ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XI (1967)
ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XI (1967)

H. Kołakowski (Warszawa)

Qualitative study of a system of three polynomial differential equations

1. Introduction. In the present paper we shall study the following system of three nonlinear ordinary differential equations:

$$(1)\qquad \frac{dx_s}{dt} = \sum_{i+j+k=l}^{n} a_s^{ijk} x_1^i x_2^j x_3^k = P_s(x) = P_s^l(x) + P_s^{l+1}(x) + \ldots + P_s^n(x), \qquad s = 1, 2, 3,$$

where $a_s^{ijk} = \text{const}$, $x = (x_1, x_2, x_3)$, l is a fixed integer such that $2 \leqslant l \leqslant n$, and $P_s^m(x)$ $(m = l, l+1, \ldots, n)$ are homogeneous polynomials of order m. We shall study the projections of integrals of these equations from the space x, t on the space x, or the so-called trajectories of the system (1). We shall assume that the point $x = 0$ is an isolated singular point, that is, there exists a neighborhood U of the origin such that for every $0 \neq x \in U$ we have $P_s(x) \neq 0$, $P_s(0) = 0$ $(s = 1, 2, 3)$. We shall discuss in detail two methods of studying system (1): one method is based on the study of the behavior of the trajectory as $\varrho(x, 0) \to \infty$ $(\varrho(x, 0) = \sqrt{x_1^2 + x_2^2 + x_3^2})$ for the purpose of the study of point $x = 0$, the second method is the "splitting" method. Quite frequently these two methods enable us to learn about the behavior of the trajectory in the whole space x. The detailed description of both methods for the two-dimensional case $(x = (x_1, x_2))$ can be found in paper [2]. We shall try to generalize some of the results of paper [2] and give several examples.

2. The study of the behavior of trajectories for $\varrho(x, 0) \to \infty$. Homogeneous equations. At first we shall transform system (1) with several successive transformations T_1, T_2, T_3. The first transformation will be a linear orthogonal transformation T_1 of the euclidean space $\dot{E}^3$. The transformation T_2 will be defined by the formulas (see [3]):

$$(2)\qquad y_1 = \frac{1}{u_1}, \qquad y_2 = \frac{u_2}{u_1}, \qquad y_3 = \frac{u_3}{u_1}.$$

The transformation T_3 corresponds to introducing new time coordinate: $t_1 = T_3(t)$. The aim of this transformation is to obtain equations with right-hand sides being polynomials in u_1, u_2, u_3. In our case T_3 can be defined as $dt_1 = u^{1-n}dt$ (for the sake of convenience in the sequel we shall write t instead of t_1). Clearly, for even n, the motion along the trajectory for $u_1 < 0$ changes its direction. To avoid new notations we shall assume that the transformation T_1 leads to equations (1) with x changed to y. Applying transformation T_3 (T_2) to such equations we get for $u \neq 0$ the following system

$$
(3)\qquad
\begin{aligned}
\frac{du_1}{dt} &= -\sum_{i+j+k=l}^{n} a_1^{ijk} u_1^{n-1-i-j-k} u_2^j u_3^k,\\
\frac{du_2}{dt} &= -\sum_{i+j+k=l}^{n} a_1^{ijk} u_1^{n-i-j-k} u_2^{j+1} u_3^k + \sum_{i+j+k=l}^{n} a_2^{ijk} u_1^{n-i-j-k} u_2^j u_3^k,\\
\frac{du_3}{dt} &= -\sum_{i+j+k=l}^{n} a_1^{ijk} u_1^{n-i-j-k} u_2^j u_3^k + \sum_{i+j+k=l}^{n} a_3^{ijk} u_1^{n-i-j-k} u_2^j u_3^k.
\end{aligned}
$$

We make the additional convention that the above formulas hold also for $u_1 = 0$. It can be seen easily that the plane $u_1 = 0$ is an integral surface. Application of various transformations of the type T_1, transformation T_3 (T_2) and extending of definition of system (3) for $u_1 = 0$ allows us to treat system (1) as a system of equations in projective space. Thus, for instance, if T_1 is the identical transformation, then the extension of definition of system (3) for $u_1 = 0$ corresponds to a continuous extension of the definition of (1) to all points of the plane at infinity except for the points of the line at infinity of the plane $x_1 = 0$. In practice, in order to extend the definition of (1) to all points of the plane at infinity it is enough to apply only three transformations of the type T_1, the first being the identity, and the remaining two consisting of the change of coordinate axes.

DEFINITION. The singular points of system (3) will be called *generalized singular points of system* (1).

Due to the fact that the transformations T_i ($i = 1, 2, 3$) do not change the topological structure of the partition of space E^3 into trajectories we have

THEOREM 1. *Every singular point of system* (1) *is a generalized singular point of this system.*

The generalized singular points with coordinates $u = (0, \alpha, \beta) \overset{\mathrm{df}}{=} (\alpha, \beta)$ correspond to the singularities of the behavior of trajectories for $\varrho(x, 0) \to \infty$.

In the sequel, in order to obtain clearer geometrical illustration of trajectories of system (1) we shall also apply transformation T_4:

$$\bar{x} = \frac{x}{\sqrt{1+\varrho^2(x, 0)}}$$

which transforms the space E^3 into the unit ball. In view of Theorem 1 the singular points will be transformed into points from the interior of the unit ball, and the generalized singular points with coordinates $u = (\alpha, \beta)$ will be transformed into the points of the unit sphere; to each generalized singular point there correspond two points of this sphere.

Let the point $u_0 = (\alpha, \beta)$ be a singular point of the system (3). Introducing the variable $\bar{u}$:

$$u = \bar{u}+u_0$$

we obtain the equations

$$\begin{aligned}
\frac{d\bar{u}_1}{dt} &= -\sum_{i+j+k=l}^{n} a_1^{ijk}\bar{u}_1^{n+1-i-j-k}(\bar{u}_2+\alpha)^j(\bar{u}_3+\beta)^k,\\
\frac{d\bar{u}_2}{dt} &= -\sum_{i+j+k=l}^{n} a_1^{ijk}\bar{u}_1^{n-i-j-k}(\bar{u}_2+\alpha)^{j+1}(\bar{u}_3+\beta)^k+\\
&\quad+\sum_{i+j+k=l}^{n} a_2^{ijk}\bar{u}_1^{n-i-j-k}(\bar{u}_2+\alpha)^j(\bar{u}_3+\beta)^k,\\
\frac{d\bar{u}_3}{dt} &= -\sum_{i+j+k=l}^{n} a_1^{ijk}\bar{u}_1^{n-i-j-k}(\bar{u}_2+\alpha)^j(\bar{u}_3+\beta)^k+\\
&\quad+\sum_{i+j+k=l}^{n} a_3^{ijk}\bar{u}_1^{n-i-j-k}(\bar{u}_2+\alpha)^j(\bar{u}_3+\beta)^k,
\end{aligned} \tag{4}$$

where the free terms

$$A(\alpha, \beta) \overset{\mathrm{df}}{=} P_2^n(1, \alpha, \beta)-\alpha P_1^n(1, \alpha, \beta), \qquad B(\alpha, \beta) \overset{\mathrm{df}}{=} P_3^n(1, \alpha, \beta)-\beta P_1^n(1, \alpha, \beta)$$

are the polynomials in α and β of the order at most $n+1$.

Clearly, the real roots of the system of equations

$$\begin{aligned}
A(\alpha, \beta) &= 0,\\
B(\alpha, \beta) &= 0
\end{aligned} \tag{5}$$

are the coordinates of the singular points.

The roots of the characteristic equation of system (4) can be written in the following way

$$\lambda_1 = -P_1^n(1, \alpha, \beta),$$
$$\lambda_q = \frac{A'_\alpha + B'_\beta}{2} + (-1)^q \sqrt{\left(\frac{A'_\alpha + B'_\beta}{2}\right)^2 + \begin{vmatrix} A'_\alpha & A'_\beta \\ B'_\alpha & B'_\beta \end{vmatrix}}, \quad q = 2, 3.$$

Remark. As we have already pointed out, for practical purposes it is sufficient to apply only three transformations of type T_1, each of them consisting of the change of the coordinate axes. Thus, for the determining of the coordinates of all singular points which possess corresponding points on the unit sphere in the space x, it is enough to consider only three systems of equations

$$A_s(\alpha_s, \beta_s) = 0,$$
$$B_s(\alpha_s, \beta_s) = 0, \quad s = 1, 2, 3,$$

where

$$A_1(\alpha_1, \beta_1) = A(\alpha, \beta),$$
$$B_1(\alpha_1, \beta_1) = B(\alpha, \beta);$$

$$A_2(\alpha_2, \beta_2) = P_3^n(\beta_2, 1, \alpha_2) - \alpha_2 P_2^n(\beta_2, 1, \alpha_2),$$
$$B_2(\alpha_2, \beta_2) = P_1^n(\beta_2, 1, \alpha_2) - \beta_2 P_2^n(\beta_2, 1, \alpha_2);$$
$$A_3(\alpha_3, \beta_3) = P_1^n(\alpha_3, \beta_3, 1) - \alpha_3 P_3^n(\alpha_3, \beta_3, 1),$$
$$B_3(\alpha_3, \beta_3) = P_2^n(\alpha_3, \beta_3, 1) - \beta_3 P_3^n(\alpha_3, \beta_3, 1).$$

The polynomials A_s, B_s ($s = 1, 2, 3$) will be called *characteristics of infinities* (we preserve the terminology of [2]).

It is well-known that to each system (1) there corresponds a partition of the space x, or — which amounts to the same if we consider transformation T_4 — a partition of the open unit ball into trajectories. Moreover, due to the extension of the definition of system (1) for points at infinity, to each system (1) there corresponds a partition into trajectories of the closed unit ball. The unit ball filled with trajectories corresponding to system (1) will be called the *Poincaré image of system* (1).

Let us consider an isolated singular point (α_0, β_0) such that $\lambda_1(\alpha_0, \beta_0) \neq 0$. Besides, let us assume that

1) if $\operatorname{im}\lambda_2 = -\operatorname{im}\lambda_3 \neq 0$, then $\operatorname{re}\lambda_2 = \operatorname{re}\lambda_3 \neq 0$,

2) if $\operatorname{im}\lambda_2 = \operatorname{im}\lambda_3 = 0$, then $\lambda_2 \neq 0$, or $\lambda_3 \neq 0$.

Thus, the point (α_0, β_0) can be (see [4]) a node, focus, saddle, saddle-focus, saddle-node, "saddle-focus"-focus, or a singular point of type "C".

The behavior of trajectories in a sufficiently small neighborhood of each of the above-mentioned singular point is, by definition, the following (see [3], [4]):

Node. All trajectories enter the point (α_0, β_0) as $t \to +\infty$ $(t \to -\infty)$ with well-defined tangents.

Focus. All trajectories enter the point (α_0, β_0) as $t \to +\infty$ $(t \to -\infty)$. Two of them enter with a well-defined common tangent, and all other trajectories are spiral lines.

Saddle. A certain surface, called the separating surface, crosses the point (α_0, β_0). All trajectories situated on this surface enter the point (α_0, β_0) as $t \to +\infty$ $(t \to -\infty)$ with well-defined tangents. Two trajectories situated on two opposite sides of this surface (the so called separatrices) enter the point (α_0, β_0) as $t \to -\infty$ $(t \to +\infty)$ with the common tangent, and all the remaining trajectories do not enter the point (α_0, β_0) at all.

Saddle-focus. A certain surface, called the separating surface, crosses the point (α_0, β_0). All trajectories situated on this surface enter the point (α_0, β_0) for $t \to +\infty$ $(t \to -\infty)$ as spiral lines. Two trajectories, situated on two opposite sides of this surface (the so called separatrices) enter the point (α_0, β_0) as $t \to -\infty$ $(t \to +\infty)$ with a well-defined common tangent, while the remaining trajectories lie at a positive distant from the point (α_0, β_0).

Saddle-node. An integral surface F crosses the point (α_0, β_0). The surface F has the property that all trajectories situated on it enter the point (α_0, β_0) as $t \to +\infty$ $(t \to -\infty)$. All trajectories on one side of F enter (α_0, β_0) as $t \to +\infty$ $(t \to -\infty)$. One trajectory on the other side of F enters the point (α_0, β_0) as $t \to -\infty$ $(t \to +\infty)$ while all the remaining trajectories lie at a positive distance from (α_0, β_0). In other words, on one side of F we have a node domain, while on the other side we have a saddle domain.

"Saddle-focus"-focus. An integral surface crosses the point (α_0, β_0). This surface is such that all trajectories on this surface are spiral lines, while on one side of this surface we have a node domain, and on the other side we have a saddle domain.

Singular point of the type "C". There are two integral half-surfaces such that the trajectories lying on one of them enter the point (α_0, β_0) as $t \to +\infty$ while the trajectories on the other half-surface enter (α_0, β_0) for $t \to -\infty$. All remaining trajectories lie at a finite distance from (α_0, β_0).

DEFINITION. A node, focus, saddle, saddle-focus, saddle-node, "saddle-focus"-focus, or a singular point of the type "C" with coordinates (α_0, β_0) will be called A^+-*limiting* if a point moving along a trajectory γ in the Poincaré image approaches the point (α_0, β_0) as $t \to +\infty$. If it approaches (α_0, β_0) as $t \to -\infty$, the point (α_0, β_0) will be called A^--*limiting*.

Naturally, such a trajectory γ always exists.

THEOREM 2. *A singular point* (α_0, β_0) *is* A^+*-limiting* (A^-*-limiting*) *if the surface* $P_1^n(1, \alpha, \beta)$ *lies above* (*below*) *the plane* α, β *in the neighborhood of the point* (α_0, β_0). (See fig. 1.)

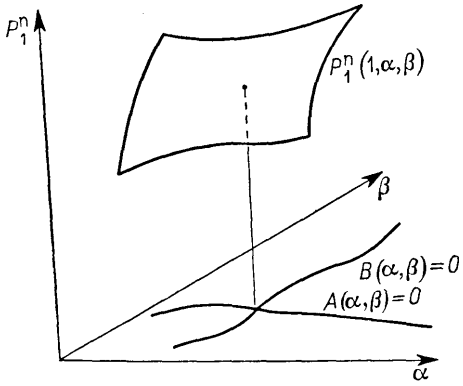

Fig. 1

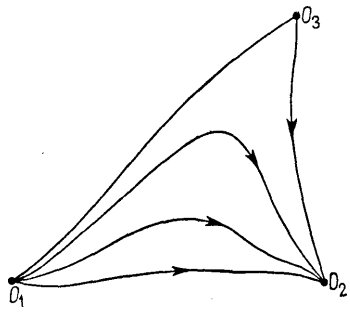

Fig. 2

Proof. We notice that the right-hand side of the first equation (4) starts with the term $\lambda_1 \bar{u}$; all other terms are of the higher order. Thus, for sufficiently small $\bar{u}$ we have

$$\operatorname{sgn} \lambda_1 = \operatorname{sgn} \frac{d\bar{u}}{dt},$$

therefore for $\lambda_1 > 0$ the curve γ enters (from the side $u > 0$) the point (α_0, β_0) as $t \to -\infty$, while for $\lambda_1 < 0$ it enters this point as $t \to +\infty$, which proves Theorem 2.

We shall study system of differential equations of the type

$$\frac{dx_s}{dt} = P_s^n(x), \quad s = 1, 2, 3, \quad x = (x_1, x_2, x_3) \tag{6}$$

for which the point $x = 0$ is, by definition, the only singular point. Obviously the rays

$$\frac{x_2}{x_1} = \alpha_0, \quad \frac{x_3}{x_1} = \beta_0, \quad x \neq 0 \tag{7}$$

are the solutions of this system.

Let us consider a surface of Poincaré image of system (6) which can be split into qualitatively identical cells, each of them filled with trajectories bounded by the sides of the curvilinear triangle $O_1 O_2 O_3$ that is trajectories joining the singular points — vertices of this triangle (fig. 2).

Such a partition can be performed, for instance if (see [1]):

1) For the generalized singular point (α_0, β_0) we have λ_2 and λ_3 real and non-vanishing.

2) The surface of Poincaré image does not contain any limiting cycles.

Let D denote the tetrahedron with base $O_1 O_2 O_3$ (fig. 2) and the vertex at the point $x = 0$, and let O_p, O_q be two of the vertices O_1, O_2, O_3. We assume that each edge of the tetrahedron D is the image of one of the rays (7), and each of its sides is the image of an integral surface of system (6).

DEFINITION. The interior of tetrahedron D filled with trajectories will be called an *elliptical region* if all trajectories from interior of D enter the point $x = 0$ for $t \to +\infty$ and for $t \to -\infty$; it will be called a *hyperbolical region* if all trajectories from the interior of D enter the point O_p as $t \to +\infty$ $(t \to -\infty)$ and enter the point O_q as $t \to -\infty$ $(t \to +\infty)$. Finally, it will be called a *parabolical region* if all trajectories from the interior of D enter the point $x = 0$ as $t \to +\infty$ $(t \to -\infty)$ and enter the point O_p as $t \to -\infty$ $(t \to +\infty)$.

THEOREM 3. *If the points O_1, O_2, O_3 are saddle points, then the interior of the tetrahedron D is an elliptical region. If the point O_1 is a saddle and points O_2, O_3 are nodes, then the interior of D is a hyperbolical region. Finally, if O_1, O_2 are saddle points and O_3 is a node then the interior of D is a parabolical region.*

The proof follows directly from the analogous theorem for two-dimensional case (see [2], Theorem 2) and from invariance of homogeneous equations under homothetic with center at the origin.

In our case we use only the topological structure of the partition into trajectories of the neighborhood of the cell $O_1 O_2 O_3$. Thus, if the replacing of points O_1, O_2, O_3 by any three of nodes, saddles, saddle-nodes, or singular points of type "C" does not lead to the change in topological structure of the partition of the neighborhood of the cell $O_1 O_2 O_3$ into trajectories, then the partition of the tetrahedron D will remain the same.

3. Applications. We shall now consider several examples.

EXAMPLE 1. Let us build the Poincaré image for following system

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1^3 + kx_1^2 x_2, \\ \frac{dx_2}{dt} &= bx_2^3 + kx_2^2 x_1, \\ \frac{dx_3}{dt} &= cx_3^3, \end{aligned} \tag{8}$$

where $a < 0$, $b > 0$, $c < 0$.

The characteristics of infinities are:

$$\begin{aligned}
A_1(\alpha_1, \beta_1) &= b\alpha_1^3 - a\alpha_1, \\
B_1(\alpha_1, \beta_1) &= c\beta_1^3 - a\beta_1 - k\alpha_1\beta_1; \\
A_2(\alpha_2, \beta_2) &= c\alpha_2^3 - b\alpha_2 - k\alpha_2\beta_2, \\
B_2(\alpha_2, \beta_2) &= a\beta_2^3 - b\beta_2; \\
A_3(\alpha_3, \beta_3) &= a\alpha_3^3 + k\alpha_3^2\beta_3 - c\alpha_3, \\
B_3(\alpha_3, \beta_3) &= b\beta_3^3 + k\alpha_3\beta_3^2 - c\beta_3.
\end{aligned}$$

The coordinates of all generalized singular points which have the corresponding points in the sphere of Poincaré image are given by real solutions of the system

$$(9) \quad \begin{cases} b\alpha_1^3 - a\alpha_1 = 0, \\ c\beta_1^3 - a\beta_1 - k\alpha_1\beta_1 = 0; \end{cases} \quad \begin{cases} c\alpha_2^3 - b\alpha_2 - k\alpha_2\beta_2 = 0, \\ \beta_2 = 0; \end{cases} \quad \begin{cases} \alpha_3 = 0, \\ \beta_3 = 0. \end{cases}$$

The table below gives the values of roots of particular equations (9) together with the roots of the corresponding characteristic equations.

TABLE 1

Roots of equations (9)	$(\alpha_1, \beta_1) =$			$(\alpha_2, \beta_2) =$	$(\alpha_3, \beta_3) =$
	$(0, 0)$	$(0, \sqrt{a/c})$	$(0, -\sqrt{a/c})$	$(0, 0)$	$(0, 0)$
λ_1	$-a$	$-a$	$-a$	$-b$	$-c$
λ_2	$-a$	$-a$	$-a$	$-b$	$-c$
λ_3	$-a$	$2a$	$2a$	$-b$	$-c$

The following surfaces: the integral surface $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, the separating surface of the saddle $(\alpha_1, \beta_1) = (0, \sqrt{a/c})$ which crosses "end-points" of the lines

$$\begin{cases} x_1 = 0, \\ x_3 = 0, \end{cases} \qquad \begin{cases} x_2 = 0, \\ x_3 = \sqrt{a/c}\, x_1 \end{cases}$$

and the separating surface of the saddle $(\alpha_1, \beta_1) = (0, -\sqrt{a/c})$ which crosses "end-points" of the lines

$$\begin{cases} x_1 = 0, \\ x_3 = 0, \end{cases} \qquad \begin{cases} x_2 = 0, \\ x_3 = -\sqrt{a/c}\, x_1 \end{cases}$$

split the space x into 16 parts. To each of these parts there corresponds (inside the unit ball) a tetrahedron with two vertices in nodes, one vertex in a saddle and one at the point $x = 0$. Thus, by Theorem 3 we have a hyperbolical region, therefore the point $x = 0$ represents a saddle point

The separatrices of the saddle $x = 0$ are the rays of the x_2-axis, for they enter $x = 0$ as $t \to -\infty$. The image of the integral surface $x_2 = 0$ is the following: The "end-points" of the axis x_3 and x_1 are plane A^--nodes (that is A^--limiting nodes), while the "end-points" of lines

$$x_3 = \sqrt{a/c}\,x_1, \qquad x_3 = -\sqrt{a/c}\,x_1$$

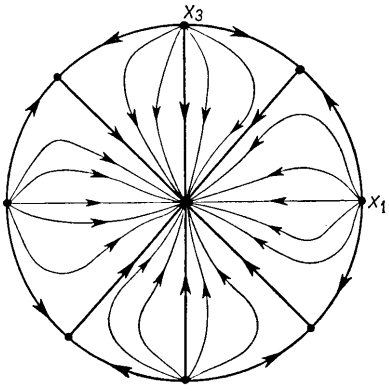

Fig. 3

are plane A^--saddles. Thus the point $(0, 0)$ of the plane $x_2 = 0$ is a node (figs. 3, 4).

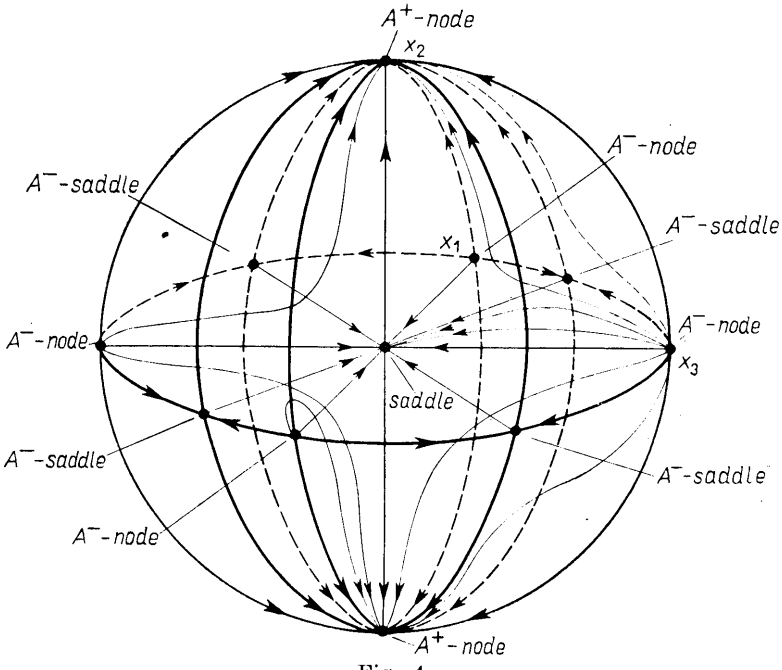

Fig. 4

EXAMPLE 2. Let as consider the system

$$
(10)\qquad
\begin{aligned}
\frac{dx_1}{dt} &= ax_1^3-bx_1x_3^2,\\
\frac{dx_2}{dt} &= -ax_1^2x_2+bx_2x_3^2,\\
\frac{dx_3}{dt} &= cx_2^3+bx_3^3,
\end{aligned}
$$

where $a<0$, $b>0$, $c>0$.

The characteristics of infinities are

$$
\begin{aligned}
A_1(\alpha_1,\beta_1) &= -2\alpha_1(a-b\beta_1^2),\\
B_1(\alpha_1,\beta_1) &= c\alpha_1^3+2b\beta_1^3-a\beta_1;\\
A_2(\alpha_2,\beta_2) &= c+a\alpha_2\beta_2^2,\\
B_2(\alpha_2,\beta_2) &= 2(a\beta_2^3-b\alpha_2^2\beta_2);\\
A_3(\alpha_3,\beta_3) &= a\alpha_3^3-2b\alpha_3-c\alpha_3\beta_3^3,\\
B_3(\alpha_3,\beta_3) &= -a\alpha_3^2\beta_3-c\beta_3.
\end{aligned}
$$

The coordinates of all generalized singular points which have corresponding points in the sphere of Poincaré image are given by real solutions of the systems

$$
(11)\qquad
\begin{cases}-2\alpha_1(a-b\beta_1^2)=0,\\ c\alpha_1^3+2b\beta_1^3-a\beta_1=0;\end{cases}
\qquad
\begin{cases}c=0,\\ \beta_2=0;\end{cases}
\qquad
\begin{cases}\alpha_3=0,\\ \beta_3=0.\end{cases}
$$

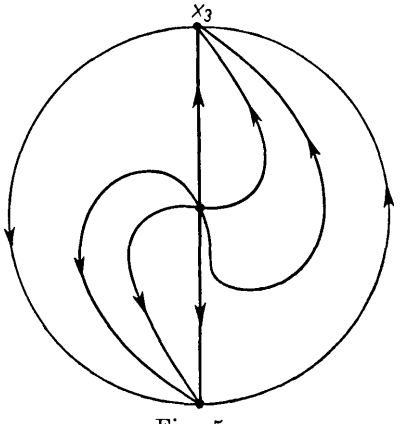

Fig. 5

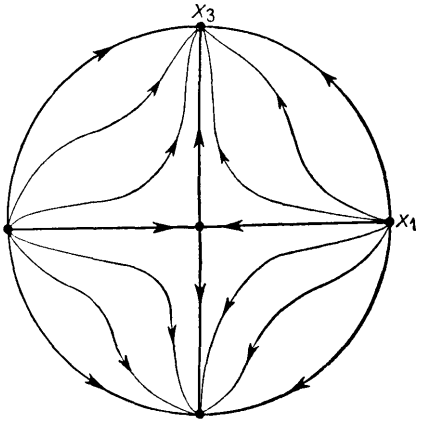

Fig. 6

Using the criteria which allow us to determine the type of singular point (see [4]) we easily obtain that the point $(\alpha_3,\beta_3)=(0,0)$ is a saddle-node. The point (α_1,β_1) is a node. The figures below illustrate the images of integral surfaces $x_1=0$ (fig. 5) and $x_2=0$ (fig. 6).

The corresponding table is

TABLE 2

Solutions of systems (11)	$(\alpha_1, \beta_1) =$	$(\alpha_3, \beta_3) =$
	$(0, 0)$	$(0, 0)$
λ_1	$-a$	$-b$
λ_2	$-2a$	$-2b$
λ_3	$-a$	0

The planes $x_1 = 0$ and $x_2 = 0$ divide the space x into four parts. In the unit ball to each of these parts there correspond a tetrahedron with vertices in one node, two saddle-nodes and at $x = 0$. The trajectories filling the interior of an arbitrary tetrahedron are such as if vertices of this tetrahedron lying on the surface of unit ball were two nodes and one saddle. In view of Theorem 3 we see that the point $x = 0$ is simply a saddle (fig. 7).

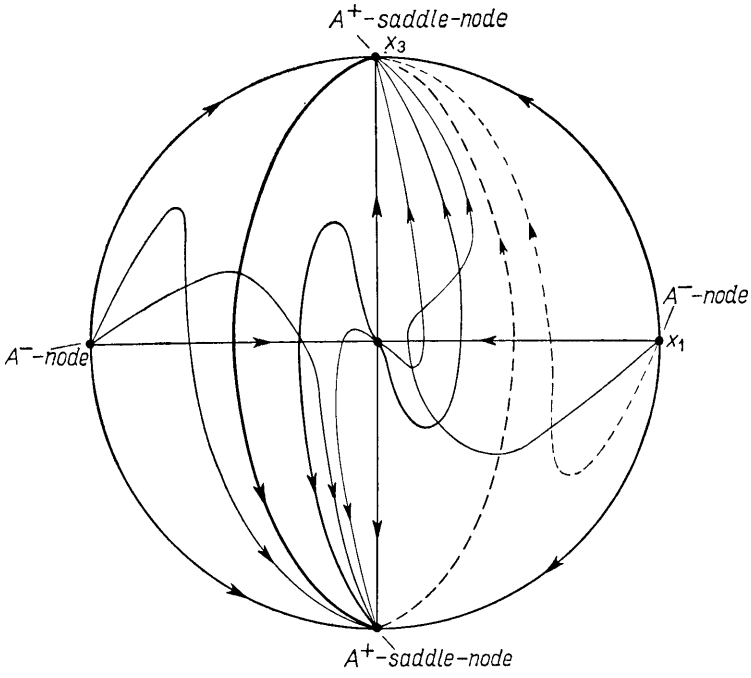

Fig. 7

The separating surface of this saddle is the plane $x_2 = 0$ (fig. 5) and the separatrices are the rays of the x_1-axis. The separatrices enter the point $x = 0$ as $t \to -\infty$ (fig. 6).

4. The method of "splitting". Let the point $x = 0$ be an isolated singular point of the system (1). To study this point we may proceed as follows: By applying the inversion S_0:

$$x = \frac{u}{\varrho^2(0, u)}$$

we transform the neighborhood of the point $x = 0$ into the neighborhood of the plane at infinity. As in section 2, it will be convenient to introduce the time coordinate $t_1 = S_1(t)$ defined by the formula

$$dt_1 = [\varrho^2(0, u)]^{-n} dt$$

(for the sake of convenience we shall write t instead of t_1). The next step will consist of the study of topological structure of the partition into trajectories of the surface and its neighborhood of the Poincaré image of the system obtained from (1) after applying transformation $S_1(S_0)$. The third step will consist of symmetric reflection of the neighborhood of the above mentioned Poincaré image with respect to its surface. Then the limiting passage of this surface to $\bar{x} = 0$ will give us the qualitative picture of the trajectories of system (1) in the neighborhood of point $x = 0$. The method described above is called "splitting" method (see [2]). The advantage of this method lies in the fact that it allows us to replace the study of generally complicated singular point $x = 0$ by the study of simpler generalized singular points determined by directions of well-defined tangents of trajectories entering the point $x = 0$.

Thus, let us consider the system (1). The transformation $S_1(S_0)$ yields:

$$\begin{aligned} \frac{du_1}{dt} &= (u_2^2+u_3^2-u_1^2)Q_1-2u_1u_2Q_2-2u_1u_3Q_3, \\ \frac{du_2}{dt} &= -2u_1u_2Q_1+(u_1^2+u_3^2-u_2^2)Q_2-2u_2u_3Q_3, \\ \frac{du_3}{dt} &= -2u_1u_3Q_1-2u_2u_3Q_2+(u_1^2+u_2^2-u_3^2)Q_3, \end{aligned} \tag{12}$$

where

$$Q_s = P_s^l(u)(u_1^2+u_2^2+u_3^2)^{n-l}+P_s^{l+1}(u)(u_1^2+u_2^2+u_3^2)^{n-l-1}+\ldots+P_s^n(u), \qquad s = 1, 2, 3.$$

Let

$$A_s^*(\alpha_s, \beta_s), \qquad B_s^*(\alpha_s, \beta_s), \qquad (s = 1, 2, 3)$$

be the characteristics of infinities of the system (12) and let $A_s(\alpha_s, \beta_s)$, $B_s(\alpha_s, \beta_s)$ $(s = 1, 2, 3)$ be the characteristics of the infinities of the system

$$\frac{dx_s}{dt} = P_s^l(x) \qquad (s = 1, 2, 3). \tag{13}$$

Let us denote by λ_s^* (λ_s) ($s = 1, 2, 3$) the roots of characteristic equations of system (12) ((13)) with respect to those generalized points O_m^* (O_m) of the system (12) ((13)) which have corresponding points on the surface of Poincaré image of this system.

DEFINITION. The characteristic equations of the system (12) with respect to the singular points O_m^* will be called the *characteristic equations of* (1) *with respect to point* $x = 0$ (see [2]).

We notice easily that the roots of the characteristic equations of system (1) with respect to $x = 0$, i.e. the numbers λ_s^*, satisfy the relations

$$\begin{aligned} \lambda_1^* &= -(1+\alpha_s^2+\beta_s^2)^{n-l+1}\lambda_1, \\ \lambda_q^* &= (1+\alpha_s^2+\beta_s^2)^{n-l+1}\lambda_q, \qquad q = 2, 3. \end{aligned} \tag{14}$$

The analogous formulas for the characteristics of infinities are the following

$$\begin{aligned} A_s^*(\alpha_s, \beta_s) &= (1+\alpha_s^2+\beta_s^2)^{n-l+1}A_s(\alpha_s, \beta_s), \\ B_s^*(\alpha_s, \beta_s) &= (1+\alpha_s^2+\beta_s^2)^{n-l+1}B_s(\alpha_s, \beta_s), \qquad s = 1, 2, 3. \end{aligned} \tag{15}$$

Thus, we have a simple method of finding the coordinates of points O_m^* and corresponding numbers λ_s^*. It is worth mentioning, that the above definition is very helpful in the case when the right-hand sides of (1) do not contain the linear terms, since in that case the roots of characteristic equations

$$\begin{vmatrix} P'_{1x_1}(0)-\lambda & P'_{1x_2}(0) & P'_{1x_3}(0) \\ P'_{2x_1}(0) & P'_{2x_2}(0-\lambda) & P'_{2x_3}(0) \\ P'_{3x_1}(0) & P'_{3x_2}(0) & P'_{3x_3}(0)-\lambda \end{vmatrix} = 0$$

are equal to zero, hence their discussion yields nothing.

From the description of the splitting method and from equalities (14) we get

THEOREM 4. *To every* A^+*-limiting* (A^-*-limiting*) *generalized singular point of the system* (1) *there corresponds an* A^-*-limiting* (A^+*-limiting*) *generalized singular point of the system* (12).

Obviously, the assertion of Theorem 4 does not require any assumptions about the polynomials $P_s(x)$. In the sequal we shall study some examples.

5. Applications of the "splitting" method.

EXAMPLE 3. Suppose we want to study the trajectories of the following system of equations

$$
(16)\qquad
\begin{aligned}
\frac{dx_1}{dt} &= ax_1^3+kx_1^2x_2+x_1\big(P_1^3(x)+P_1^4(x)+\ldots+P_1^{n-1}(x)\big),\\
\frac{dx_2}{dt} &= bx_2^3+kx_1x_2^2+x_2\big(P_2^3(x)+P_2^4(x)+\ldots+P_2^{n-1}(x)\big),\\
\frac{dx_3}{dt} &= cx_3^3+x_3\big(P_3^3(x)+P_3^4(x)+\ldots+P_3^{n-1}(x)\big),
\end{aligned}
$$

where $a<0$, $b>0$, $c<0$, in the neighborhood of the point $x=0$.

The following table gives the data concerning this system.

TABLE 3

Coordinates of points O_m^*	$(\alpha_1, \beta_1) =$			$(\alpha_2, \beta_2) =$	$(\alpha_3, \beta_3) =$
	$(0, 0)$	$(0, \sqrt{a/c})$	$(0, -\sqrt{a/c})$	$(0, 0)$	$(0, 0)$
λ_1^*	a	$a\left(1+\frac{a}{c}\right)^{n-2}$	$a\left(1+\frac{a}{c}\right)^{n-2}$	b	c
λ_2^*	$-a$	$-\left(1+\frac{a}{c}\right)^{n-2}$	$-a\left(1+\frac{a}{c}\right)^{n-2}$	$-b$	$-c$
λ_3^*	$-a$	$2a\left(1+\frac{a}{c}\right)^{n-2}$	$2a\left(1+\frac{a}{c}\right)^{n-2}$	$-b$	$-c$

Thus, all points O_m^* are saddle points. Simple geometrical considerations show that the point $x=0$ is a saddle point. The particular steps of the "splitting" method are applied only to the neighborhood of the point $(0, 0)$ of the integral surface $x_2=0$; they are presented on fig. 8. The separating surface of the saddle $x=0$ is the plane $x_2=0$ with the node at the point $(0, 0)$. The separatrices of the saddle $x=0$ are the rays of the x_2-axis, and they enter the point $x=0$ as $t\to-\infty$.

EXAMPLE 4. We shall study the behavior of the trajectories of the system

$$
(17)\qquad
\begin{aligned}
\frac{dx_1}{dt} &= ax_1^2+x_1\big(P_1^2(x)+P_1^3(x)+\ldots+P_1^{n-1}(x)\big),\\
\frac{dx_2}{dt} &= bx_2^2+x_2\big(P_2^2(x)+P_2^3(x)+\ldots+P_2^{n-1}(x)\big),\\
\frac{dx_3}{dt} &= cx_3^2+x_3\big(P_3^2(x)+P_3^3(x)+\ldots+P_3^{n-1}(x)\big),
\end{aligned}
$$

where $a>0$, $b>0$, $c>0$ in the neighborhood of the point $x=0$.

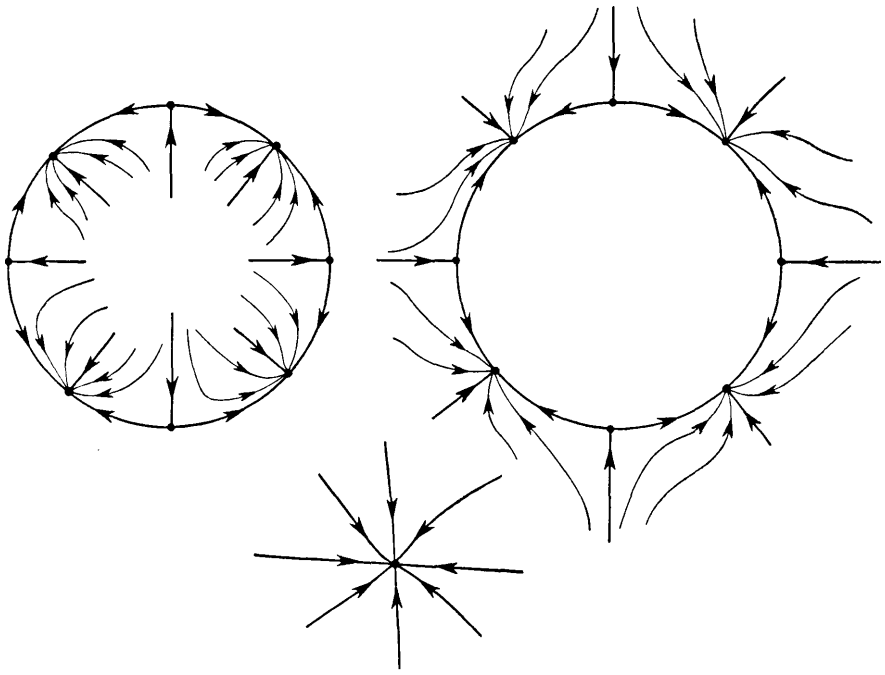

Fig. 8

The corresponding table is of the form

TABLE 4

Coordinates of points O_m^*	$(\alpha_1, \beta_1) =$				$(\alpha_2, \beta_2) =$		$(\alpha_3, \beta_3) =$
	$(0, 0)$	$(a/b, a/c)$	$(0, a/c)$	$(a/b, 0)$	$(0, 0)$	$(b/c, 0)$	$(0, 0)$
λ_1^*	a	$a\left(1+\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)^{n-1}$	$a\left(1+\frac{a^2}{c^2}\right)^{n-1}$	$a\left(1+\frac{a^2}{b^2}\right)^{n-1}$	b	$b\left(1+\frac{b^2}{c^2}\right)^{n-1}$	c
λ_2^*	$-a$	$a\left(1+\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)^{n-1}$	$-a\left(1+\frac{a^2}{c^2}\right)^{n-1}$	$a\left(1+\frac{a^2}{b^2}\right)^{n-1}$	$-b$	$b\left(1+\frac{b^2}{c^2}\right)^{n-1}$	$-c$
λ_3^*	$-a$	$a\left(1+\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)^{n-1}$	$a\left(1+\frac{a^2}{c^2}\right)^{n-1}$	$-a\left(1+\frac{a^2}{b^2}\right)^{n-1}$	$-b$	$-b\left(1+\frac{b^2}{c^2}\right)^{n-1}$	$-c$

We see easily that all points O_m^* except the point $(\alpha_1, \beta_1) = (a/b, a/c)$ are saddle points, while the point $(\alpha_1, \beta_1) = (a/b, a/c)$ is a node point. The successive steps of the "splitting" method applied to the neighbor-

hood of the point (0, 0) of each of the integral surfaces $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ give us (fig. 9):

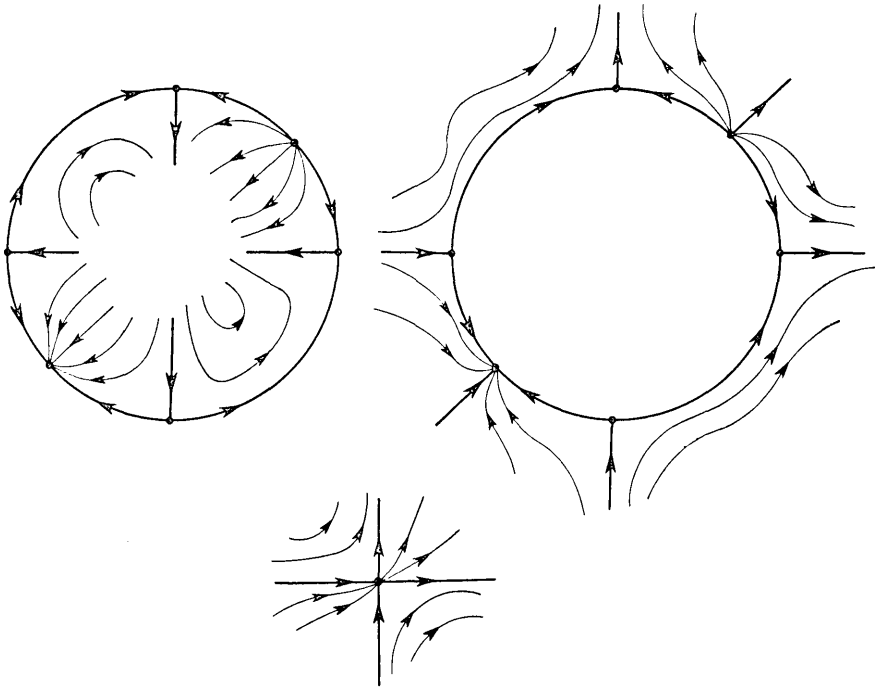

Fig. 9

The data in Table 4 and corresponding geometrical considerations lead to the following conclusion: all trajectories in the sufficiently small neighborhood of the point $x = 0$ from the side $x_1 > 0$, $x_2 > 0$, $x_3 > 0$ ($x_1 < 0$, $x_2 < 0$, $x_3 < 0$) enter this point as $t \to -\infty$ ($t \to +\infty$). All other trajectories, which do not lie on the integral planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ do not enter the point $x = 0$ at all.

EXAMPLE 5. Let us consider the behavior of the trajectories of the system

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_1x_1^3+a_2x_1x_2^2+x_1\big(P_1^3(x)+P_1^4(x)+\ldots+P_1^n(x)\big),\\
\frac{dx_2}{dt} &= b_1x_2^3+b_2x_1^2x_2+x_2\big(P_2^3(x)+P_2^4(x)+\ldots+P_2^n(x)\big),\\
\frac{dx_3}{dt} &= c_1x_3^3+c_2x_2^2x_3+c_3x_1^2x_3+x_3\big(P_3^3(x)+P_3^4(x)+\ldots+P_3^n(x)\big),
\end{aligned}
\tag{18}
$$

where $a_1 > 0$, $b_1 > 0$, $c_1 > 0$, $c_2 - b_1 > 0$, $a_2 - b_1 < 0$, $b_2 - a_1 > 0$, $c_3 - a_1 > 0$, in the neighborhood of the point $x = 0$.

The table, analogous to the previous tables, is:

TABLE 5

Coordinates of points O_m^*	$(\alpha_1, \beta_1) = (0, 0)$	$(\alpha_2, \beta_2) = (0, 0)$	$(\alpha_3, \beta_3) = (0, 0)$
λ_1^*	a_1	b_1	c_1
λ_2^*	b_2-a_1	c_2-b_1	$-c_1$
λ_3^*	c_3-a_1	a_2-b_1	$-c_1$

From this table, and from the conditions imposed on the parametres $a_1, a_2, b_1, b_2, c_1, c_2, c_3$ it can be seen that the points $(\alpha_1, \beta_1) = (0, 0)$, $(\alpha_2, \beta_2) = (0, 0)$, $(\alpha_3, \beta_3) = (0, 0)$ are respectively: node, saddle, saddle. Thus the point $x = 0$ is a node, and the trajectories entering the point $x = 0$ enter as $t \to -\infty$.

I wish to express my thanks to Dr J. Bratkowski for the valuable help in preparing this paper.

References

[1] А. А. Андронов, А. А. Витт и С. Э. Хайкин, *Теория колебаний*, Москва 1959.

[2] Ю. Г. Братковский, *Некоторые вопросы качественной теории алгебраических дифференциальных систем в малом и в большом*, Учен. зап. Рязанск. гос. пед. ин-та. 15 (1958), pp. 88-108.

[3] Р. М. Минц, *Исследование траекторий системы трех дифференциальных уравнений в бесконечности*, Сборник памяти академика А. А. Андронова, Москва 1955, pp. 499-534.

[4] — *О характере состояния равновесия системы трех дифференциальных уравнений в случае, когда один из корней характеристического уравнения равен нулю*, ДАН СССР 111 (3) (1956), pp. 535-537.