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On bilinear series in Banach spaces, I

1. Introduction. The purpose of this paper is to inwestigate the convergence of bilinear series of the form

$$\sum_{n=1}^{\infty} U(x_n, y_n) \tag{1}$$

where $U: X\times Y \to Z$ is a (bounded) bilinear operator from the product of Banach spaces X, Y into a Banach space Z. By the *convergence* we shall always mean the norm convergence in the space in question. In particular, if X is the set R of real numbers, $Z = Y$ and $U(x, y) = xy$, then (1) is just the series

$$\sum_{n=1}^{\infty} x_n y_n. \tag{2}$$

If $X = Y$ and $x_n = y_n$, (1) is a series $\sum V(x_n)$, where $V: X \to Z$ is a homogeneous quadratic operator.

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2. Comparison functions. The symbols $\mathfrak{M}^+$, $\mathfrak{M}^0$, and $\mathfrak{M}^-$ will denote the set of all positive monotone sequences $\{\mu(n)\}_{n=1,2,\dots}$ such that $\mu(n) \to \infty$, $\mu(n) = c > 0$, and $\mu(n) \to 0$, respectively; also let $\mathfrak{M} = \mathfrak{M}^+ \cup \mathfrak{M}^0 \cup \mathfrak{M}^-$.

DEFINITION. Let $\{x_n\} \subset X$. A sequence $\mu = \{\mu(n)\}$ is said to be a *comparison function* for the series $\sum x_n$ provided that $\mu \in \mathfrak{M}$ and

$$\sup_n \frac{\|\sigma_n\|}{\mu(n)} < \infty, \quad \text{i.e.} \quad \|\sigma_n\| = O\big(\mu(n)\big), \tag{3}$$

where $\sigma_n = x_1 + \dots + x_n$, and is said to be a *comparison function of positive order*, *of order* 0, *of negative order*, provided that $\mu \in \mathfrak{M}^+$, $\mu \in \mathfrak{M}^0$, $\mu \in \mathfrak{M}^-$, respectively.

Let us notice that every series $\sum x_n$ has at least one comparison function of positive order, e.g.,

$$\mu(n) = \begin{cases} \|x_1\| + \ldots + \|x_n\| & \text{if} \quad \sum \|x_n\| = \infty, \\ n & \text{if} \quad \sum \|x_n\| < \infty. \end{cases}$$

A series $\sum x_n$ has a comparison function of negative order if and only if $\lim \sigma_n = 0$, and has a comparison function of order 0 if and only if it is bounded. Let us also notice that if $x_n \neq 0$ for $n = 1, 2, \ldots$, then there exists a comparison function of negative order which is not a comparison function for the series $\sum x_n$. If $\varphi_1, \varphi_2, \ldots$ is an orthonormal system in a Hilbert space, then $\mu(n) = \sqrt{n}$ is a comparison function for $\sum \varphi_n$.

If $\mu_1, \mu_2 \in \mathfrak{M}$, let $\mu_1 \prec \mu_2$ mean that $\mu_1(n) = O(\mu_2(n))$; μ_1, μ_2 are called *equivalent* (written $\mu_1 \sim \mu_2$) if $\mu_1 \prec \mu_2$ and $\mu_2 \prec \mu_1$.

3. Criteria of convergence.

LEMMA 1. *Let $\mu \in \mathfrak{M}^- \cup \mathfrak{M}^0$. If the series*

$$\sum_{n=1}^{\infty} \mu(n) \Delta y_n, \quad \text{where} \quad \Delta y_n = y_n - y_{n+1}, \tag{4}$$

is weakly unconditionally convergent, i.e., if

$$\sum_{n=1}^{\infty} \mu(n) |f(\Delta y_n)| < \infty \tag{5}$$

for every f in Y^*, then the sequence $\mu(n)\|y_{n+1}\|$ is bounded.

Proof. If $\mu \in \mathfrak{M}^-$ and $f \in Y^*$, then by Kronecker's theorem ([6], p. 131),

$$\lim_{n\to\infty} \big(|f(y_1 - y_2)| + \ldots + |f(y_n - y_{n+1})|\big)\mu(n) = 0$$

and hence $|f(y_1) - f(y_{n+1})|\mu(n) \to 0$. Since $\mu(n) \downarrow 0$, this yields $f(y_{n+1})\mu(n) \to$ $\to 0$, which means that $\mu(n) y_{n+1}$ tends weakly to 0 and $\mu(n)\|y_{n+1}\| = O(1)$.

Now, if $\mu \in \mathfrak{M}^0$, i.e., $\mu(n) = c > 0$, then $\sum |f(\Delta y_n)| < \infty$. Consequently, $\{y_1 - y_n\}$ is weakly convergent and therefore both $\{y_1 - y_n\}$ and $\{y_n\}$ are bounded.

LEMMA 2. *Suppose that $\mu \in \mathfrak{M}^-$ and*

$$\sum_{n=1}^{\infty} \mu(n) \|\Delta y_n\| < \infty. \tag{6}$$

Then $\lim_{n\to\infty} \mu(n)\|y_{n+1}\| = 0$.

The proof is analogous to that of Lemma 1.

LEMMA 3. *Let* $\mu \in \mathfrak{M}^+$. *If series* (4) *is weakly unconditionally convergent and* $\{y_n\}$ *is weakly convergent to* 0, *then* $\{\mu(n) y_n\}$ *is weakly convergent to* 0.

Proof. If $f \in Y^*$ and (4) is weakly unconditionally convergent, then there exists an N such that

$$\mu(k)|f(y_k - y_l)| = \mu(k)\Big|f\Big(\sum_{n=k}^{l-1} \Delta y_n\Big)\Big|$$
$$\leqslant \mu(k) \sum_{n=k}^{l-1} |f(\Delta y_n)| \leqslant \sum_{n=k}^{l-1} \mu(n)|f(\Delta y_n)| < \varepsilon$$

for $l > k \geqslant N$. Keeping k fixed and passing to the limit with $l \to \infty$ we infer that $\mu(k)|f(y_k)| \leqslant \varepsilon$ for $k \geqslant N$.

LEMMA 4. *If* $\mu \in \mathfrak{M}^+$, $y_n \to 0$, *and* (6) *holds, then* $\mu(n) y_n \to 0$.

The proof is analogous to that of Lemma 3.

THEOREM 1. *Let* $x_n \in X$, $y_n \in Y$, *let* $\mu(n)$ *be a comparison function for the series* $\sum x_n$, *let* $y_n \to 0$, *and let*

$$\sum_{n=1}^{\infty} \mu(n) \|\Delta y_n\| < \infty.$$

Then series (1) *is convergent in* Z. *Moreover,*

$$\sum_{n=}^{\infty} U(x_n, y_n) = \sum_{n=1}^{\infty} U(\sigma_n, \Delta y_n)$$

and

$$\Big\|\sum_{n=1}^{\infty} U(x_n, y_n)\Big\| \leqslant K \sum_{n=1}^{\infty} \mu(n) \|\Delta y_n\|, \tag{7}$$

where $K = \|U\| \sup \dfrac{\|\sigma_n\|}{\mu(n)}$ *and* $\sigma_n = x_1 + \ldots + x_n$.

Proof. Applying Abel's formula ([1], p. 68) to the sequence $\{S_k\}$ of partial sums of (1) we get

$$S_k = \sum_{n=1}^{k-1} U(\sigma_n, \Delta y_n) + U(\sigma_k, y_k).$$

The series $\sum U(\sigma_n, \Delta y_n)$ is absolutely convergent because

$$\sum_{n=1}^{\infty} \|U(\sigma_n, \Delta y_n)\| \leqslant K \sum_{n=1}^{\infty} \mu(n) \|\Delta y_n\|.$$

Since $\|U(\sigma_k, y_k)\| \leqslant K\mu(k)\|y_k\|$, in virtue of Lemmas 4 and 2, the sequence S_k is convergent. Moreover

$$\Big\|\sum_{n=1}^{\infty} U(x_n, y_n)\Big\| = \Big\|\sum_{n=1}^{\infty} U(\sigma_n, \Delta y_n)\Big\| \leqslant K\sum_{n=1}^{\infty}\mu(n)\|\Delta y_n\|.$$

Remark 1. If $\mu \in \mathfrak{M}^-$, then the assumption $y_n \to 0$ in Theorem 1 is superflous.

THEOREM 2. *Suppose that* $x_n \in X$, $y_n \in Y$, $\mu \in \mathfrak{M}^-$, *that the series* $\sum x_n$ *is convergent to* x_0,

$$\|x_0 - \sigma_n\| = O\big(\mu(n)\big)$$

where $\sigma_n = x_1 + \ldots + x_n$, *and that* (6) *holds. Then* (1) *is convergent and for every* m

$$\Big\|\sum_{n=m}^{\infty} U(x_n, y_n)\Big\| \leqslant \|U\| \sup_n \frac{\|x_0-\sigma_n\|}{\mu(n)}\Big[\mu(m-1)\|y_m\| + \sum_{n=m}^{\infty}\mu(n)\|\Delta y_n\|\Big].$$

Proof. Denote $r_0 = x_0, r_n = x_0 - \sigma_n$ $(n = 1, 2, \ldots)$. Then $\|r_n\| \to 0$. Applying Abel's transformation we get

$$(8) \qquad \sum_{n=m}^{p} U(x_n, y_n) = \sum_{n=m}^{p-1} U(r_n, y_{n+1} - y_n) + U(r_{m-1}, y_m) - U(r_p, y_p).$$

The first term on the right-hand side is convergent as $p \to \infty$ because

$$\sum_{n=m}^{p-1}\|U(r_n, \Delta y_n)\| \leqslant \|U\| \sup_n \frac{\|r_n\|}{\mu(n)} \sum_{n=m}^{p-1}\mu(n)\|\Delta y_n\|.$$

Moreover, $\|U(r_p, y_p)\| \leqslant \|U\|\mu(p)\|y_p\| \sup_n \|r_n\|/\mu(n)$ and therefore, by Lemma 2, $\|U(r_p, y_p)\| \to 0$ as $p \to \infty$. Thus, series (1) is convergent. In virtue of (8) we also have

$$\Big\|\sum_{n=m}^{p} U(x_n, y_n)\Big\| \leqslant \|U\| \sup_n \frac{\|r_n\|}{\mu(n)}\Big[\sum_{n=m}^{p-1}\mu(n)\|\Delta y_n\| + \mu(m-1)\|y_m\|\Big] + \|U(r_p, y_p)\|.$$

Passing to the limit as $p \to \infty$ we get the desired conclusion.

THEOREM 3. *Suppose that series* (4) *is weakly unconditionally convergent. Suppose also that either* (i) $\mu \in \mathfrak{M}^{0-}$ *or* (ii) $\mu \in \mathfrak{M}^+$ *and* $\{y_n\}$ *tends weakly to* 0. *Furthermore, suppose that* $\{x_n\}$ *is a sequence of numbers such that* $\lim \sigma_n/\mu(n) = 0$, *where* $\sigma_n = x_1 + \ldots + x_n$. *Then series* (2) *is convergent in* Y *and for every* f *in* Y^* *the inequality*

$$\Big|\sum_{n=1}^{\infty} x_n f(y_n)\Big| \leqslant \sup_n \frac{|\sigma_n|}{\mu(n)} \sum_{n=1}^{\infty}\mu(n)|f(\Delta y_n)|$$

holds.

Proof. Denote $\alpha_k = \sum_{n=1}^{k-1} \sigma_n \Delta y_n$ and $\beta_k = \sigma_k y_k$. Abel's formula gives $\alpha_k + \beta_k = s_k = \sum_{n=1}^{k} x_n y_n$. By Lemmas 1 and 3, the sequence $\mu(k)\|y_k\|$ is bounded and therefore

$$\|\beta_k\| = \frac{|\sigma_k|}{\mu(k)} \mu(k) \|y_k\| \to 0.$$

Since $\sum \mu(n) \Delta y_n$ is weakly unconditionally convergent, for every sequence $\{t_n\}$ of numbers convergent to 0 the series $\sum t_n \mu(n) \Delta y_n$ is convergent in Y; in particular, setting $t_n = \sigma_n/\mu(n)$ we infer that the series $\sum \sigma_n \Delta y_n$ is convergent. Consequently, $s_k = \alpha_k + \beta_k$ is the sum of two convergent sequences. If $f \in Y^*$, then

$$\Big|\sum_{n=1}^{k} x_n f(y_n)\Big| \leqslant |f(\alpha_k)| + |f(\beta_k)| \leqslant \sum_{n=1}^{k} \frac{|\sigma_n|}{\mu(n)} \mu(n) |f(\Delta y_n)| + o(1);$$

passing to the limit we get the desired inequality.

4. Examples and applications.

EXAMPLE 1. *Let $\mu \in \mathfrak{M}$, $\sum [1/\mu(n)]^2 < \infty$, let* (4) *be weakly unconditionally convergent, and let $\{y_n\}$ tend weakly to* 0. *Then the series $\sum y_n r_n(t)$, where r_n denotes the n-th Rademacher function* ([1], p. 51-52), *is convergent* (*in norm*) *almost everywhere on* $\langle 0, 1\rangle$.

Proof. By Rademacher's theorem ([1], p. 52) the series

$$\sum_{n=1}^{\infty} \frac{r_n(t)}{\mu(n)}$$

is convergent a.e. on $\langle 0, 1\rangle$. Applying Kronecker's theorem ([6], p. 131) we infer that

$$\frac{r_1(t) + \ldots + r_n(t)}{\mu(n)} \to 0$$

a.e. on $\langle 0, 1\rangle$ and we apply Theorem 3.

EXAMPLE 2. *Suppose that $\mu \in \mathfrak{M}$,*

$$\sum_{n=2}^{\infty} \left[\frac{\log n}{\mu(n)}\right]^2 < \infty,$$

(4) *is weakly unconditionally convergent and $\{y_n\}$ tends weakly to* 0. *Then for every orthonormal system $\{\varphi_n\}$ in $L_2(0, 1)$ the series $\sum y_n \varphi_n(t)$ is convergent a.e. on* $\langle 0; 1\rangle$.

This follows from the Menchoff-Rademacher theorem ([1], p. 76) in a way analogous to that above.

EXAMPLE 3. *If $\{y_n\}$ is a sequence of numbers tending to* 0 *and*

$$\sum_{n=1}^{\infty} n^{1/p}|\Delta y_n| < \infty$$

where $1 \leqslant p < \infty$, then

$$\Big(\sum_{n=1}^{\infty} |y_n|^p\Big)^{1/p} \leqslant \sum_{n=1}^{\infty} n^{1/p}|\Delta y_n|. \tag{9}$$

Proof. Let $X = Z = l_p$, $Y = R$, $U(x, y) = yx$ and let x_n be the nth unit vector $(0, \ldots, 1, 0, \ldots)$. Since

$$\|\sigma_n\|_{l_p} = n^{1/p} \quad \text{and} \quad \|U\| = 1,$$

and the series $\sum y_n x_n$ is l_p-convergent to $(y_1, y_2, \ldots)$, we may apply Theorem 1 with $K = 1$.

Remark 2. Inequality (9) may be compared with Example 1. Indeed, suppose that $\{y_n\}$ tends weakly to 0 and

$$\sum_{n=1}^{\infty} \sqrt{n}\,|f(\Delta y_n)| < \infty.$$

Then, by (9), $\sum |f(y_n)|^2 < \infty$ and therefore for every f in Y^* the series $\sum f(y_n)\, r_n(t)$ is convergent a.e. on $\langle 0, 1\rangle$. This conclusion is weaker than that of Example 1, but the assumptions are also weaker (e.g. consider $\mu(n) = \sqrt{n}$).

EXAMPLE 4. *There is no orthonormal system $\{\varphi_n\}$ in $L_2(0, 1)$ such that each φ_n is bounded and that*

$$\|\varphi_1 + \ldots + \varphi_n\|_{L_\infty} = O\big(\mu_0(n)\big),$$

where $\mu_0(n) = n^{\frac{1}{2}-\varepsilon} (\log n)^k$, $0 < \varepsilon < 1/2$, $k > 0$.

Indeed, if such a system existed, then for every sequence $\{a_n\}$ tending to 0 and such that $\sum \mu_0(n)\,|\Delta a_n| < \infty$ the series $\sum a_n \varphi_n$ would be uniformly convergent a.e. (in virtue of Theorem 1). This, however, is not true for $a_n = 1/\sqrt{n}$ whereas

$$\sum_{n=2}^{\infty} \mu_0(n)\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big) = \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^{\varepsilon}\sqrt{n+1}\,(\sqrt{n}+\sqrt{n+1})} < \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^{1+\varepsilon}} < \infty.$$

EXAMPLE 5. *Suppose that $a > 1$ and*

$$a_n^2 + a_{n+1}^2 + \ldots = O\big((\log n)^{-2a}\big),$$

then for every orthonormal system $\{\varphi_n\}$ in $L_2(0, 1)$ the series $\sum a_n \varphi_n(t)$ is convergent a.e. for any rearrangement of its terms. (This is a special case of a theorem of Leindler [7], p. 113, Theorem 1; cf. also [2], p. 345).

Indeed, denote $\mu(n) = (\log n)^{-a}$, $y_n = (\log n)^{(k+1)/k}$, where k is an integer and $k > 1$, and

$$M = \sup\left\{\frac{\log(n+1)}{\log n};\ n = 2, 3, \ldots\right\}.$$

Applying the elementary identity $a^{k+1}-b^{k+1} = (a-b)\sum a^{k-i}b^i$ we get

$$\begin{aligned}
|\Delta y_n| &= \left[\left(\log(n+1)\right)^{1/k}\right]^{k+1} - [(\log n)^{1/k}]^{k+1} \\
&= \left[\left(\log(n+1)\right)^{1/k} - (\log n)^{1/k}\right] \sum_{i=0}^{k} \left(\log(n+1)\right)^{(k-i)/k} (\log n)^{i/k} \\
&\leqslant (k+1)\log(n+1)\left[\left(\log(n+1)\right)^{1/k} - (\log n)^{1/k}\right] \\
&= (k+1)\log(n+1)\,\frac{\log(n+1)-\log n}{\sum\limits_{i=0}^{k-1}\left(\log(n+1)\right)^{(k-1-i)/k}(\log n)^{i/k}} \\
&\leqslant (k+1)\log(n+1)\,\frac{\log(1+1/n)}{k(\log n)^{1/k}} \\
&\leqslant \left(1+\frac{1}{k}\right) M\,\frac{(\log n)^{1/k}}{n};
\end{aligned}$$

therefore, if $a > 1+1/k$, then $\sum\limits_{n=2}^{\infty} \mu(n)|\Delta y_n| < \infty$. In virtue of the assumption above,

$$\Big\|\sum_{m=n+1}^{\infty} a_m \varphi_m\Big\|_{L^2} (\log n)^a = O(1);$$

hence, by Theorem 2, the series $\sum a_n y_n \varphi_n$ is convergent in L_2. Therefore $\sum a_n^2 (\log n)^{2+2/k} < \infty$ and our assertion follows from a theorem of Orlicz ([1], p. 108).

Example 6. (Plessner's theorem, cf. [2], p. 341). *If* $a > \frac{1}{2}$ *and*

$$\varrho_n^2 + \varrho_{n+1}^2 + \ldots = O\left((\log n)^{-2a}\right),$$

where $\varrho_n^2 = a_n^2 + b_n^2$, *then* $\sum (a_n \cos nt + b_n \sin nt)$ *is convergent a.e.*

Indeed, denoting $\mu(n) = (\log n)^{-a}$, $y_n = (\log n)^{1/2}$ and applying the Kolmogorov-Selivestrov-Plessner theorem ([2], p. 332), we get the conclusion in a way similar to that used in the preceding proof.

5. The space Λ_μ for $\mu \in \mathfrak{M}^{0+}$. Let $\mu \in \mathfrak{M}^{0+}$ and let Λ_μ denote the set of all sequences $\lambda = \{\lambda_n\}$ tending to 0 and such that

$$\|\lambda\| = \sum_{n=1}^{\infty} \mu(n)|\Delta\lambda_n| < \infty.$$

It is clear that Λ_μ is a Banach space. We shall show that the unit vectors $e^{(1)}, e^{(2)}, \ldots$ form a basis in it. Indeed, if $\lambda = \{\lambda_n\} \in \Lambda_\mu$, then (applying Theorem 1 to $X = Z = Y = R$, $x_n = e^{(n)}$, $y_n = \lambda_n$) we infer that the series $\sum \lambda_n e^{(n)}$ is convergent in Λ_μ; a straightforward computation shows that this series is convergent to λ and that the expansion is unique.

THEOREM 4. *The general form of a linear functional on Λ_μ is*

$$f(\lambda) = \sum_{n=1}^{\infty} \lambda_n q_n \tag{10}$$

where q_n are numbers and

$$\|f\| = \sup_n \frac{|q_1 + \ldots + q_n|}{\mu(n)} < \infty.$$

Proof. If $f \in \Lambda_\mu^*$, denote $q_n = f(e^{(n)})$. Since $\{e^{(n)}\}$ is a basis, f is of the form (10). Moreover,

$$|q_1 + \ldots + q_n| = |f(e^{(1)} + \ldots + e^{(n)})| \leqslant \|f\| \mu(n)$$

and hence $\sup |q_1 + \ldots + q_n| / \mu(n) \leqslant \|f\|$. The converse inequality follows from Theorem 1.

LEMMA 5. *A sequence μ in $\mathfrak{M}$ is a comparison function for a series $\sum x_n$ if and only if for every f in X^**

$$\sup_n \frac{|f(x_1 + \ldots + x_n)|}{\mu(n)} < \infty.$$

This follows from [5], p. 255.

LEMMA 6. *Let q_n be a sequence of numbers. Then $\sum \lambda_n q_n$ is convergent for every $\lambda = \{\lambda_n\}$ in Λ_μ if and only if μ is a comparison function for $\sum q_n$.*

Proof. The sufficiency follows from Theorem 1. In order to prove the necessity let us consider the sequence $\{f_m\}$ of linear functionals on Λ_μ defined as

$$f_m(\lambda) = \sum_{n=1}^{\infty} \lambda_n q_n^{(m)}, \qquad \text{where} \qquad q_n^{(m)} = \begin{cases} q_n & \text{for} \quad n \leqslant m, \\ 0 & \text{for} \quad n > m. \end{cases}$$

Then

$$\|f_m\| = \sup_k \frac{|q_1^{(m)} + \ldots + q_k^{(m)}|}{\mu(k)} = \sup_{k \leqslant m} \frac{|q_1 + \ldots + q_k|}{\mu(k)}.$$

By assumption, for every λ in Λ_μ the sequence $\{f_m(\lambda)\}$ is convergent as $m \to \infty$. Hence (see [5], p. 229, Theorem 1)

$$\sup_m \|f_m\| = \sup_k \frac{|q_1 + \ldots + q_k|}{\mu(k)} < \infty,$$

and the desired conclusion follows.

THEOREM 5. *In order that $\sum \lambda_n x_n$ be convergent for every λ in Λ_μ it is necessary and sufficient that $\mu(n)$ be a comparison function for $\sum x_n$.*

Proof. The sufficiency follows from Theorem 1, and we shall show the necessity. Let us notice that if for every $\lambda = \{\lambda_n\}$ in Λ_μ the series $\sum \lambda_n x_n$ is convergent in X, then the series $\sum \lambda_n f(x_n)$, where f is any element of X^*, is convergent as well. Thus, the desired conclusion is a consequence of Lemmas 5 and 6.

COROLLARY 1. *If the series $\sum \lambda_n x_n$ is weakly convergent for every λ in Λ_μ, then it is strongly convergent for every λ in Λ_μ.*

Proof. Since the series $\sum \lambda_n f(x_n)$ is convergent for every λ in Λ^μ and every f in X^*, μ is a comparison function for the series $\sum x_n$ (by Lemmas 5 and 6); therefore our assertion follows from Theorem 1.

Now, let X be a Banach space with a basis $\{\eta_n\}$. If $\lambda \in \Lambda_\mu$, denote

$$U(\lambda, x) = \sum_{n=1}^{\infty} \lambda_n t_n(x) \eta_n$$

where $\{t_n\} \subset X^*$ is the system biorthogonal to $\{\eta_n\}$.

PROPOSITION. *The operator* $U\colon \Lambda_1 \times X \to X$ *is bilinear* (here Λ_1 means $\Lambda_{\{1,1,\ldots\}}$).

Proof. For every x in X the series $\sum t_n(x)\eta_n$ has a comparison function of order 0 ($\mu(n) = 1$ for $n = 1, 2, \ldots$). Thus, by Theorem 1, U is well defined on $\Lambda_1 \times X$ and

$$\|U(\lambda, x)\| \leqslant \sup_k \Big\| \sum_{n=1}^{k} t_n(x)\eta_n \Big\| \sum_{n=1}^{\infty} |\Delta \lambda_n|.$$

Since there exist positive constants c_1 and c_2 such that

$$c_1 \|x\| \leqslant \sup_k \Big\| \sum_{n=1}^{k} t_n(x)\eta_n \Big\| \leqslant c_2 \|x\|$$

([8], p. 151-154), we get $\|U(\lambda, x)\| \leqslant c_2 \|\lambda\| \|x\|$.

6. The space Λ_μ for μ in $\mathfrak{M}^-$. Let μ be a fixed sequence in $\mathfrak{M}^-$ and let Λ_μ denote the space of all sequences $\lambda = \{\lambda_n\}$ such that

$$\|\lambda\| = \mu(0)|\lambda_1| + \sum_{n=1}^{\infty} \mu(n) |\Delta \lambda_n| < \infty.$$

In this case $\{\lambda_n\}$ need not tend to 0, but again Λ_μ is a separable Banach space.

THEOREM 6. *The general form of a linear functional on Λ_μ is* (10) *with*

$$\|f\| = \sup_n \frac{|q_{n+1} + q_{n+2} + \ldots|}{\mu(n)} < \infty.$$

Proof. We begin as in the proof of Theorem 4 and estimate

$$|q_{n+1}+\ldots+q_m| = |f(e^{(n+1)}+\ldots+e^{(m)})| \leqslant \|f\|\,\|e^{(n+1)}+\ldots+e^{(m)}\| = \|f\|\big(\mu(n)+\mu(m)\big).$$

If $m\to\infty$, then $\mu(m)\to 0$ and therefore

$$\sup_n \frac{|q_{n+1}+q_{n+2}+\ldots|}{\mu(n)} \leqslant \|f\|.$$

The converse inequality follows from Theorem 2.

THEOREM 7. *Let $\mu\in\mathfrak{M}^-$. The series $\sum\lambda_n x_n$ is convergent for every λ in Λ_μ if and only if*

$$\sup_n \frac{\|x_{n+1}+x_{n+2}+\ldots\|}{\mu(n)} < \infty.$$

The proof is similar to that of Theorem 5. We can also conclude that if $\sum\lambda_n x_n$ is weakly convergent for each λ in Λ_μ, then it is strongly convergent as well.

7. The Haar series of Hölder functions. Let $\omega(h)$ be a positive continuous decreasing function defined for $h>0$ and such that $\omega(h)\to 0$ as $h\to 0$, and let

$$\limsup_{h\to 0} \frac{h}{\omega(h)} < \infty.$$

A function f is said to *satisfy condition* H_ω (a generalized Hölder condition) if $|f(t+h)-f(t)| = O\big(\omega(h)\big)$ as $h\to 0$.

The *Haar function* (cf. [1], p. 48) are defined as folows:

$$\chi_0^{(0)}(t) = 1$$

and

$$\chi_n^{(k)}(t) = \begin{cases} \sqrt{2^n} & \text{for} \quad (k-1)/2^n < t < (2k-1)/2^n, \\ -\sqrt{2^n} & \text{for} \quad (2k-1)/2^{n+1} < t < k/2^n, \\ 0 & \text{elsewhere in } \langle 0,1\rangle, \end{cases}$$

where $n=0,1,\ldots;\ k=1,2,\ldots,2^n$. This double sequence is rearranged into a single one by setting $\chi_0^{(0)}(t)=\chi_1$ and $\chi_{2^n+k}=\chi_n^{(k)}$ for $n=0,1,2,\ldots;$ $k=1,2,\ldots,2^n$. The series

$$(11)\qquad \sum_{n=1}^{\infty} c_n(f)\chi_n(t), \quad \text{where} \quad c_n(f) = \int_0^1 f(t)\chi_n(t)\,dt,$$

is the *Haar series* for f. It is known that for every continuous function f on $\langle 0,1\rangle$, series (11) is uniformly convergent to f (cf. [1], p. 49), and f satisfies condition H_ω if and only if

$$\Big\|f-\sum_{k=1}^{n} c_k(f)\chi_k\Big\|_{C\langle 0,1\rangle} = O\big(\omega(1/n)\big)$$

(cf. Golubov [4], p. 1274-1276). Combining this with Theorem 7 we get

THEOREM 8. *Let $f\in C\langle 0,1\rangle$. In order that f satisfy condition H_ω it is necessary and sufficient that the series*

$$\sum_{n=1}^{\infty}\lambda_n c_n(f)\chi_n(t)$$

be uniformly convergent for every λ in Λ_μ, where $\mu(n) = \omega(1/n)$.

Combining a theorem of Ciesielski ([3], p. 156) we get a similar theorem for the Franklin series $\sum_{n=1}^{\infty} b_n(f)\varphi_n(t)$ of f:

THEOREM 9. *A continuous function f on $\langle 0,1\rangle$ satisfies the Hölder condition with an exponent a $(0<a<1)$ if and only if the series*

$$\sum_{n=1}^{\infty}\lambda_n b_n(f)\varphi_n(t)$$

is uniformly convergent for every λ in Λ_μ, where $\mu(n) = n^{-a}$.

COROLLARY 2. *If f satisfies Hölder condition with an exponent a, then*

$$\Big\|\sum_{n=m}^{\infty}(\log n)\, b_n(f)\varphi_n\Big\| = O\left(\frac{\log m}{m^a}\right),$$

$$\Big\|\sum_{n=m}^{\infty}n^{\beta} b_n(f)\varphi_n\Big\| = O\left(\frac{1}{m^{a-\beta}}\right), \quad \textit{where} \quad 0\leqslant\beta<a.$$

COROLLARY 3. *If $f\in C\langle 0,1\rangle$, then $\sum n^{-a}b_n(f)\varphi_n$ satisfies Hölder's condition with exponent a since*

$$\Big\|\sum_{n=m}^{\infty}n^{-a} b_n(f)\varphi_n\Big\| = O(1/m^a).$$

Indeed, the Franklin system is a basis for $C\langle 0,1\rangle$ and therefore for every f in $C\langle 0,1\rangle$ there exists a μ_f in $\mathfrak{M}^-$ such that $\|r_n(f)\|\leqslant K\mu_f(n)$. Since $\lambda = \{n^{-a}\}\in\Lambda_{\mu_f}$, Theorem 2 yields the estimation

$$\Big\|\sum_{n=m}^{\infty}n^{-a} b_n(f)\varphi_n\Big\| \leqslant K\sup_n \mu_f(n)\left[m^{-a}+\sum_{n=m}^{\infty}\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)\right] = O(m^{-a}),$$

and we may apply Theorem 9.

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