$$
Let $\varepsilon$ be an arbitrary positive number. If the coordinates of a point $X$ satisfy the inequalities $|x_1-x_1^0| < \varepsilon, \ldots, |x_{n-1}-x_{n-1}^0| < \varepsilon, \ 0 < x_n < \varepsilon,$ then we have

$$\Omega \left\{ \left( |x_1-x_1^0| + |t_1|x_n \right)^2 + \ldots + \left( |x_{n-1}-x_{n-1}^0| + |t_{n-1}|x_n \right)^2 \right\}^{1/2} \leq \Omega (n \varepsilon |T|) \leq \varphi (n \varepsilon) \Omega (|T|).$$

In view of the convergence of the integral

$$J = \sum_{n=1}^{\infty} \frac{\Omega (|T|)}{|T|^n} \ dT$$

we obtain the estimation

$$|g(X)| \leq \frac{\varphi (n \varepsilon)}{a_n} \sum_{n=1}^{\infty} \frac{\Omega (|T|)}{|T|^n} = C_1 \frac{\varphi (n \varepsilon)}{a_n},$$

where $C_1$ is a positive constant. In fact, if we apply the transformation $(T)$, then we have

$$J = C_2 \int_0^\infty \frac{\Omega \left( (r^2+1)^{1/2} \right) r^{n-2}}{(r^2+1)^{n/2}} \ dr, \quad C_2 = \text{const},$$

$$\int_0^\infty \frac{\Omega \left( (r^2+1)^{1/2} \right) r^{n-2}}{(r^2+1)^{n/2}} \ dr \leq \int_0^\infty \frac{\Omega \left( (r^2+1)^{1/2} \right) (r^2+1)^{(n-2)/2}}{(r^2+1)^{n/2}} \ dr = \int_0^\infty \frac{\Omega \left( (r^2+1)^{1/2} \right)}{r^2+1} \ dr$$

$$\leq \varphi (2) \int_0^\infty \frac{\Omega (r)}{r^2} \ dr \quad (R > 1).$$

So we conclude that $g(X) \to 0$ if $X \to (x_1^0, \ldots, x_{n-1}^0, 0)$.

**Theorem.** If $f(X)$ is continuous and belongs to $K \subset L$ for $S \varepsilon E_{n-1}$ then the function $u(X)$ defined by formula (1) is a solution of the problem of Neumann in the half space $x_n > 0$.

This follows from Lemma 1 (since the function $\varphi^{2-n}$ is harmonic).

**References**