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An estimate for the Lebesgue $(C, 1)$-functions of some polynomial-like systems

1. Introduction. Let $\{\varphi_n(x)\}$ be an orthonormal system with respect to a positive Lebesgue-integrable weight-function $\varrho(x)$ in $\langle -1, 1\rangle$, i.e.

$$\int_{-1}^{1} \varphi_n(x)\varphi_k(x)\varrho(x)\,dx = \begin{cases} 0 & \text{when} \quad n \neq k, \\ 1 & \text{when} \quad n = k. \end{cases}$$

The system $\{\varphi_n(x)\}$ is called *polynomial-like* in $\langle -1, 1\rangle$ if the kernel

$$K_\nu(t, x) = \sum_{j=0}^{\nu} \varphi_j(t)\varphi_j(x)$$

is of the form

$$K_\nu(t, x) = \sum_{k=1}^{r} F_k(t, x) \sum_{i,j=-p}^{p} \gamma_{ijk}^{(\nu)} \varphi_{\nu+i}(t)\varphi_{\nu+j}(x), \tag{1}$$

where p and r are positive integers independent of ν, the coefficients $\gamma_{ijk}^{(\nu)}$ are uniformly bounded for all i, j, k and all ν, and $F_k(t, x)$ are measurable functions of two variables satisfying the condition

$$F_k(t, x) = O\left(\frac{1}{|t-x|}\right) \tag{2}$$

uniformly in $t, x \in \langle -1, 1\rangle$, $t \neq x$; if $\nu+i < 0$, we set $\varphi_{\nu+i}(x) \equiv 0$. The definition of the polynomial-like orthonormal system was given by G. Alexits (see [1], p. 158).

In this note we deduce a certain estimate for the Lebesgue $(C, 1)$-functions

$$L_n^{(1)}(x) = \frac{1}{n+1} \int_{-1}^{1} \left| \sum_{\nu=0}^{n} K_\nu(t, x) \right| \varrho(t)\,dt$$

of a given polynomial-like system. This estimate is important for $(C, 1)$-summability problems of Fourier series with respect to the orthonormal system $\{\varphi_n(x)\}$. In our considerations we restrict ourselves to the interval

$\langle -1, 1\rangle$; the case of an interval $\langle a, b\rangle$ can easily be reduced to the previous by the substitution $u = -1+2(x-a)/(b-a)$.

2. We shall present an extension of Theorem 3.4.3 of [1], p. 186.

THEOREM. *Let $\{\varphi_n(x)\}$ be a polynomial-like orthonormal system with weight function $\varrho(x)$ in the interval $\langle -1, 1)\rangle$. Suppose that there exist constants $C_1 > 0$, $0 \leqslant C_2 < 1$, $C_3 > 0$, $C_4 > 0$ such that for all $x \in (-1, 1)$*

$$(3) \qquad \varrho(x) \leqslant \frac{C_1}{(1-x^2)^{C_2}}, \qquad \sum_{k=0}^{n} \varphi_k^2(x) \leqslant C_3 \lambda_n \qquad (n = 0, 1, 2, \ldots),$$

where $\lambda_n > 0$, $\lambda_{n+1}/\lambda_n < C_4$ and $\lambda_k \to \infty$ as $k \to \infty$.

Then

$$(4) \qquad L_n^{(1)}(x) \leqslant C\lambda_n^{(1-C_2)/4} n^{(C_2-3)/4} \Big(\sum_{\nu=0}^{n} \lambda_\nu\Big)^{(1+C_2)/4}$$

for every $x \in (-1, 1)$ and $n \geqslant 1$, where C is a positive constant depending neither on x nor on n.

Proof. Let ε_n be positive numbers tending to zero ($\varepsilon_n < \frac{1}{3}$). A specific definition of ε_n will be given later. The characteristic functions of sets of points (t, x) for which $\sum_{\nu=0}^{n} K_\nu(t, x) \geqslant 0$ or $\sum_{\nu=0}^{n} K_\nu(t, x) < 0$ will be denoted by $P_n(t, x)$ and $N_n(t, x)$, respectively.

Then

$$L_n^{(1)}(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} \int_{-1}^{1} P_n(t, x) K_\nu(t, x) \varrho(t) dt - \\ - \frac{1}{n+1} \sum_{\nu=0}^{n} \int_{-1}^{1} N_n(t, x) K_\nu(t, x) \varrho(t) dt = A_n + B_n.$$

1° We shall consider in detail the case $0 \leqslant x \leqslant 1-2\varepsilon_n$. Write

$$\int_{-1}^{1} P_n(t, x) K_\nu(t, x) \varrho(t) dt$$
$$= \Big(\int_{-1}^{x-\varepsilon_n} + \int_{x+\varepsilon_n}^{1}\Big) P_n(t, x) K_\nu(t, x) \varrho(t) dt + \int_{x-\varepsilon_n}^{x+\varepsilon_n} P_n(t, x) K_\nu(t, x) \varrho(t) dt = I_{\nu 1} + I_{\nu 2}$$

and set

$$g_k(t, x) = \begin{cases} P_n(t, x) F_k(t, x) & \text{for} \quad t \in \langle -1, x-\varepsilon_n\rangle \cup \langle x+\varepsilon_n, 1\rangle, \\ 0 & \text{for} \quad t \in (x-\varepsilon_n, x+\varepsilon_n). \end{cases}$$

By Schwarz inequality, (1) and (3), we get

$$\sum_{\nu=0}^{n}|I_{\nu 1}| \leqslant \sum_{k=1}^{r}\sum_{i,j=-p}^{p}\Big\{\sum_{\nu=0}^{n}[\gamma_{ijk}^{(\nu)}]^2\varphi_{\nu+j}^2(x)\sum_{\nu=0}^{n}\Big[\int_{-1}^{1} g_k(t,x)\varphi_{\nu+i}(t)\varrho(t)\,dt\Big]^2\Big\}^{1/2}$$

$$= O(1)\lambda_n^{1/2}\sum_{k=1}^{r}\sum_{i=-p}^{p}\Big\{\sum_{\nu=0}^{n}\Big[\int_{-1}^{1} g_k(t,x)\varphi_{\nu+i}(t)\varrho(t)\,dt\Big]^2\Big\}^{1/2}.$$

The condition $|t-x| \geqslant \varepsilon_n$ implies $|g_k(t,x)| \leqslant P_n(t,x)|F_k(t,x)| = O(1/\varepsilon_n)$. Therefore, if n is fixed, the function $g_k(t,x)$ is bounded. Consequently, the integrals under the sign of summation are Fourier coefficients of the square-integrable function $g_k(t,x)$. Applying Bessel inequality together with (2) and (3), we obtain

$$\sum_{\nu=0}^{n}\Big[\int_{-1}^{1} g_k(t,x)\varphi_{\nu+i}(t)\varrho(t)\,dt\Big]^2 \leqslant \int_{-1}^{1} g_k^2(t,x)\varrho(t)\,dt$$

$$= O(1)\Big(\int_{-1}^{x-\varepsilon_n} + \int_{x+\varepsilon_n}^{1}\Big)\frac{dt}{(t-x)^2(1-t^2)^{C_2}}.$$

Further,

$$\int_{x+\varepsilon_n}^{1}\frac{dt}{(t-x)^2(1-t^2)^{C_2}} = O(1)\Big(\int_{x+\varepsilon_n}^{1-\varepsilon_n} + \int_{1-\varepsilon_n}^{1}\Big)\frac{dt}{(t-x)^2(1-t)^{C_2}} = O(1)\varepsilon_n^{-(1+C_2)}.$$

Analogously,

$$\int_{-1}^{x-\varepsilon_n}\frac{dt}{(t-x)^2(1-t^2)^{C_2}} = O(1)\varepsilon_n^{-(1+C_2)}.$$

Hence,

$$\sum_{\nu=0}^{n}|I_{\nu 1}| = O(1)\lambda_n^{1/2}\varepsilon_n^{-(1+C_2)/2}.$$

Similarly, the Schwarz inequality, and (3) yield

$$I_{\nu 2}^2 \leqslant \int_{x-\varepsilon_n}^{x+\varepsilon_n} P_n^2(t,x)\varrho(t)\,dt\int_{x-\varepsilon_n}^{x+\varepsilon_n} K_\nu^2(t,x)\varrho(t)\,dt$$

$$\leqslant \int_{x-\varepsilon_n}^{x+\varepsilon_n}\varrho(t)\,dt\int_{-1}^{1} K_\nu^2(t,x)\varrho(t)\,dt = O(1)\int_{x-\varepsilon_n}^{x+\varepsilon_n}\frac{dt}{(1-t^2)^{C_2}}\sum_{k=0}^{\nu}\varphi_k^2(x)$$

$$= O(1)\lambda_\nu\int_{x-\varepsilon_n}^{x+\varepsilon_n}\frac{dt}{\varepsilon_n^{C_2}} = O(1)\lambda_\nu\varepsilon_n^{1-C_2}.$$

Hence,

$$\sum_{\nu=0}^{n} |I_{\nu 2}| \leqslant n^{1/2}\Big(\sum_{\nu=0}^{n} I_{\nu 2}^2\Big)^{1/2} = O(1)\, n^{1/2} \varepsilon_n^{(1-C_2)/2}\Big(\sum_{\nu=0}^{n} \lambda_\nu\Big)^{1/2}.$$

Now let

$$\varepsilon_n = \left(\frac{\lambda_n}{n \sum\limits_{\nu=0}^{n} \lambda_\nu}\right)^{1/2}$$

for n large enough. Then

$$\lambda_n^{1/2} \varepsilon_n^{-(1+C_2)/2} = O(1)\, n^{1/2} \varepsilon_n^{(1-C_2)/2}\Big(\sum_{\nu=0}^{n} \lambda_\nu\Big)^{1/2}.$$

Combining the results we get inequality (4) for A_n. The same estimate also holds for B_n.

2° If $1-2\varepsilon_n < x < 1$, we write $\int\limits_{-1}^{1} P_n K_\nu \varrho\, dt$ as the sum of the integrals

$$\tilde{I}_{\nu 1} = \int\limits_{-1}^{1-3\varepsilon_n} P_n(t,x) K_\nu(t,x) \varrho(t)\, dt, \qquad \tilde{I}_{\nu 2} = \int\limits_{1-3\varepsilon_n}^{1} P_n(t,x) K_\nu(t,x) \varrho(t)\, dt.$$

Reasoning as in 1°, it can easily be observed that

$$\sum_{\nu=0}^{n} |\tilde{I}_{\nu 1}| = O(1) \lambda_n^{1/2} \varepsilon_n^{-(1+C_2)/2}$$

and

$$\sum_{\nu=0}^{n} |\tilde{I}_{\nu 2}| = O(1)\, n^{1/2} \varepsilon_n^{(1-C_2)/2}\Big(\sum_{\nu=0}^{n} \lambda_\nu\Big)^{1/2}.$$

Thus, estimate (4) is established.

In case of $-1 < x < 0$ the proof runs analogously. We may notice that conditions (3) are satisfied by the normalized Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ $(\alpha > -1,\ \beta > -1)$ and $\varrho(x) = (1-x)^\alpha (1+x)^\beta$ with $\lambda_n = n^{2\sigma+2}$, where $\sigma = \max(\alpha, \beta) \geqslant -\frac{1}{2}$ (see [2], p. 418).

References

[1] G. Alexits, *Konvergenzprobleme der Orthogonalreihen*, Berlin 1960.

[2] И. П. Натансон, *Конструктивная теория функций*, Москва-Ленинград 1949.